Dp-finite II, Section 4

The goal of the section is to prove the following

Proposition 4.5. *Let $\mathbb{M}$ be a dp-finite monster model. Let M be a small substructure. Given $a, b \in \mathbb{M}$, we can find $a' \equiv_M a$ and $b' \equiv_M b$ such that* $\text{dp-rk}(a'b'/M) = \text{dp-rk}(a'/M) + \text{dp-rk}(b'/M)$ *and* $\text{tp}(b'/Ma')$ *is finitely satisfiable in M.*

This is going to be used (uniquely!) to show

Proposition 5.6. *Let K be a small model defining a critical coordinate configuration. Let $X \subseteq \mathbb{K}$ be a heavy K-definable set. Then there is $a \in K$ such that for any $\epsilon \in I_K$,*

$$\boxed{\text{dp-rk}(\epsilon/K) \geq \rho} \implies a + \epsilon \in X.$$

$\mathbb{K}$ unstable suf. sat, dp-finite field.

X is infinite K-definable $\Rightarrow \exists a \in X(K)$ s.t. $\underbrace{a + I_K \subseteq X.}$

$$a + \epsilon \in X$$

- ρ is the critical rank.

Proposition 4.5. *Let $\mathbb{M}$ be a dp-finite monster model. Let M be a small substructure. Given $a, b \in \mathbb{M}$, we can find $a' \equiv_M a$ and $b' \equiv_M b$ such that* $\text{dp-rk}(a'b'/M) = \text{dp-rk}(a'/M) + \text{dp-rk}(b'/M)$ *and* $\text{tp}(b'/Ma')$ *is finitely satisfiable in M.*

We will need the following lemma:

Lemma 4.4. *Let $\mathbb{M}$ be a monster model and $M \preceq M' \preceq \mathbb{M}$ be two small submodels. Let $\Sigma(x)$ be a partial type over M with* $\text{dp-rk}(\Sigma(x)) \geq r$. *Then there is an ict-pattern $\{\phi(x; b_{ij})\}_{i<r, j\in\mathbb{Z}}$ and witnesses $\{a_\eta\}_{\eta: r\to\mathbb{Z}}$ such that*

- *The b_{ij} are mutually indiscernible over M'.*
- $\text{tp}(a_\eta/M')$ *is independent of η.* ∥ $a_\eta \models \Sigma(x)$
- $\text{tp}(a_\eta/M')$ *and* $\text{tp}(b_{ij}/M')$ *are finitely satisfiable in M.*

Proof 4.5 (assuming 4.4): set

$r = \text{dp-rk}(a/M)$ $\qquad$ $s = \text{dp-rk}(b/M)$.

$\Downarrow$

there is ict-pattern $\{\varphi(x, c_{ij})\}_{i<r, j\in\mathbb{Z}}$

and $\{a_\eta\}_{\eta: r\to\mathbb{Z}}$ each $\text{tp}(a_\eta/M) = \text{tp}(a/M)$.

Let M' be small containing M, a_η, c_{ij}.

Apply 4.4 to $M \preceq M' \preceq \mathbb{M}$ and $\text{tp}(b/M)$. We obtain $\{\psi(y, d_{ij})\}_{i<s, j\in\mathbb{Z}}$

$\{b_\eta\}_{\eta: s\to\mathbb{Z}}$ s.t.:

- d_{ij} are mutually ind. over M'
- $tp(b_\eta/M')$ is indep. of η.
- $tp(b_\eta/M')$ and $tp(d_{ij}/M')$ are fin. satisfiable over M.

Claim: for any $\eta: r \to \mathbb{Z}$, $\eta': s \to \mathbb{Z}$

$$a_\eta b_{\eta'} \equiv_M a_{\vec{0}} b_{\eta'} \equiv_M a_{\vec{0}} b_{\vec{0}}$$

Proof of Claim:

We have $a_\eta \equiv_M a \equiv_M a_{\vec{0}}$.

We use $tp(b_\eta / M a_\eta a_{\vec{0}})$ is fin. satis. in M

(since $tp(b_\eta/M')$ is fin. satis. in M).

$\Rightarrow$ $a_\eta b_{\eta'} \equiv_M a_{\vec{0}} b_{\eta'}$

If not, $\models \theta(a_\eta, b_{\eta'}, m) \wedge \neg\theta(a_{\vec{0}}, b_{\eta'}, m)$

$\Rightarrow \exists m' \in M$ s.t.

$$\models \theta(a_\eta, m', m) \wedge \neg\theta(a_{\vec{0}}, m', m).$$

contradicts that $a_\eta \equiv_M a_{\vec{0}}$.

