Section 5: RIDDLES ANSWERED (ctd.)

Recall (from last week) our current aim:

unstable dp-finite field

Prop 5.6: Let K be a small model defining a critical coordinate configuration. Let $X \subseteq \mathbb{K}$ be a heavy K-definable set. Then there ex. $a \in K$ s.th. for all $\varepsilon \in I_K$, we have

$\mathbb{K} \succ K$ suff. sat.

$$\underbrace{\text{dp-rk}(\varepsilon/K) \geq \rho}_{\text{critical rank}} \Rightarrow a + \varepsilon \in X.$$

This generalizes the following fact from the dp-minimal world:

[Johnson, On dp-minimal fields, Prop 4.12]
If X is infinite K-def.ble, then there is $a \in K$ s.th. $a + I_K \subseteq X$.

same as full dp-rank.

Prop 5.6 will be used to show

(A) $\rho = \text{dp-rank}(\mathbb{K})$ — TODAY!

$\rightsquigarrow$ heavy = full dp-rank

(B) The canonical topology is a field topology.

In analogy to the use of the above fact in the dp-minimal world.

LAST TIME:

RECALL

Def: • A coordinate configuration on $\mathbb{M}$ is a tuple $(X_1, \ldots, X_n, Y)$ s.th

• $X_1, \ldots, X_n \subseteq \mathbb{M}$ are rank-minimal, def.ble

• $Y \subseteq X_1 \times \ldots \times X_n$ is broad and the map $Y \to T \subseteq \mathbb{M}$ target

$(x_1, \ldots, x_n) \longmapsto x_1 + \ldots + x_n$

has finite fibres.

• The critical rank of $\mathbb{M}$ is the max. $\mathbb{M}$ rank of a coord. configuration on K

"$\text{dp-rk}(X_1 \times \ldots \times X_n) = \text{dp-rk}(Y) = \text{dp-rk}(T)$"

• critical set: target of a crit. coord. conf.

Cor. 5.2: Let $(Q_1, \ldots, Q_n, P)$ be a coord. conf. of rank r. Then, there is a coord. conf. $(Q_1', \ldots, Q_n', P')$ s.th.

(1.) $(Q_1', \ldots, Q_n', P')$ has rank r

(2.) for all i, $Q_i' \subseteq Q_i$

(2.) for all i, $Q_i' \subseteq Q_i$

(3.) $P' = P \cap (Q_1' \times \ldots \times Q_n')$

(4.) The complement $(Q_1' \times \ldots \times Q_n') \setminus P$ is narrow,

so $\text{dp-rk}((Q_1' \times \ldots \times Q_n') \setminus P') < r$.

§5.1 Near interior.

Lemma 5.3: Let $X \subseteq \mathbb{K}$ be a heavy def.ble set. Then, there is a critical set W and a $\delta \in \mathbb{K}$ s.th.

$$\text{dp-rk}(\{x,y\} \in W \times W : x - y \notin X + \delta\}) < \text{dp-rk}(W \times W) = 2 \cdot \delta.$$

IDEA: $x - y \in X + \delta$ for "almost all" $(x,y) \in W \times W$.

RECALL

If $(X_1, \ldots, X_n, Y)$ is a critical coord configuration, then we call $S \subseteq \mathbb{K}$ heavy if there is $c \in \mathbb{K}$ s.th. $\text{dp-rk}(T \cap (S + c)) = \text{dp-rk}(T)$.

$\rightarrow$ critical sets are trivially heavy.

Proof: Let $(Q_1, \dots, Q_n, P)$ be a crit. coord. conf

5.2 $\Rightarrow$ wma $\quad$ dp-rk $((\prod Q_i) \setminus P) < \rho$

dp-rk $((\prod Q_i) \cap P) = \rho$

RECALL

[dp I, 4.15] Y critical, $Q_1, \dots, Q_n$ quasi-min.
then ex. $\delta \in \mathbb{K}$ s.th.
$\{ \bar{x} \in \prod Q_i : x_1 + \dots + x_n \in Y + \delta \}$
is broad in $\prod Q_i$.

Take $Y \subseteq X$ critical

(★) By def. of heavy, there is Y critical s.th.
dp-rk $((X+a) \cap Y) =$ dp-rk(Y)
$\Rightarrow$ $(X+a) \cap Y$ critical
$\Rightarrow$ $X \cap (Y-a)$ critical.

