Dp-finite fields reading seminar paper II, §5.2, §5.3, §5.4

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Disclaimer: all of this work is due to Will Johnson and can be found in his first two papers on Dp-finite fields, [J1] and [J2].

Theorem 5.9

1. Critical rank = dp-rank.
2. A definable set $X \subseteq \mathbb{K}$ is heavy iff $\text{dp-rk}(X) = \text{dp-rk}(\mathbb{K})$.
3. The property of a definable set having full dp-rank is definable in families.

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3. The property of a definable set having full dp-rank is definable in families.

Proof.

1. $\longrightarrow$ Franzi's talk.
2. We know that definable sets of full dp-rank are heavy. Conversely, if X is heavy, then X contains a critical set, which has critical dp-rank, i.e. $\text{dp-rk}(\mathbb{K})$.
3. We already know heaviness is definable in families. □

Corollary 5.10

Let K be unstable with $\text{dp-rk}(K) = n$. Each of the following families is a neighbourhood basis of 0 in the canonical topology.

1. $X -_\infty X = \{\delta \in K \mid X \cap (X + \delta) \text{ is heavy}\}$, for definable X with $\text{dp-rk}(X) = n$.
2. $X \ominus X = \{\delta \in K \mid X \cap (X + \delta) \text{ is infinite}\}$, for definable X with $\text{dp-rk}(X) = n$.
3. $X - X = \{\delta \in K \mid X \cap (X + \delta) \text{ is nonempty}\}$, for definable X with $\text{dp-rk}(X) = n$.

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Proof.

1. $\{X -_\infty X \mid X \text{ definable}, \text{dp-rk}(X) = n\}$ is neighbourhood basis of 0 by definition.

Trivially $X -_\infty X \subseteq X \ominus X \subseteq X - X$. Therefore both other families really are families of neighbourhoods of 0 in the canonical topology. For the cofinality:

2. We use the following claim.

□

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For any heavy set X there is a heavy set Y such that

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Proof.

First note $I_K - I_K = I_K \subseteq X -_{\infty} X$. Therefore the 2-type

$$x \in I_K,\ y \in I_K,\ x - y \in X -_{\infty} X$$

is inconsistent. There exists $Y \supseteq I_K$ which is K-definable such that

$$Y - Y \subseteq X -_{\infty} X.$$

Since Y contains a K-definable basic neighbourhood, it is heavy. □

Proposition 5.12

Let K be a small model defining a critical coordinate configuration, and let I_K be the group of K-infinitesimals. Then the group U_K of multiplicative K-infinitesimals is exactly $1 + I_K$.

2. Dp-finite II, §5.3: additive vs multiplicative infinitesimals

Proposition 5.12

Let K be a small model defining a critical coordinate configuration, and let I_K be the group of K-infinitesimals. Then the group U_K of multiplicative K-infinitesimals is exactly $1 + I_K$.

Proof.

- $U_K \subseteq 1 + I_K$ by Theorem 3.12.3.
- $\text{dp-rk}(I_K) = \text{dp-rk}(\mathbb{K}) = \rho$ by Lemma 5.8 and Proposition 5.7.

Let $\varepsilon_0 \in I_K$ and K-definable heavy set $X \subseteq \mathbb{K}$. We must show that $X \cap (1+\varepsilon)^{-1}X$ is heavy.

- Choose $a \in K$ such that for any $\varepsilon \in I_K$, $\text{dp-rk}(\varepsilon/K) = \rho = a + \varepsilon \in X$, by Proposition 5.6.

Claim

If ε_1 is a K-infinitesimal with $\text{dp-rk}(\varepsilon_1/K\varepsilon_0) = \rho$, then $(a+\varepsilon_1) \in X \cap (1+\varepsilon_0)^{-1}X$.

Proof.

Claim

If ε_1 is a K-infinitesimal with dp-rk$(\varepsilon_1/K\varepsilon_0) = \rho$, then $(a + \varepsilon_1) \in X \cap (1 + \varepsilon_0)^{-1}X$.

Proof.

Claim

If ε_1 is a K-infinitesimal with $\text{dp-rk}(\varepsilon_1/K\varepsilon_0) = \rho$, then $(a + \varepsilon_1) \in X \cap (1 + \varepsilon_0)^{-1}X$.

Proof of claim.

- $\varepsilon_2 := a\varepsilon_0 + \varepsilon_1 + \varepsilon_1\varepsilon_0$ is K-infinitesimal by [J1, Remark 6.9.3, Theorem 6.17, Corollary 10.5].
- $\varepsilon_2 = (a + \varepsilon_1)(1 + \varepsilon_0) - a$, so ε_1 and ε_2 inter-definable over $K\varepsilon_0$.
- $\text{dp-rk}(\varepsilon_2/K\varepsilon_0) = \text{dp-rk}(\varepsilon_1/K\varepsilon_0) = \rho$.
- $\text{dp-rk}(\varepsilon_2/K) = \text{dp-rk}(\varepsilon_1/K) = \rho$.
- $a + \varepsilon_1 \in X$.
- $(a + \varepsilon_1)(1 + \varepsilon_0) = a + \varepsilon_2 \in X$.
- $a + \varepsilon_1 \in X \cap (1 + \varepsilon_0)^{-1}X$. □

□

Proof cont.

Let $S := \{\varepsilon \in I_K \mid a + \varepsilon \notin X \cap (1 + \varepsilon_0^{-1}X\}$.

