Dp-finite fields II, section 8, after Will Johnson

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18 March 2021
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$\mathbb{K}$ an unstable dp-finite field of dp-rank n.

Definition 8.2. Let $G \leq (\mathbb{K}, +)$ be type-definable.

(1) G is *heavy* if every definable set containing G is heavy.

(2) G is *bounded* if for every heavy subgroup $G' \leq \mathbb{K}$, there is a non-zero $a \in \mathbb{K}$ such that $G \leq a \cdot G'$.

(3) A small $K_0 \preceq \mathbb{K}$ is *magic* if whenever $G \leq (\mathbb{K}^m, +)$ is type-definable, then $K_0 \cdot G \subset G$ implies $G = G^{00}$. In other words, all type-definable K_0-vector spaces are connected. That magic fields exist comes from dpI, 8.4 and 8.7.

The aim of this talk is to show

Corollary 8.9. *If K is a small submodel, then I_K is bounded.*

8.1 – Recall the following : let $G \leq \mathbb{K}$ be type-definable. TFAE:

(1) dp−rk$(G) = n$.
(2) Every definable set $D \supseteq G$ has rank n.
(3) Every definable set $D \supseteq G$ is heavy.
(4) G contains I_K for some small $K \preceq \mathbb{K}$.

Recall also the notion of *strict r-cube* and *reduced rank*. Given a modular lattice M, a strict r-cube is an injective homomorphism from the power set of r to M, with unbounded relative indices. The *base* of the cube is the image of $\emptyset$. The reduced rank $\mathrm{rk}_0(M)$ is the maximum r such that a strict r-cube exists. If $a \geq b$, then $\mathrm{rk}_0(a/b)$ is the reduced rank of the sublattice $[b, a]$. If M is a sublattice of subgroups of $(\mathbb{K}, +)$, then we know that $\mathrm{rk}_0(M) \leq n$.

We fix a magic subfield K_0, and let $\Lambda = \Lambda_{K_0}$ be the lattice of type-definable K_0-linear subspaces of $\mathbb{K}^1$. So, if $M \in \Lambda$, then $M = M^{00}$, by magic. Let $r = \mathrm{rk}_0(\Lambda)$. A K_0-*pedestal* (or simply *pedestal* since K_0 is fixed) is an element of $\Lambda^+ := \Lambda \setminus \{0\}$ which is the basis of a strict r-cube.

Lemma 8.5. *Let $G \in \Lambda$ be such that $\mathrm{rk}_0(\mathbb{K}/G) = r$. Then G is bounded.*
Proof. Wma $G \neq (0)$. Let H be a heavy subgroup of $\mathbb{K}$, and choose a strict r-cube in $[G, \mathbb{K}] \subseteq \Lambda$, with base J. Let $K \prec \mathbb{K}$ be small and chosen large enough so G and J are type-definable over K, $K \supseteq K_0$ and $I_K \subseteq H$.

Then $I_K \cdot J \subseteq I_K$ (dpI, 10.4.3), so that $I_K \cdot G \subseteq I_K \cdot J \subseteq I_K \subseteq H$. As $I_K \neq (0)$ (dpI, 6.9.1), if $0 \neq \varepsilon \in I_K$, then $\varepsilon \cdot G \subseteq I_K \cdot G \subseteq H$, as desired.

Lemma 8.7. *Let* $\{U_x\}$ *be a* 0*-definable family of basic neighbourhoods. Then there is a* 0*-definable family of basic neighbourhoods* $\{V_x\}$ *satisfying*

$$\exists b \, \forall c \, \exists d \;\; V_b \cdot V_d \subseteq U_c.$$

Proof. Recall that bases of neighbourhoods can be given by $\{X -_\infty X \mid X \text{ heavy}\}$, $\{X \ominus X \mid X \text{ heavy}\}$ or $\{X - X \mid X \text{ heavy}\}$ (5.10).
Fix a (non-zero) pedestal J; choose $K_1 \succeq K_0$ such that J is type-defined over K_1, and let $K_2 \succeq K_1$ be $|K_1|^+$-saturated. Then:

$$I_{K_1} \subseteq J, \quad I_{K_2} \cdot J \subseteq I_{K_2}$$

whence for any $c \in \mathrm{dcl}(K_2)$, $I_{K_2} \cdot I_{K_1} \subseteq I_{K_2} \cdot J \subseteq I_{K_2} \subseteq U_c$ (the last equality is almost by definition of I_{K_2}).