$\Rightarrow$ ex. $\delta \in \mathbb{K}$ s.th.

(∗) $\{ (\bar{x}, \bar{y}) \in \prod Q_i \times \prod Q_i : \sum x_i - \sum y_i \in \underbrace{Y + \delta}_{\subseteq X + \delta} \}$

is broad in $\prod Q_i \times \prod Q_i$

Why? Q_i rank-min $\Rightarrow$ $-Q_i$

4.15 $\Rightarrow$ ex. $\delta \in \mathbb{K}$ s.th.

$\{(\bar{x}, \bar{z}) \in \prod Q_i \times \prod(-Q_i) : \sum x_i + \sum z_i \in Y + \delta\}$

is broad in $\prod Q_i \times \prod(-Q_i)$ $\Rightarrow$ $(*)$

LAST TIME

Thm 5.1: Let $Q_1, \dots, Q_n$ be quasi-minimal and let $P \subseteq Q_1 \times \dots \times Q_n$ have full rank, i.e.

$$\text{dp-rk}(P) = \text{dp-rk}(Q_1 \times \dots \times Q_n).$$

Then there are smaller ~~quasi~~ rank- minimal $Q_i' \subseteq Q_i$ s.th.

$$\text{dp-rk}((Q_1' \times \dots \times Q_n') \cap P) = \text{dp-rk}(Q_1 \times \dots \times Q_n)$$

$$\text{dp-rk}((Q_1' \times \dots \times Q_n') \setminus P) < \text{dp-rk}(Q_1 \times \dots \times Q_n)$$

$\Rightarrow$ ex. $Q_i', Q_i'' \subseteq Q_i$ infinite def·ble s.th.

$$A = \{(\bar{x}, \bar{y}) \in \prod Q_i' \times \prod Q_i'' : \sum x_i - \sum y_i \notin X + \delta\}$$

is narrow in $\underline{\prod Q_i' \times \prod Q_i''}$, so

$\text{dp-rk}(A) < 2\rho$.

Define $P' = P \cap \prod Q_i'$ $\quad P'' = P \cap \prod Q_i''$

$\Rightarrow (Q_1', \dots, Q_n', P')$ and $(Q_1'', \dots, Q_n'', P'')$ are crit. coord. conf., say with targets T' and T''.

$B = \{(\bar{x}, \bar{y}) \in P' \times P'' : \sum x_i - \sum y_i \notin X + \delta\}$

$B = \{(\bar{x},\bar{y}) \in P^n \times P^n : \sum x_i - \sum y_i \notin X + \delta\}$

$\subseteq \{(\bar{x},\bar{y}) \in \underbrace{\prod Q_i \times \prod Q_i : \sum x_i - \sum y_i \notin X + \delta\}}_{\text{narrow}}$

so dp-rk $(B) < 2\rho$

$\Rightarrow$ dp-rk$(\{(x,y) \in T' \times T'' : x - y \notin X + \delta\}) < 2\rho$

images of the summing map $\bar{z} \mapsto \sum z_i$

T', T'' critical $\Rightarrow$ heavy

RECALL [dp I, 4.20 (8): X, Y heavy, then so is $X -_\infty Y = \{\delta \in \mathbb{K} : X \cap (Y + \delta) \text{ is heavy}\}$
in part, $X -_\infty Y \neq \emptyset$.

$\Rightarrow$ ex. some $\tau \in \mathbb{K}$ s.th.

$W := T' \cap (T'' + \tau)$ is heavy

[dp-rk$(\{(x,y) \in W \times W : \underbrace{x}_{\in T'} - \underbrace{(y+\tau)}_{\in T''} \notin X + \delta\})$

$< 2\rho$

Now: $\text{dp-rk}(W) \leq \text{dp-rk}(T') = \rho$

W heavy $\Rightarrow \text{dp-rk}(W) \geq \rho$ (*)

$\Rightarrow \text{dp-rk}(W) = \rho$, so W critical [dpI, 4.8]

Indeed:

$$\text{dp-rk}(\{(x,y) \in W \times W : x - y \notin X + \overset{\delta+\tau.}{\overset{\shortparallel}{\delta'}}\}) < \text{dp-rk}(W \times W)$$

☑

"Problem": In 5.3, we have no control over the field of definition of W and δ!