To show $a_{\vec{0}} b_{\eta'} \equiv_M a_{\vec{0}} b_{\vec{0}}$, thus follows since

$$\mathrm{tp}(b_\eta / M') = \mathrm{tp}(b_{\vec{0}} / M'),\ \text{and}$$

$$M a_0 a_\eta \subseteq M'$$

One concludes noting that the following is ict-pattern in $\mathrm{tp}(a_{\vec{0}} b_{\vec{0}} / M)$:

$$\begin{array}{lll}
\cdots \varphi(x, c_{1,1}) & \varphi(x, c_{1,2}) & \cdots \\
\cdots \varphi(x, c_{2,1}) & \varphi(x, c_{2,2}) & \cdots \\
\vdots & & \\
\cdots \varphi(x, c_{r,1}) & \varphi(x, c_{r,2}) & \cdots \\
\cdots \psi(y, d_{1,1}) & \cdots & \\
\vdots & & \\
\cdots \psi(y, d_{s,1}) & \cdots &
\end{array}$$

This is witnessed by $\{a_\eta b_{\eta'}\}_{r+s \to \mathbb{Z}}$.

$\Rightarrow$ $\text{dp-rk}(a_{\vec{0}} b_{\vec{0}} / M) = r+s$.

Proposition 4.5. *Let $\mathbb{M}$ be a dp-finite monster model. Let M be a small substructure. Given $a, b \in \mathbb{M}$, we can find $a' \equiv_M a$ and $b' \equiv_M b$ such that* $\text{dp-rk}(a'b'/M) = \text{dp-rk}(a'/M) + \text{dp-rk}(b'/M)$ *and* $\mathrm{tp}(b'/Ma')$ *is finitely satisfiable in M.*

Note that $\mathrm{tp}(b_{\vec{0}} / M a_{\vec{0}}) \subseteq \mathrm{tp}(b_{\vec{0}} / M')$ which is fin. sat. in M.

Lemma 4.4 uses 3 lemmas.

Lemma 4.1. *Let M be an $|A|^+$-saturated structure for some $A \subseteq M$. Suppose $\{\phi(x; b_{ij})\}_{i<r, j\in\mathbb{Z}}$ is an ict-pattern of depth r in some partial type $\Sigma(x)$ over A. Then there exists $\{b'_{ij}\}_{i<r,j\in\mathbb{Z}}$ and $\{a_\eta\}_{\eta:r\to\mathbb{Z}}$ such that*

(1) • $\{\phi(x; b'_{ij})\}$ *is an ict-pattern of depth r in $\Sigma(x)$.* ✓

(2) • *The a_η are witnesses:*

$$\left| \begin{array}{l} M \models \Sigma(a_\eta) \checkmark \\ M \models \phi(a_\eta; b'_{ij}) \iff j = \eta(i). \end{array} \right]$$

(3) • *The array $\{b'_{ij}\}$ is mutually indiscernible over A.* ✓

(4) • *The type $\mathrm{tp}(a_\eta/A)$ is independent of η.* !!

Proof: (1)-(3) standard yoga of ict-patterns and indiscernibles.

Assume we have $\{a_\eta\}_{\eta:r\to\mathbb{Z}}$ and $\{b_{ij}\}_{i<r,j\in\mathbb{Z}}$ satisfying (1)-(3).

By saturation, we can work in a larger model (since the data of the problem is of small cardinality).

WLOG, we may assume M is $|A|^+$ strongly homogeneous.

Fix $a_{\vec{0}}$, $a_{\vec{0}} \models \Sigma(x)$

$$M \models \varphi(a_{\vec{0}}, b_{ij}) \iff j = 0$$

For $\eta : r \to \mathbb{Z}$, let $\sigma_\eta \in \mathrm{Aut}(M/A)$

s.t. $b_{ij} \longmapsto b_{i, j+\eta(i)}$

(this exists, since $\{b_{ij}\}$ mutually ind. + strong homogeneity)

define $a'_\eta = \sigma_\eta(a_{\vec{0}})$

$a'_{\vec{0}} = a_{\vec{0}}$.

$$M \models \varphi(a_{\vec{0}}, b_{ij}) \iff j = 0$$
$$\Updownarrow$$
$$M \models \varphi(\sigma_\eta(a_{\vec{0}}), \sigma_\eta(b_{ij})) \iff j = 0$$
$$\Updownarrow$$
$$M \models \varphi(a'_\eta, b_{i,j+\eta(i)}) \iff j = \eta(i).$$

$\blacksquare$

Lemma 4.2. *Let M be a structure and $A \subseteq M$ be a subset. Suppose M is $|A|^+$-saturated. Suppose that* in some reduct M_0 *of* M,

- *There is a partial type* $\Sigma(x)$ *over* A.
- dp-rk$(\Sigma(x)) \geq r$ *for some finite* r.

