Linear PDE with Constant Coefficients

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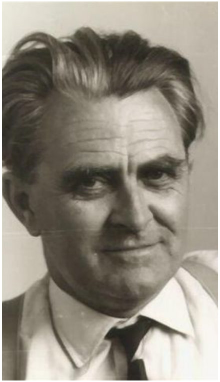

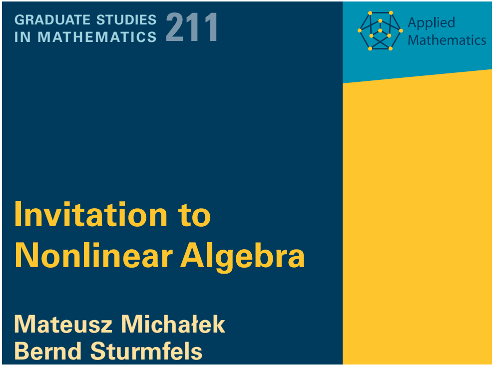

Prologue

A polynomial is exactly the same thing as a homogeneous linear partial differential equation (PDE) with constant coefficients.

Polynomials in one variable are ODE:

Exercise: Find all functions $\phi(z)$ that satisfy the equation

$$\phi''''(z) - 6\phi'''(z) + 10\phi''(z) - 6\phi'(z) + 9\phi(z) \;=\; 0.$$

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Solution: ODE in operator form is the characteristic polynomial:

$$(\partial^4 - 6\partial^3 + 10\partial^2 - 6\partial + 9) \bullet \phi(z) \quad = \quad 0$$

$$x^4 - 6x^3 + 10x^2 - 6x + 9 \; = \; (x-3)^2 \cdot (x^2+1)$$

$$\text{Basis of solutions:} \quad \{\, e^{3z} \,,\; z \cdot e^{3z} \,,\; e^{iz} \,,\; e^{-iz} \,\}$$

$$\text{Basis of solutions:} \quad \{ e^{3z}, z \cdot e^{3z}, \sin(z), \cos(z) \}$$

Geometry: The ODE represents an affine scheme of length 4.

18th Century

Undergraduates study the one-dimensional wave equation

$$\phi_{tt}(z,t) \;=\; c^2\,\phi_{zz}(z,t), \qquad \text{where } \; c \in \mathbb{R}\backslash\{0\}.$$

for functions $\phi : \mathbb{R}^2 \to \mathbb{R}$. The corresponding polynomial is

$$x_1^2 - c^2 x_2^2 \;=\; (x_1 - c x_2)(x_1 + c x_2).$$

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$$x_1^2 - c^2 x_2^2 \;=\; (x_1 - cx_2)(x_1 + cx_2).$$

In 1747, Jean Le Rond D'Alembert found that the general solution is the superposition of traveling waves:

$$\phi(z,t) \;=\; f(z+ct) \,+\, g(z-ct),$$

where f and g are twice differentiable functions in one variable.

Question: How to deal with the special parameter value $c = 0$?

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where f and g are twice differentiable functions in one variable.

Question: How to deal with the special parameter value $c = 0$?
A: Replace $g(z-ct)$ with $\frac{1}{2c}(h(z+ct) - h(z-ct))$ and take limit:

$$\phi(z,t) \;=\; f(z) \,+\, t \cdot h'(z).$$

Geometry: A pair of lines becomes a double line.

20th Century

In his 1938 article on foundations of algebraic geometry, Gröbner introduced differential operators to characterize **membership in a polynomial ideal**. He derived this for zero-dimensional ideals (Macaulay's inverse systems), and he envisioned it for all ideals. Gröbner wanted algorithmic solutions. *We provide them.*

Wolfgang Gröbner: Über die algebraischen Eigenschaften der Integrale von linearen Differentialgleichungen mit konstanten Koeffizienten, *Monatshefte für Mathematik und Physik*, 1939

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In the 1960s, Ehrenpreis and Palamodov studied solutions to linear partial differential equations (PDE) with constant coefficients. A main step was the characterization of membership in a primary ideal by *Noetherian operators*.

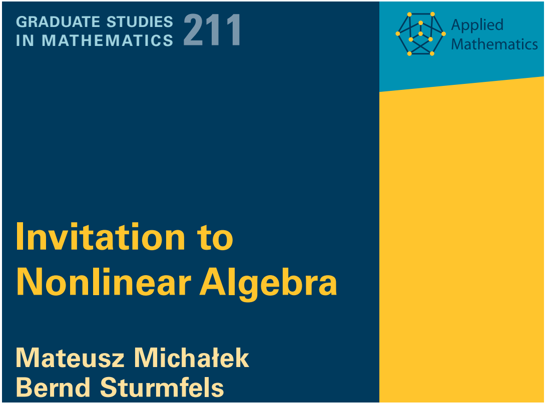
Their celebrated **Fundamental Principle** appears in the books

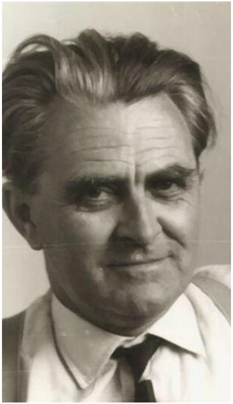
Leon Ehrenpreis: Fourier Analysis in Several Complex Variables, 1970

