Dp-finite fields reading seminar
paper I, 3.10
& paper II, §3

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Disclaimer: all of this work is due to Will Johnson and can be found in his first two papers on Dp-finite fields, [J1] and [J2].

1. Dp-finite I, Proposition 10.1

- $\mathbb{M}$ dp-finite but not finite Morley rank.
- M_0 small model
- $\mathcal{P}_n$ poset of type-definable M_0-linear subspaces of $\mathbb{M}^n$.
- $\mathcal{P} := \mathcal{P}^1$ and $\mathcal{P}^+ := \mathcal{P} \setminus \{0\}$.

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Proposition 10.1

1. For each n, $\mathcal{P}_n$ is a bounded lattice.
2. For any small $M \subseteq M_0$, $I_M \in \mathcal{P}$. Thus $\mathcal{P} \supset \{0, \mathbb{M}\}$.
3. If $J \in \mathcal{P}$, $J \neq 0$, then every definable $D \supseteq J$ is heavy.
4. If $J \in \mathcal{P}^+$ is type-definable over some small $M \supseteq M_0$, then $J \supseteq I_M$.
5. If $J \in \mathcal{P}_n$, then $J = J^{00}$.
6. $\mathcal{P}^+$ is sublattice of $\mathcal{P}$. Thus $\mathcal{P}^+$ is a bounded-above lattice.
7. $\mathcal{P}$ has reduced rank r for some $0 < r \leq \mathrm{dp}-\mathrm{rk}(\mathbb{M})$. The reduced rank of $\mathcal{P}^+$ is r. The reduced rank of $\mathcal{P}^n$ is rn.

Proof.

1. By Carol's talk: $G \wedge H = G \cap H = (G \cap H)^{00}$.
2. I_M is neither 0 nor $\mathbb{M}$ and is M_0-linear.
3. Suppose $0 \neq J \subseteq D$. Re-scaling if necessary, $1 \in J$. Thus $M_0 \subseteq J$. Heaviness goes up.
4. Now follows from Corollary 6.19 – Silvain's talk.
5. From Corollary 8.7 – Carol's talk.
6. Only have to show that $\mathcal{P}^+$ is closed under meet (i.e. intersection) since for $J_1, J_2 \in \mathcal{P}^+$, choose one small model $M \supseteq M_0$ over which both are type-definable. Then $J_1 \wedge J_2 = J_1 \cap J_2$ is type-definable over M and M_0-linear. By (4), $I_M \leq J_1 \cap J_2$ and we already know $I_M \neq 0$.

□

Recall: the reduced rank $\mathrm{rk}_0(P)$ of a modular lattice $(P, \leq)$ is supremum of $n \in \mathbb{N} \cup \{\infty\}$ such that a strict n-cube exists in P, i.e. $\exists$ embedding $\mathcal{P}ow([n]) \longrightarrow P$. For $a \leq b$, the rank $\mathrm{rk}_\perp(a/b)$ is supremum of $n \in \mathbb{N} \cup \{\infty\}$ such that a strict n-cube exists in $[b, a]$ with bottom b.

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Proof.

7. Simply denote $r = \mathrm{rk}_0(\mathcal{P})$. By 9.31 (and 8.1), $r \leq n$. Clearly $0 < \mathrm{rk}_0(\mathcal{P}^+) \leq \mathrm{rk}_0(\mathcal{P}) = r \leq n$. The '$<$' is from (2). Suppose that $\mathrm{rk}_0(\mathcal{P}^+) < \mathrm{rk}_0(\mathcal{P})$. Then there is a strict r-cube in $\mathcal{P}$ which does not lie in $\mathcal{P}^+$, so it must include 0, in fact as the bottom of the cube, i.e. as the image of $\emptyset \in \mathcal{P}ow([r])$. In particular $r \leq \mathrm{rk}_\perp(\mathcal{P})$. But also (more quirkily), $\mathrm{rk}_\perp(\mathcal{P}) \leq 1$ by (6)! Therefore $r \leq 1 \leq \mathrm{rk}_0(\mathcal{P}^+)$ – contradiction. So $\mathrm{rk}_0(\mathcal{P}^+) = r$ also.

