LOOKING BACK: THE DP-FINITE FINALE

Aim for today:

GIVE AN OVERVIEW OVER THE PROOF OF

THEOREM (Johnson): Let K be an infinite dp-finite field. Then K is algebraically closed, real closed or admits a non-trivial henselian valuation.

BASIC IDEAS:

(1) Let K be a henselian field, neither separably closed nor real closed.
Then all non-trivial henselian valuations on K induce the same topology, and this topology is $\mathcal{L}_{ring}$-def·ble:

Let $f \in K[X]$ be separable and irreducible with $\deg(f) > 1$, and $b \in K$ with $f'(b) \neq 0$.

Define $U_{f,b} := \left\{ \frac{1}{f(x)} - \frac{1}{f(b)} : x \in K \right\}$

Then, the sets $c \cdot U_{f,b}$ for $c \in K^{\times}$ form a basis of open nbhds of 0 of the (unique) henselian V-topology on K.

⇝ When showing the theorem above, it makes sense to find a def·ble V-topology first.

(2.) Let (K,v) be non-henselian. Then there is $L \supseteq K$ finite s.th. v does not extend uniquely to L,

say $(K,v) \subseteq (L, w_1)$
$(K,v) \subseteq (L, w_2)$

Then (L, w_1, w_2) is interpretable in (K,v).

Thus,

"any dp-finite valued field is henselian"

follows from

"any dp-finite field admits at most one def·ble V-topology".

§ Topologies

In [dpI] - [dpVI], two topological constructions occur:

(A) Constructing a W-topology from a golden lattice [dp V & VI]

(B) Constructing the canonical topology via heavy sets. [dp I & II]

Historically, step (B) comes first. Following Will's suggestion, we started with step (A).

Step (A) is necessary to obtain a (unique) V-topology.

NOW: A chronological fast-track through the proof.

[Dp I] The foundation

Main results:
- Construction of the infinitesimals I_K
- these induce a Hausdorff non-discrete ring-topology on K.

CONJECTURE: I_K is an ideal in a ring R s.th. $\mathrm{Frac}(R) = K$, $R = \underbrace{\mathcal{O}_1 \cap \dots \cap \mathcal{O}_n}_{\text{val. rings}}$

HOPE: In fact $n = 1$.

Dictionary:

Let $(K, +, \cdot, \dots)$ be a field, $\tau \subseteq K$ a family of def.ble sets satisfying
- $\forall U, V \in \tau \ \exists W \in \tau \ W \subseteq V \cap U$

Then properties of the topology generated by using τ as a basis correspond (via compactness) to properties of the τ-infinitesimals:

Take $\mathbb{K} \succ K$ monster,

$$I_K = \bigcap_{U \in \tau} U(\mathbb{K})$$

e.g. $0 \in I_K \iff \forall U \in \tau : 0 \in U$

or $I_K - I_K \subseteq I_K \iff \forall U \in \tau \ \exists V \in \tau : V - V \subseteq U$ } group topology

Motivation:

In the dp-min case,

$$\tau = \{ \underbrace{X -_\infty X}_{\{c \in K \mid (X-c)\cap X \text{ infinite}\}} : X \subseteq K \text{ infinite \& } K\text{-def'ble} \}$$

In the dp-finite case, we need the 'right' notion of big to replace infinite!

Machinery:

- broad & narrow sets
- heavy & light sets (using critical coordinate configurations)

$\hookrightarrow$ heaviness is well-defined (4.18) & def'ble in families (4.20)

$$\tau = \{ \underbrace{X -_\infty X}_{\{c \in K : (X-c)\cap X \text{ heavy}\}} : X \subseteq K \text{ heavy \& } K\text{-def'ble} \}$$

$$I_K = \bigcap_{U \in \tau} U(\mathbb{K})$$

Step 1: $(I_K, +) \leq (\mathbb{K}, +)$ subgroup (6.17), so τ is a basis of nbhds of zero for a Hausdorff non-discrete grp. top. on $(K, +)$ (6.18) and scalar multiplication is cts.

Step 2 (More difficult): Multiplication is cts.

Note: $I_K = I_K^{00}$ (6.20)

However, unlike in the dp-min. case,
00-connected subgroups of $(K,+)$ are not ordered by inclusion.
$\rightsquigarrow$ get a lattice, complexity (reduced rank) is bounded by the dp-rank!

If $K_0 \leqslant \mathbb{K}$ is magic then all type-def.ble K_0-linear subspaces are 00-connected

Magic subfields exist [8.7]

Define P = lattice of tp-def.ble K_0-linear subspaces of $\mathbb{K}$

Then:
[10.1]: For any small $K \supseteq K_0$, $I_K \in P \setminus \{0, \mathbb{K}\}$

[10.5]: Multiplication is continuous.