Lemma 5.4: K small model, defining a critical coord. conf. Let X be heavy and K-def-ble. Then there is a K-def-ble critical set W and a $\delta \in K$ s.th.

$$\text{dp-rk}(\{(x,y) \in W \times W : x - y \notin X + \delta\})$$

$< \text{dp-rk}(W \times W)$

$< \text{dp-rk}(W \times W) = 2\rho$

Proof: Take $(Q_1, \ldots, Q_n, P)$ K-def-ble crit. coord. conf. with target T.
Note
$$\text{dp-rk}(T \times T) = 2\rho.$$

CLAIM: If $\{D_b\}$ is a K-def-ble family of subsets of $T \times T$, then
$$\{b : \text{dp-rk}(D_b) = 2\rho\}$$
is K-def-ble.

Proof of claim:

The map
$$S : P \times P \twoheadrightarrow T \times T$$
$$(\bar{x}, \bar{y}) \mapsto (\Sigma x_i, \Sigma y_i)$$
is a surjection with finite fibres.

$$\Rightarrow \text{dp-rk}(D_b) = \text{dp-rk}(S^{-1}(D_b))$$

suffices to prove the set
$$\{b : \text{dp-rk}(\underline{S^{-1}(D_b)}) = \text{dp-rk}(\underline{\Pi Q_i \times \Pi Q_i})\}$$
is K-def-ble.

This follows from (the proof of)

RECALL

[dp I, 3.24]: T NIP, eliminates $\exists^\infty$, $Q_1', \ldots, Q_n'$ quasi-min of finite rank.

$r = \text{dp-rk}(\prod Q_i')$

Given $\{D_b'\}_{b \in Y}$ def-ble family in $\prod Q_i'$, the set

$\{b : \text{dp-rk}(D_b') = \text{dp-rk}(\prod Q_i')\}$

is def-ble.

- full dp-rank iff broad
- broadness needs no (new) parameters

$\boxtimes$ claim

Use 5.3 to find δ_0 and critical W s.th.

$\text{dp-rk}(\{(x,y) \in W \times W : x - y \notin X + \delta_0\} < 2\rho$.

[dp I, 4.9]: Translations of critical are critical

[dp I, 4.18] Heaviness is well-defined

↳ intersection with some translate always has full rank

↳ WMA $\text{dp-rk}(W \cap T) = \rho$.

[dp I, 4.8]: Def-ble subsets of critical sets of full rank are critical.

$\Rightarrow W' = W \cap Y$ is critical and

$\text{dp-rk}(\{(x,y) \in W' \times W' : x - y \notin X + \delta_0\}) < 2\rho$

Write $W' = \varphi(\mathbb{K}, b_0)$ and consider $\mathbb{K}$-definable conditions on b, δ:

(A) $\varphi(\mathbb{K}, b) \subseteq T$

(B) $\text{dp-rk}(\varphi(\mathbb{K}, b)) = \rho$ (by ✿)

(C) $\text{dp-rk}(\{(x,y) \in \varphi(\mathbb{K}, b) \times \varphi(\mathbb{K}, b) : x - y \notin X + \delta\}) < 2\rho$

(by Claim)

So we find $b, \delta \in \mathbb{K}$ satisfying (A)-(C)

$W'' = \varphi(\mathbb{K}, b)$ is critical.

(C) $\Rightarrow$ Lemma holds $\boxtimes$

Recall: I_K = "group of additive K-infinitesimals"
$= \bigcap \{ X - \infty X : X \subseteq \mathbb{K} \text{ heavy \& } K\text{-def'ble} \}$
↑ not necessary (next time)

Prop. 5.6: Let K be a small model defining a critical coord. configuration.
Let $X \subseteq \mathbb{K}$ heavy K-def'ble set.
Then ex. $a \in K$ s.th. for all $\varepsilon \in I_K$,
$\text{dp-rk}(\varepsilon / K) \geq \rho \Rightarrow a + \varepsilon \in X$.

Proof: Use 5.4 to find K-def'ble critical W and $a \in K$ s.th.
$\text{dp-rk}(\{(x,y) \in W \times W : x - y \notin X - a\}) < 2\rho$.

Take $\varepsilon \in I_K$ with $\text{dp-rk}(\varepsilon / K) \geq \rho$.
Then $W(\mathbb{K}) \subseteq D_0 \cup D_1$
$D_0 = \{ x \in W : x + \varepsilon \in W \}$
$D_1 = \{ x \in W : x + \varepsilon \notin W \}$

[dp I, 4.21: X K-def'ble & heavy,
$X = D_0 \dot\cup D_1$
$\Rightarrow$ ex. K-def'ble heavy $Y \subseteq X$ with
$Y(\mathbb{K}) \subseteq D_i$ for some $i \in \{0,1\}$.