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Consider $\mathcal{P}_n$, and for each $i \in \{1, \dots, n\}$ let $J_i := 0^{\oplus(i-1)} \oplus \mathbb{M} \oplus 0^{\oplus(n-i)}$. Then $J_i \in \mathcal{P}_n$ and the collection $(J_i)_i$ is independent (= relatively independent over 0), i.e.

$$0 = J_i \wedge \bigvee_{j \neq i} J_j,$$

for each i. Also $\bigvee_i J_i = \mathbb{M}^n$. Since rk_0 is sub-additive and $(J_i)_i$ is independent, by 9.25 we have

$$\mathrm{rk}_0(\mathbb{M}^n) = \sum_i \mathrm{rk}_0(J_i) = \sum_i r = rn,$$

since $\mathcal{P} \longrightarrow [0, J_i]$, $X \longmapsto 0^{\oplus(i-1)} \oplus X \oplus 0^{\oplus(n-i)}$ is isomorphism. □

2. Dp-finite II, Section 3

K small model. Symmetries are invertible!

An affine symmetry

$$\begin{aligned} f : \mathbb{K} &\longrightarrow \mathbb{K} \\ x &\longmapsto ax + b \end{aligned}$$

is *K-deformation* if for every K-definable heavy set X the intersection

$$X \cap f^{-1}(X)$$

is heavy.

Roughly: K-deformations are thought of as K-infinitesimally close to the identity.

- $x \longmapsto x + \varepsilon$ is K-deformation iff ϵ is K-infinitesimal (sense check!).
- All affine symmetries preserve heaviness.
- Set of K-deformations closed under compositional conjugation by K-definable affine symmetries, i.e. for f, g K-definable affine symmetries, if f is a K-deformation then so is $g^{-1} \circ f \circ g$.

The third of the above points shows the 'normal' part of the statement — to be proved later — that the K-deformations form a normal subgroup of the group of K-definable affine symmetries.

Definition

Let $X \subseteq \mathbb{K}$ be K-definable. An affine symmetry f *K-displaces* X if

$$x \in X \wedge x \in K \implies f(x) \notin X.$$

Writing $X(K)$ for the K-points of X, this says that $f(X(K)) \cap X = \emptyset$.

Lemma (Lemma 3.5)

Let $K \preceq K'$. Let $f = aX + b$ and $f' = a'X + b'$ be two affine symmetries of $\mathbb{K}$. Suppose $\mathrm{tp}(a'b'/K')$ is an heir of $\mathrm{tp}(ab/K)$.

1. *If f is a K-deformation, then f' is a K'-deformation.*
2. *If $X \subseteq \mathbb{K}$ is K-definable and is K-displaced by f, then X is K'-displaced by f'.*

Proof.

Reduction: since $ab \equiv_K a'b'$, can assume $(a, b, f) = (a', b', f')$.

1. Suppose f not a K'-deformation. There exists a K'-definable heavy set X such that $X \cap f^{-1}(X)$ is not heavy. Suppose X is defined by $\varphi(x, c')$ for some (tuple) $c' \in K'$, i.e. $X = \varphi(\mathbb{K}, c')$. Re-writing: $\varphi(\mathbb{K}, c')$ is heavy and $\varphi(\mathbb{K}, c') \cap f^{-1}(\varphi(\mathbb{K}, c'))$ is light. Viewing $\varphi(x, y)$ as fixed, this is a Kab-definable property of c'. By the law of inheritance, there exists $c \in K$ with the same property, i.e. $\varphi(\mathbb{K}, c)$ is heavy and $\varphi(\mathbb{K}, c) \cap f^{-1}(\varphi(\mathbb{K}, c))$ is light. Therefore f is not a K-deformation.

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2. Suppose X not K'-displaced by f. Then there exists $c' \in K'$ such that $c' \in X$ and $f(c') \in X$. Again, these latter two conditions on c' are actually Kab-definable. By the law of inheritance, there exists $c \in K$ with the same two properties. Thus f fails to K-displace X.

□

From now on, K small.

Lemma (Lemma 3.6)

Suppose K defines a critical coordinate configuration. Let f be a K-deformation. Let X be K-definable and K-displaced by f. Then f is light.

Proof.

Write $f = aX + b$. Build sequence $(a_i, b_i, K_i)_{i<\omega}$ such that

1. $(K_i)_i$ increasing elementary chain, with $K = K_0$,
2. $a_i, b_i \in K_{i+1}$, for all i,
3. $\mathrm{tp}(a_i b_i / K_i)$ is an heir of $\mathrm{tp}(ab/K)$, for all i.