So there are a K_1-definable neighbourhood W_1 and a K_2-definable neighbourhood W_2 such that $W_1 \cdot W_2 \subseteq U_c$. This is by compactness, as each J_{K_i} is an intersection of K_i-definable basic neighbouhoods. Let $\theta(x, z)$ define U_z, and consider the type

$$q(z) = \{\forall y \ \rho(y) \to (\exists x \in (\varphi(\mathbb{K}) \cdot \psi(\mathbb{K}, y)) \setminus \theta(\mathbb{K}, z))\},$$

where $\varphi(x)$ ranges over all $\mathcal{L}(K_1)$-formulas defining basic neighbourhoods, $\psi(x, y)$ over all $\mathcal{L}$-formulas, and $\rho(y)$ is the formula expressing that $\psi(\mathbb{K}, y)$ is a basic neighbourhood. By the above, this type is not realized in K_2, hence is inconsistent. So, there are K_1-definable basic neighbourhoods $X_1, \ldots, X_m$, and a 0-definable family $\{V_y\}$ of basic neighbourhoods such that for all $c \in \mathrm{dcl}(K_2)$, there is $i \in \{1, \ldots, m\}$ and $d \in \mathrm{dcl}(K_2)$ such that $X_i \cdot V_d \subset U_c$. Setting $X = \bigcap_i X_i$, wma $X_i = X$.

So we have

$$\forall c \in \mathrm{dcl}(K_2) \exists d \in \mathrm{dcl}(K_2)\ X \cdot V_d \subseteq U_c.$$

??? Enlarging V_d, wma $X = V_b$ for some $b \in \mathrm{dcl}(K_1) \subseteq \mathrm{dcl}(K_2)$. The property then passes to $\mathbb{K}$: $\forall c\, \exists d\ V_b \cdot V_d \subseteq U_c$.
I don't understand what he is saying. I would do the following: let X be defined by Y_a, where Y_y is a 0-definable family of basic neighbourhoods. Consider the 0-definable family $(Y \cap V)_{e,d}$.
??? I don't see the point of writing $\mathrm{dcl}(K_2)$ if $K_2 \preceq \mathbb{K}$.

Lemma 8.8. *If* $K_1 \preceq K_2 \preceq \mathbb{K}$*, then* $I_{K_1} \cdot I_{K_2} \subseteq I_{K_2}$*.*

Proof. In 8.7, we proved the conclusion under certain hypotheses on K_1 - which we don't have here. Let U be a K_2-definable basic neighbourhood, $U = U_c$, where $\{U_x\}$ is a 0-definable family of basic neighbourhoods and $c \in \text{dcl}(K_2)$. Let V_x be a 0-definable family of basic neighbourhoods as in 8.7, and b such that $\forall x \, \exists y \, V_b \cdot V_y \subseteq U_x$. As $K_1 \preceq \mathbb{K}$, we may take $b \in K_1$, and then there is d such that $V_b \cdot V_d \subseteq U_c$. Moreover we can take $d \in \text{dcl}(K_2)$ because $b, c \in \text{dcl}(K_2)$ and $K_2 \preceq \mathbb{K}$. So we get

$$I_{K_1} \cdot I_{K_2} \subseteq V_b \cdot V_d \subseteq U_b = U,$$

i.e., $I_{K_1} \cdot I_{K_2} \subseteq I_{K_2}$ because U was an arbitrary K_2-definable basic neighbourhood.

Corollary 8.9. *For any small submodel $K \preceq \mathbb{K}$, the group I_K is bounded.*

Proof. Take a non-zero pedestal J, which is type-definable over some small $K_1 \preceq \mathbb{K}$ containing K and K_0. Thus by dpI(10.4), we have $I_{K_1} \subseteq J$, and because J is bounded (8.6), we get I_{K_1} bounded. If $0 \neq \epsilon \in I_{K_1}$, then $\epsilon \cdot I_K \subseteq I_{K_1} \cdot I_K \subseteq I_{K_1}$, and therefore I_K is bounded.

Proposition 8.12. *If G_1 and G_2 are two bounded type-definable subgroups of $(\mathbb{K}, +)$, then $G_1 + G_2$ is also bounded.*

Proof. Fix non-zero pedestal J, which is heavy. There are non-zero $a_1, a_2 \in \mathbb{K}$ such that $G_1 \subseteq a_1 \cdot J$ and $G_2 \subseteq a_2 \cdot J$; then $a_1 \cdot J$ and $a_2 \cdot J$ are pedestals. Let K be a small model containing K_0 and over which $a_1 \cdot J$ and $a_2 \cdot J$ are type-defined. Then by dpI(10.4), $I_K \cdot (a_1 \cdot J) \subseteq I_K$ and $I_K \cdot (a_2 \cdot J) \subseteq I_K$. If $0 \neq \varepsilon \in I_K$, then we get $\varepsilon \cdot G_1 \subseteq I_K$ and $\varepsilon \cdot G_2 \subseteq I_K$, so that $G_1 + G_2 \subseteq \varepsilon^{-1} \cdot I_K$, and $G_1 + G_2$ is bounded.

Proposition 10.4 of dpI.

(1) Λ has a pedestal.

(2) Let $J \in \Lambda$ be a pedestal, and $A_1, \ldots, A_r$ be the atoms of a strict r-cube with basis J. If $G \in \Lambda$ is such that $G \cap A_i \not\subseteq J$ for all i, then $G \supseteq J$.

(3) If J is a pedestal and K is a small model containing K_0, then $I_K \cdot J \subseteq I_K \subseteq J$.

(5) If J is a pedestal, and $\alpha \neq 0$, then $\alpha \cdot J$ is a pedestal.