Take such Y.

if $i=1$: $x \in Y(K) \Rightarrow x \in D_1 \Rightarrow x+\varepsilon \notin W$
$\Rightarrow x+\varepsilon \notin Y$
i.e. Y is K-displaced by ε

3.6 $\Rightarrow$ Y is light. $\lightning$.

So $i=0$, $Y(K) \subseteq D_0$, and we have

(☼) $(x \in K \wedge x \in Y) \Rightarrow x+\varepsilon \in W$

Note: $\text{dp-rk}(Y) = \rho$
("$\geq$" Y is heavy)

Pablo's talk, 4.5:
Let $\mathbb{M}$ be a dp-finite monster model, $M \preccurlyeq \mathbb{M}$ small. Given $a, b \in \mathbb{M}$, we find $a \equiv_M a'$, $b \equiv_M b'$ s.th.

$$\text{dp-rk}(a'b'/M) = \text{dp-rk}(a'/M) + \text{dp-rk}(b'/M)$$

and

$tp(b'/Ma')$ is fin. sat. in M.

RECALL

[? Why can we assume ε remains fixed?]

Recall: I_K = "group of additive K-infinitesimals"
$= \bigcap \{X - X : X \subseteq \mathbb{K}$ heavy & K-def.ble$\}$
↑not necessary (next time)

Prop. 5.6: Let K be a small model defining a critical coord. configuration.
Let $X \subseteq \mathbb{K}$ heavy K-def.ble set.
Then ex. $a \in K$ s.th. for all $\varepsilon \in I_K$,
$\text{dp-rk}(\varepsilon / K) \geq \rho \Rightarrow a + \varepsilon \in X$.

AIM: $\rho = \text{dp-rk}(\mathbb{K})$

§5.2 The critical rank

Prop 5.7: Let K be a small model defining a critical coord. conf., and I_K = group of K-infinitesimals.
Then $\text{dp-rk}(I_K) \leq \rho$.

Pf: Assume $\text{dp-rk}(I_K) = \rho' > \rho$.

Take $\varepsilon \in I_K$ with $\text{dp-rk}(\varepsilon/K) = \rho'$.

Let Y be a critical, K-def·ble set
$\rightsquigarrow$ Y heavy, $\text{dp-rk}(Y) = \rho$

5.6. $\Rightarrow$ ex. $a \in K$ s.th. $a + \varepsilon \in Y$

Now: $\rho' = \text{dp-rk}(\varepsilon/K) = \text{dp-rk}(a+\varepsilon/K)$ $\lightning$
$\leq \text{dp-rk}(Y) = \rho$ $\boxtimes$

Lemma 5.8: Let K be a small model.
Then $\text{dp-rk}(I_K) = \text{dp-rk}(\mathbb{K})$.

Proof: Take $n = \text{dp-rk}(\mathbb{K})$.

Choose an ICT-pattern $\{\varphi(x, b_{ij})\}_{i<n, j<\omega}$
of depth n in $\mathbb{K}$.
For each $\eta : n \to \omega$, take $a_\eta \in \mathbb{K}$ s.th

$$\models \varphi(a_\eta, b_{ij}) \iff j = \eta(i).$$

Let $K' \succcurlyeq K$ small containing all a_η's.
Let $\varepsilon \in I_{K'} \setminus \{0\}$ $\quad$ $b_{ij} \in K'$

Then $a_\eta \in \varepsilon^{-1} I_{K'}$ for all η.

[dp I, 6.9.3: M small $\Rightarrow$ $I_M \cdot M \subseteq I_M$

So there is an ICT-pattern of depth n in $\varepsilon^{-1} I_{k'}$

$$\begin{aligned} \text{dp-rk}(I_k) &\geq \text{dp-rk}(I_{k'}) \\ &= \text{dp-rk}(\varepsilon^{-1} I_{k'}) \\ &\geq n \end{aligned}$$

since $I_{k'} \subseteq I_k$

$\boxtimes$

Theorem 5.9 (first part)

$\rho = \text{dp-rk}(\mathbb{K})$.

Pf: $\rho \leq \text{dp-rk}(\mathbb{K})$ by definition. $\boxtimes$