- S type-definable over $K\varepsilon_0$
- If dp-rk$(S) =$ dp-rk$(I_K) = \rho$ then there exists $\varepsilon \in S$ with dp-rk$(\varepsilon / K\varepsilon_0) = \rho$. Contradiction to claim.
- So dp-rk$(S) <$ dp-rk$(I_K) = \rho$, so dp-rk$(I_K \setminus S) = \rho$.
- $I_K \setminus S \longrightarrow X \cap (1 + \varepsilon_0)^{-1}X$ via $\varepsilon \longmapsto a + \varepsilon$.
- So dp-rk$(X \cap (1 + \varepsilon)^{-1}X) \geq \rho$, so $X \cap (1 + \varepsilon_0)^{-1}X$ is heavy. □

...working over arbitrary small model

Theroem 5.14

Let $K \preceq \mathbb{K}$ small model. Then $1 + I_K = U_K$.

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Proof.

By Proposition 5.12, we may assume there is $K \preceq K'$ with $1 + I_{K'} = U_{K'}$. Again, by Theorem 3.12.3, we have $1 + I_K \supseteq U_K$. Let X be K-definable heavy set. Consider the K-definable set

$$Y = \{\mu \in \mathbb{K} \mid X \cap (\mu X) \text{ is heavy}\}.$$

Suffices to show $1 + I_K \subseteq Y$. In particular Y is K'-definable, so $1 + I_{K'} = U_{K'} \subseteq Y$. There exists a K'-definable heavy $N \supseteq I_{K'}$ such that $1 + N \subseteq Y$. By [J1, Proposition 6.5], we may assume N is of the form $Z -_\infty Z$, a basic neighbourhood. Using definability of heaviness in families (this is 'pulling parameters down' trick again), there is K-definable heavy Z' such that $1 + I_K \subseteq 1 + (Z' -_\infty Z') \subseteq Y$. □

Corollary 5.15

Canonical topology is a field topology.

Proposition 5.16

Let K small model. Then

$$U_K = \bigcap \{ X \cdot X^{-1} \mid X \subseteq \mathbb{K}^{\times} \text{ is } K\text{-definable heavy} \}.$$

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Proof.

By Lemma 5.8 and Theorem 5.14, $\text{dp-rk}(U_K) = \text{dp-rk}(\mathbb{K})$. Denote

$$X \div X := \{\mu \in \mathbb{K}^\times \mid X \cap (\mu X) \text{ non-empty}\}$$
$$X \div_\infty X := \{\mu \in \mathbb{K}^\times \mid X \cap (\mu X) \text{ heavy}\}.$$

Note $X \div_\infty X \subseteq X \div X = X \cdot X^{-1}$. Note also $X \div_\infty X$ is definable and

$$\begin{aligned} U_K &= \bigcap \{X \div_\infty X \mid X \subseteq \mathbb{K}^\times \text{ heavy and } K\text{-definable}\} \qquad \text{by Thm 3.12. (proof)} \\ &\subseteq \bigcap \{X \div X \mid X \subseteq \mathbb{K}^\times \text{ heavy and } K\text{-definable}\} \\ &\subseteq \bigcap \{X \div X \mid U_K \subseteq X \subseteq \mathbb{K}^\times, \text{ and } X \text{ is } K\text{-definable}\} \\ &= U_K \qquad \text{by Thm 3.12.4 (proof).} \end{aligned}$$

□

3. Dp-finite II, §5.4: algebraic properties

We're collecting facts. Let $\mathbb{K}$ saturated unstable dp-finite field. Let $K \preceq \mathbb{K}$ small model.

Proposition 5.17

1. I_K is a subgroup of $\mathbb{K}$.
2. $I_K = I_K \cdot I_K \quad \big(= \{\sum_{i<n} x_i y_i \mid x_i, y_i \in I_K\}\big)$.
3. $1 + I_K$ is a subgroup of $\mathbb{K}^\times$.
4. For every $n \geq 1$

$$\begin{aligned} 1 + I_K &\longrightarrow 1 + I_K \\ x &\longmapsto x^n \end{aligned}$$

   is surjective.

5. In $\mathrm{char}(K) = p$, the Artin–Schreier map

$$\begin{aligned} I_K &\longrightarrow I_K \\ x &\longmapsto x^p - x \end{aligned}$$

   is surjective.

Proof.

1. [J1, Theorem 6.17]
2. Inclusion $I_K \cdot I_K \subseteq I_K$ by [J1, Corollary 10.5].

$$\begin{aligned} J - J &= \bigcap \{X - X \mid X \supseteq J,\ X \text{ is } K\text{-definable}\} \\ &\supseteq \bigcap \{X - X \mid X \supseteq J,\ X \text{ is } K\text{-definable, full rank}\} \\ &\supseteq \bigcap \{X -_{\infty} X \mid X \supseteq J,\ X \text{ is } K\text{-definable}\} \\ &= I_K. \end{aligned}$$

   Thus $I_K = \{xy - uv \mid u, v, x, y \in I_K\} = I_K \cdot I_K$.
3. Theorems 2.12.1 and 5.14.
4. $\mathbb{K}^{\times n}$ has full dp-rank, thus $1 + I_K \subseteq \mathbb{K}^{\times n}$ by Theorems 3.12.4 and 5.9.2.
5. Use [J1, Corollary 6.19]

□

Bibliography

Will Johnson.
Dp-finite fields I: infinitesimals and positive characteristic.
Preprint, 2020. (arXiv:1903.11322 [math.LO])

Will Johnson.
Dp-finite fields II: the canonical topology and its relation to henselianity.
Preprint, 2019. (arXiv:1910.05932 [math.LO])