Then in M *there is an ict-pattern* $\{\phi(x; b_{ij})\}_{i<r, j\in\mathbb{Z}}$ *and witnesses* $\{a_\eta\}_{\eta:r\to\mathbb{Z}}$ *realizing* $\Sigma(x)$, *such that*

- *The formula* $\phi(x; y)$ *comes from the reduct language.*
- *The array* b_{ij} *is mutually indiscernible over* A, in the expansion.
- *The type of* a_η *over* A in the expansion *is independent of* η.

Proof: The reduct M_0 is also $|A|^+$-sat.
there is an ict-pattern $\{\varphi(x, b_{ij})\}_{\substack{i<r \\ j\in\mathbb{Z}}}$
with φ a formula in the reduct
language. We apply 4.1 in the
expansion. $\square$

Lemma 4.3. *Let $\mathbb{M}$ be a monster model and $M \preceq M' \preceq \mathbb{M}$ be two small submodels. There is a small submodel N containing M with the following properties:*

- $\mathrm{tp}(N/M')$ *is finitely satisfiable in* M.
- *The expansion of N by all externally M'-definable sets is an $|M'|^+$-saturated structure.*

Proof: Consider the theory of the pair (M', M), and let $(M', M) \preceq (N', N)$ $|M'|^+$-saturated. (Note: $M \preceq M' \preceq N'$).

We take N as in (N', N).

Suppose $\varphi(x, m') \in \mathrm{tp}(N/M')$.

$(N', N) \models \exists x\, \varphi(x, m') \wedge P(x)$

$(M', M) \models \exists x\, \varphi(x, m') \wedge P(x)$

$\Rightarrow$ this shows $\mathrm{tp}(N/M')$ is fin. sat. in M.

For the 2nd point, we note that every M'-ext. def. set is definable in (N', N):

$\{a \in N : \varphi(a, m')\} = \{a \in N' : \varphi(a, m') \wedge P(a)\}$

$m' \in M'$.

Lemma 4.4. *Let $\mathbb{M}$ be a monster model and $M \preceq M' \preceq \mathbb{M}$ be two small submodels. Let $\Sigma(x)$ be a partial type over M with* dp-rk$(\Sigma(x)) \geq r$. *Then there is an ict-pattern $\{\phi(x; b_{ij})\}_{i<r, j\in\mathbb{Z}}$ and witnesses $\{a_\eta\}_{\eta:r\to\mathbb{Z}}$ such that*

- *The b_{ij} are mutually indiscernible over M'.*
- tp(a_η/M') *is independent of η.*
- tp(a_η/M') *and* tp(b_{ij}/M') *are finitely satisfiable in M.*

Proof: Take N as in lemma 4.3.

- $tp(N/M')$ is fin. sat. in M
- $N^{M'}_{ext}$ is $|M'|$- saturated.

We apply lemma 4.2 to $\Sigma(x)$ and $\{\varphi(x, b'_{ij})\}_{i<r, j\in\mathbb{Z}}$ (witnessing dp-rk$(\Sigma(x)) \geq r$)

to find an ict-pattern

$\{\varphi(x, b_{ij})\}_{i<r, j\in\mathbb{Z}}$, $\{a_\eta\}_{\eta: r\to\mathbb{Z}}$

s.t.:

- $\{b_{ij}\}$ are mut. ind. in $N^{M'}_{ext}$.

$\Rightarrow \{b_{ij}\}$ are mut. ind. in M'.

- $tp_{ext}(a_\eta/M)$ is indep. of η.

(if $tp(a_\eta/M') \neq tp(a_{\eta'}/M')$

$\models \underbrace{\varphi(a_\eta, m')} \wedge \neg\varphi(a_{\eta'}, m')$

$N \models R_{\varphi(x,m')}(a_\eta) \wedge R_{\varphi(x,m')}(a_{\eta'})$)

contr..

$\Rightarrow tp(a_\eta/M')$ is indp. of η.

- $tp(a_\eta/M')$ and $tp(b_{ij}/M')$ are fin. sat. in M

since $a_\eta, b_{ij} \in N$

$tp(N/M')$ is fin. sat. in M. $\square$