Section 5: RIDDLES ANSWERED

Recall (from last week) our current aim:

unstable dp-finite field

Prop 5.6: Let K be a small model defining a critical coordinate configuration.
Let $X \subseteq \mathbb{K}$ be a heavy K-def'ble set.
Then there ex. $a \in K$ s.th. for all $\varepsilon \in I_K$, we have

$\mathbb{K} \succ K$ suff. sat.

$$\text{dp-rk}(\varepsilon/K) \geq \rho \Rightarrow a + \varepsilon \in X.$$

critical rank

This generalizes the following fact from the dp-minimal world:

[Johnson, On dp-minimal fields, Prop 4.12]
If X is infinite, K-def'ble, then there is $a \in K$ s.th. $a + I_K \subseteq X$.

same as full dp-rank.

Prop 5.6 will be used to show
(A) $\rho = \text{dp-rank}(\mathbb{K})$
$\rightsquigarrow$ heavy = full dp-rank

NEXT TIME

(B) The canonical topology is a field topology.

In analogy to the use of the above fact in the dp-minimal world.

RECALL [An infinite def.ble set Q is called <u>quasi-minimal</u> if $\text{dp-rk}(D) \in \{0, \text{dp-rk}(Q)\}$ for all $D \subseteq Q$ def.ble.

also called "rank-minimal"

<u>Thm 5.1</u>: Let $Q_1, \ldots, Q_n$ be quasi-minimal and let $P \subseteq Q_1 \times \ldots \times Q_n$ have full rank, i.e.

$$\text{dp-rk}(P) = \text{dp-rk}(Q_1 \times \ldots \times Q_n).$$

Then there are smaller quasi-minimal $Q_i' \subseteq Q_i$ s.th.

- $\text{dp-rk}((Q_1' \times \ldots \times Q_n') \cap P) = \text{dp-rk}(Q_1 \times \ldots \times Q_n)$
- $\text{dp-rk}((Q_1' \times \ldots \times Q_n') \setminus P) < \text{dp-rk}(Q_1 \times \ldots \times Q_n)$

<u>Pf</u>: If $n=1$, $Q_1' = P$. Then P is infinite and $\text{dp-rk}(Q_1') = \text{dp-rk}(Q_1)$ by quasi-min.

$\Rightarrow$ If $n>1$, then P is broad in $Q_1 \times \ldots \times Q_n$ [dp I, 3.23: because P has full rank]

RECALL

Definition: Let $X_1, \dots, X_n \subseteq \mathbb{M}$ be def.ble infinite subsets, $Y \subseteq X_1 \times \dots \times X_n$.
We say that Y is broad (in $\prod_{i=1}^n X_i$) if for every $m \in \omega$ ex. $S_1, \dots, S_n$, $|S_i| = m$ and $\prod_{i=1}^n S_i \subseteq Y$.

broad = contains arb. large boxes

Moreover, if $Y \subseteq X_1 \times \dots \times X_n$ is broad and the X_i's are quasimin, then
$$\text{dp-rk}(Y) = \text{dp-rk}(X_1 \times \dots \times X_n)$$

dp I, 3.10
$\Rightarrow$ ex. inf. def.ble subsets $Q_i' \subseteq Q_i$ s.th.
$$H := \prod Q_i' \setminus P$$
is a "hyperplane", i.e. for every $b \in Q_n'$ the set
$$\{(a_1, \dots, a_{n-1}) \in Q_1' \times \dots \times Q_{n-1}' : (\bar{a}, b) \notin P\}$$
is $\underbrace{\text{narrow}}_{\text{not broad}}$ in $Q_1' \times \dots \times Q_n'$.

dp I, 3.8(1) $\Rightarrow$ H is narrow in $Q_1' \times \dots \times Q_n'$

Now,
$$\text{dp-rk}((\prod Q_i') \setminus P) = \text{dp-rk}(H) \underset{\text{dp I, 3.23}}{<} \text{dp-rk}(\prod Q_i')$$

$\Rightarrow \quad \text{dp-rk}\left(\left(\prod Q_i'\right) \cap P\right) = \text{dp-rk}\left(\prod Q_i'\right)$

On the other hand

$$\text{dp-rk}(Q_i') = \text{dp-rk}(Q_i)$$

so

$$\text{dp-rk}\left(\prod Q_i'\right) = \text{dp-rk}\left(\prod Q_i\right) \qquad \boxtimes$$

Def: • A coordinate configuration on $\mathbb{M}$ is a tuple $(X_1, \dots, X_n, Y)$ s.th

- $X_1, \dots, X_n \subseteq \mathbb{M}$ are rank-minimal, defble
- $Y \subseteq X_1 \times \dots \times X_n$ is broad and the map

$$Y \to T \subseteq \mathbb{M} \quad \text{target}$$
$$(x_1, \dots, x_n) \mapsto x_1 + \dots + x_n$$

has finite fibres.

• The critical rank of $\mathbb{M}$ is the max. rank of a coord. configuration on K

"$\text{dp-rk}(X_1 \times \dots \times X_n) = \text{dp-rk}(Y) = \text{dp-rk}(T)$"

RECALL

- If $(X_1, \dots, X_n, Y)$ is a critical coord configuration, then we call $S \subseteq M$ heavy if there is $c \in M$ s.th. $\text{dp-rk}(T \cap (S+c)) = \text{dp-rk}(T)$.
- We call T critical if it is the target of a critical coord. conf.

Note: If T is critical, then $\text{dp-rk}(T) = \rho$.
Any critical set is trivially heavy (set $S = T$, $c = 0$)

Cor. 5.2: Let $(Q_1, \dots, Q_n, P)$ be a coord. conf. of rank r. Then, there is a coord. conf. $(Q_1', \dots, Q_n', P')$ s.th.

(1.) $(Q_1', \dots, Q_n', P')$ has rank r

(2.) For all i, $Q_i' \subseteq Q_i$

(3.) $P' = P \cap (Q_1' \times \dots \times Q_n')$

(4.) The complement $(Q_1' \times \dots \times Q_n') \setminus P'$ is narrow,
so $\text{dp-rk}((Q_1' \times \dots \times Q_n') \setminus P') < r$.

Pf: Take $Q_i' \subseteq Q_i$ as given by 5.1.
$P' = P \cap (Q_1' \times \dots \times Q_n')$

5.1.

$$\begin{array}{ccc} \text{dp-rk}\left(\left(\prod Q_i'\right) \setminus P\right) & < & \text{dp-rk}\left(\prod Q_i'\right) \\ \parallel & & \parallel \\ \text{dp-rk}\left(\left(\prod Q_i'\right) \setminus P'\right) & & \text{dp-rk}\left(\prod Q_i\right) \end{array}$$

So in part, $(Q_1', \ldots, Q_n', P')$ is a coord. conf. with narrow complement. $\boxtimes$