[Dp II]: Field topology & riddle resolution

Main results:
- heaviness = full dp-rank
- canonical top. is a field top.
- I_K is bounded

Machinery:
- Deformations
- Multipl. infinitesimals
- Simultaneous coheir & dp-rank independence
- heavy & bounded groups

K small (sometimes: also defines a crit. coord. conf.)

K-deformation: Affine symmetry $f: \mathbb{K} \to \mathbb{K}$ s.th. for all K-def-ble heavy X, $X \cap f^{-1}(X)$ is heavy.

Intuition: $f: x \mapsto x + \varepsilon$ K-deform. $\Leftrightarrow$ $\varepsilon \in I_K$

(3.10): K-deformations are a subgrp. of the $\mathbb{K}$-def-ble affine symmetries of $\mathbb{K}$

Multiplicative infinitesimal:
$\mu \in \mathbb{K}^\times$ s.th. $x \mapsto \mu \cdot x$ is a K-deform.

$\rightsquigarrow$ type-def-ble subgrp of $(\mathbb{K}^\times, \cdot)$

Independence : $a, b \in \mathbb{K}$

- coheir- indep. : $tp(a/Kb)$ fin. sat. in K
- dp-rk - indep. : $\text{dp-rk}(ab/K) = \text{dp-rk}(a/K) + \text{dp-rk}(b/K)$

$\leadsto$ can be achieved simultaneously! (4.5)

Q: Is that obvious in dp-min theories ?

Now: $\mathbb{K}$ dp-finite field, suff. sat.

In the dp-min. setting, showing that the canonical top. is a field top. required

"infinite def.ble sets have non-empty interior":

X inf., K-def.ble $\Rightarrow$ ex. $a \in X(K)$ s.th. $a + I_K \subseteq X$.

(uses independence)

Dp-finite analogue (5.6)

X heavy, K-def.ble $\Rightarrow$ ex. $a \in K$ s.th. $(\varepsilon \in I_K,\ \text{dp-rk}(\varepsilon/K) \geq \rho) \Rightarrow a + \varepsilon \in X$.

$\Downarrow$

(5.9) • $\rho = \text{dp-rk}(\mathbb{K})$
• heavy = full dp-rank
in part, full rank is def.ble in families.

(5.14) $1 + I_K = U_K$

$\Rightarrow$ canonical top. is a field top. (5.15)

Boundedness: $G \leq (\mathbb{K},+)$ type-def.ble

heavy: $\text{dp-rk}(G) = \text{dp-rk}(\mathbb{K})$.

bounded: $\forall\ G' \leq \mathbb{K}$ heavy, ex. $a \in \mathbb{K}^{\times}$ with $G \leq a \cdot G'$

(8.9) K small $\Rightarrow$ I_K bounded.

[Dp III] Understanding reduced rank

Paper deals with inflators & directories (which turn out to be unnecessary)

(6.5) $R = \mathcal{O}_1 \cap \ldots \cap \mathcal{O}_n$ intersection of n pairwise incompatible val. rings on a field K. Then

$$rk_0(Sub_R(K)) = n$$

[Dp IV]

Main results: • Canonical top. is not always a V-topology
• Proof of conjecture in dp-rk 2

Machinery (of relevance to us)
• W_n- topologies

M R-Module has property W_n if
$\forall a_0, \ldots, a_n \in M \ \exists\, 0 \le i \le n$ s.th
$a_i \in R \cdot a_1 + \ldots + R \cdot a_{i-1} + R \cdot a_{i+1} + \ldots + R \cdot a_n$

(7.3) M has property W_n
$\iff rk_0(Sub_R(M)) \le n$.

[DpV] Henselianity conj. $\Rightarrow$ Shelah's conj.
(for dp-finite fields)

Main result: • $\mathbb{K}$ unstable & dp-finite
$\Rightarrow$ $\mathbb{K}$ admits a def·ble V-topology

Machinery: • W_n-rings & W_n-topologies
• Coembeddability
• Golden lattices

Terminology: R integral domain, $K = \mathrm{Frac}(R)$
• $wt(R) = rk^\circ(\mathrm{Sub}_R(K)) = rk^\circ(\mathrm{Sub}_R(R))$
W_n-ring $:\Leftrightarrow wt(R) \leq n$
$\Leftrightarrow \mathrm{Sub}_R(K)$ has property W_n
• **W_n-set**: $S \subseteq K$ s.th. $\forall x_1, \dots, x_{n+1} \in K\ \exists i \leq n+1$
$x_i \in x_1 \cdot S + \dots + x_{i-1} \cdot S + x_{i+1} \cdot S + \dots + x_{n+1} \cdot S$
• **W_n-topology**: Hausdorff non-discrete locally bdd ring top s.th. for every nbhd $U \ni 0$ ex. $c \in K^\times$ s.th. $c \cdot U$ is a W_n-set.