Define $f_i := a_i X + b_i$. By previous, f_i is a K_i-deformation and X is K_i-displaced by f_i.
For $\alpha \in \{0, 1\}^{<\omega}$, define X_α recursively as follows:

- $X_\emptyset = X_{\{\}} = X$,
- $X_{\alpha 0} := \{x \in X_\alpha \mid f_n(x) \notin X\}$,
- $X_{\alpha 1} := \{x \in X_\alpha \mid f_n(x) \in X\}$,

where $\mathrm{length}(\alpha) = n$. NIP $\implies$ not all X_α non-empty. □

Proof.

Suppose $X = X_\emptyset$ is heavy.

Claim

X_α heavy $\implies$ $X_{\alpha 1}$ heavy.

Proof of claim. Since f_n is a K_n-deformation and X_α is heavy and K_n-definable, $X_\alpha \cap f_n^{-1}(X_\alpha)$ is heavy. Then $X_{\alpha 1} = X_\alpha \cap f_n^{-1}(X) \supseteq X_\alpha \cap f_n^{-1}(X_\alpha)$ is also heavy. $\square_{\text{claim}}$

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Proof of claim. Note that X_α is K_n-definable and $X_\alpha(K_n) \subseteq X(K_n)$. Since X is K_n-displaced by f_n we have

$$x \in X(K_n) \implies f_n(x) \notin X(K_n).$$

Therefore $X_\alpha(K_n) \subseteq X_{\alpha 0}$. Heaviness goes up, by 4.22 from paper I, and so $X_{\alpha 0}$ is heavy. $\square_{\text{claim}}$

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Therefore $X_\alpha(K_n) \subseteq X_{\alpha 0}$. Heaviness goes up, by 4.22 from paper I, and so $X_{\alpha 0}$ is heavy. $\square_{\text{claim}}$
Therefore all X_α are heavy, so non-empty. Contradiction. Therefore X is light. $\square$

Theorem (Theorem 3.10)

The K-deformations form a subgroup of the K-definable affine symmetries of $\mathbb{K}$.

Proof.

Let f_1, f_2 be two K-deformations. We must show that $f_1 \circ f_2^{-1}$ is a K-deformation. Let $K' \succeq K$ be a small model over which f_1, f_2 are both definable. Let $K'' \succeq K$ be a small model defining a critical coordinate configuration. *Move K'' over K so that $\mathrm{tp}(K''/K')$ is finitely satisfiable in K.*

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An element $\mu \in \mathbb{K}^\times$ is a *multiplicative K-infinitesimal* if the map

$$x \longmapsto \mu x$$

is a K-deformation. Denote by U_K the set of multiplicative K-infinitesimals.

Theorem (Theorem 3.12)

1. *$U_K \leq K^\times$*
2. *U_K type-definable over K*
3. *If μ is multiplicative infinisimal then $\mu - 1$ is additive infinitesimal.*
4. *Let $G \leq K^\times$ be type-definable over K. Suppose that for all K-definable $D \supseteq G$, D is heavy. Then $U_K \leq G$.*

Proof.

1. 3.10!
2. Recall that heaviness is definable in families. Now, for fixed K-definable heavy X, the set

$$\{\mu \in \mathbb{K}^\times \mid X \cap (\mu X) \text{ is heavy}\}$$

   is K-definable. The intersection of such sets (for all such X) is exactly the group of multiplicative K-infinitesimals.
3. Write $f = \mu X$ and $g = X + 1$. Since f is K-deformation and g is a K-definable affine symmetry, as we already saw above, the map $g^{-1} \circ f \circ g$ is a K-deformation. By 3.10, the composition $g^{-1} \circ f \circ g \circ f^{-1}$ (in fact it's a commutator!) is a K-deformation. But this commutator is the affine symmetry $X + (\mu - 1)$. Therefore $\mu - 1$ is an additive K-infinitesimal.
4. Suppose G is such a subgroup with the stated property. Then $G = \bigcap\{D \cdot D^{-1} \mid D \text{ is } K\text{-definable and } D \supseteq G\}$ – using compactness for the non-obvious direction. Let $\mu \in U_K$. Then $D \cap (\mu D)$ is heavy, so nonempty. Therefore $\mu \in D \cdot D^{-1}$ for all D. So $\mu \in G$.

□

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