Ex: $R = \mathcal{O}_1 \cap \dots \cap \mathcal{O}_n \Rightarrow R$ is a W_n-ring.

R W_n-ring $\Rightarrow$ **$\{cR : c \in K^\times\}$** nbhd basis of Hausdorff non-discrete locally bdd field top. on K.
(3.6) This is a **W_n-topology**.

(for ω-complete W_n-top., converse also holds)

Definability & relation to V-topologies

(4.1) τ W_n-top. on $\mathbb{K}$, $U \subseteq \mathbb{K}$ s.th.
- U is a bdd nbhd wrt τ
- U is v-def.ble or tp-def.ble
- $U \leq (\mathbb{K}, +)$

Then U is co-embeddable with a def.ble set, and τ is a def.ble top.

$X, Y \subseteq \mathbb{K}$ **co-emb.**: $\exists\, a, b \in \mathbb{K}^\times$ $a \cdot X \subseteq Y$, $b \cdot Y \subseteq X$.

$\rightsquigarrow$ If canonical top W_n-top $\underset{U = I_K}{\overset{4.1}{\Longrightarrow}}$ def.ble.

Note: τ W_n-top. $\Rightarrow$ τ V-topology

(4.10) (K, τ) W_n-top. field.
Then there is at least one and at most n V-top coarsenings of τ.
If τ is def.ble, then any V-top. coarsening is def.ble

Golden lattices: K field, Λ bounded sublattice of $Sub_{\mathbb{Z}}(K)$ is called golden if
- $G \in \Lambda$, $c \in K^\times \Rightarrow cG \in \Lambda$
- Λ has finite reduced rank
- $\Lambda^{+} = \Lambda \setminus \{0\}$ is closed under $\cap$.
- $\Lambda \notin \{0, K\}$

Example: If $K \preccurlyeq \mathbb{K}$ is magic, the lattice of tp-def'ble K-lin. subspaces is golden. [Dp I]

(5.9) Λ golden, then Λ^+ is a nbhd base for a W-top. τ on K
$rk^\circ(\Lambda) = r \implies \tau$ W_r-topology.

Thus: Every (unstable) dp-finite field admits a def'ble V-top. !

In fact, Will shows even more:

(6.6) K unstable, dp-finite. The def'ble V-top. on K are exactly the V-top. coarsenings of the can. topology.

Sketch: "$\Rightarrow$" all the above
"$\Leftarrow$" τ V-top, τ_0 can. top., B def'ble bdd nbhd (wrt τ)
τ W_1-top.
$\implies \forall x, y\, (x \in By \vee y \in Bx)$

for $y = 1$: $\forall x\, (x \in B \vee 1/x \in B)$
$\implies$ dp-rk(B) = dp-rk(K)
so $B - B$ is a nbhd wrt τ_0
In fact: B-B bounded in τ
$\implies \{a(B-B) : a \in K^\times\}$ nbhd base for τ
$\implies \tau$ coarsens τ_0 $\boxtimes$

[Dp VI] Uniqueness of V-topologies

Main result: Dp-finite Shelah conjecture

Machinery:
- local W_n-topologies
- squaring & AS-maps

(K,τ) W_n-top. field is local if for every $B \subseteq K$ bounded ex. $C \subseteq K$ bounded s.th.
$\forall x \in B \ (1/x \in C \ \vee \ 1/(1-x) \in C)$

Non-standard view: (K,τ) W-top., $(K^*,\tau^*) \succ (K,\tau)$
$$R_\tau = \bigcup_{B \in \tau \text{ bdd}} B^*$$
Then τ local $\iff$ R_τ is a local ring.

(4.15) (K,τ) W-top. field.
(1.) If $\mathrm{char}(K) \neq 2$ and squaring $K^* \to (K^*)^2$ is an open map, then τ is local & has a unique V-top. coarsenings.
(2.) same with AS-map.

follows from algebraic properties of the (multiplicative) infinitesimals

THIS MEANS WILL HAS WON!

Addendum

Dp-finite Shelah Conjecture: Any dp-finite field is either finite or alg. closed or real closed or admits a non-trivial henselian valuation.

Dp-finite Henselianity Conjecture:

(K,v) dp-finite $\Rightarrow$ v henselian

Definition not featured above:

- (K,τ) top. field. $B \subseteq K$ **bounded** if $\forall U \,\exists W$ $W \cdot B \subseteq U$