Victor Palamodov: Linear Differential Operators w Constant Coeffs, 1970

Section 3.3

Wolfgang Gröbner





Theorem 3.27. *Let I be a zero-dimensional ideal in $\mathbb{C}[x_1, \ldots, x_n]$, here interpreted as a system of linear PDEs. The space of holomorphic solutions has dimension equal to the degree of I. There exist nonzero polynomial solutions if and only if the maximal ideal $M = \langle x_1, \ldots, x_n \rangle$ is an associated prime of I. In that case, the polynomial solutions are precisely the solutions to the system of PDEs given by the M-primary component $(I : (I : M^\infty))$.*

Quiz on Power Sums

Given three distinct integers $a, b, c > 0$, describe the space of all functions $\phi = \phi(z_1, z_2, z_3)$ that satisfy the three PDE

$$\frac{\partial^a \phi}{\partial z_1^a} + \frac{\partial^a \phi}{\partial z_2^a} + \frac{\partial^a \phi}{\partial z_3^a} = \frac{\partial^b \phi}{\partial z_1^b} + \frac{\partial^b \phi}{\partial z_2^b} + \frac{\partial^b \phi}{\partial z_3^b} = \frac{\partial^c \phi}{\partial z_1^c} + \frac{\partial^c \phi}{\partial z_2^c} + \frac{\partial^c \phi}{\partial z_3^c} = 0.$$

$$\langle\, \partial_1^a + \partial_2^a + \partial_3^a \,,\, \partial_1^b + \partial_2^b + \partial_3^b \,,\, \partial_1^c + \partial_2^c + \partial_3^c \,\rangle$$

Example: Consider $(a, b, c) = (1, 2, 3)$. The solution space is six-dimensional. It consists of a cubic and all its derivatives:

$$\phi \,=\, (z_1 - z_2)(z_1 - z_3)(z_2 - z_3).$$

The ideal $I = \langle x_1 + x_2 + x_3, x_1^2 + x_2^2 + x_3^2, x_1^3 + x_2^3 + x_3^3 \rangle$ is Gorenstein.

Example: Consider $(a, b, c) = (2, 5, 8)$. What happens now?

H. Melánová, BSt, R. Winter: Recovery from power sums, 2106.13981

Prime Ideals

Let P be a prime ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and $V(P)$ its variety in $\mathbb{C}^n$. A polynomial f is in the ideal P if and only if f vanishes on $V(P)$.

Setting $x_i = \partial_{z_i}$, view P as PDE for an unknown function $\phi(z_1, \ldots, z_n)$.

Remark

For $u \in \mathbb{C}^n$, the exponential function

$$z \mapsto \exp(u \cdot z) = \exp(u_1 z_1 + \cdots + u_n z_n)$$

satisfies the PDE given by P if and only if $u \in V(P)$.

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Proposition

Each solution to P admits an integral representation

$$\phi(z) \quad = \quad \int_{V(P)} \exp(x \cdot z)\, d\mu(x),$$

where μ is a measure on the irreducible variety $V(P)$.

Primary Ideals

Set $\mathbb{C}[x] = \mathbb{C}[x_1, \ldots, x_n]$. An ideal Q is *primary* if it has only one associated prime P. The variety $V(Q) = V(P)$ in $\mathbb{C}^n$ is irreducible.

Theorem (Ehrenpreis-Palamodov)

Fix a prime ideal P in $\mathbb{C}[x]$. For any P-primary ideal Q, there exist polynomials $B_1, \ldots, B_m$ in $2n$ variables such that the function

$$\phi(z) \quad = \quad \sum_{i=1}^{m} \int_{V(P)} B_i(x,z)\, exp(x \cdot z)\, d\mu_i(x)$$

is a solution to the PDE, for any measures $\mu_1, \ldots, \mu_m$ on $V(P)$.

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is a solution to the PDE, for any measures $\mu_1, \ldots, \mu_m$ on $V(P)$. Conversely, every solution $\phi(z)$ of the PDE given by Q admits such an integral representation. The minimal number is

$$m \;=\; \mathrm{length}\big(R_P/QR_P\big) \;=\; \frac{\mathtt{degree}(Q)}{\mathtt{degree}(P)}.$$

The polynomials $B_1(x, z), \ldots, B_m(x, z)$ are *Noetherian multipliers*. They depend only on the ideal Q and encode all solutions $\phi(z)$.

Palamodov's Example

Let $n = 3, P = \langle x_1, x_2 \rangle$. Then $Q = \langle x_1^2, x_2^2, x_1 - x_2 x_3 \rangle$ is P-primary of multiplicity $m = 2$. We seek functions $\phi(z_1, z_2, z_3)$ that satisfy

$$\frac{\partial^2 \phi}{\partial z_1^2} = \frac{\partial^2 \phi}{\partial z_1^2} = \frac{\partial \phi}{\partial z_1} - \frac{\partial^2 \phi}{\partial z_2 \partial z_3} = 0.$$

Writing ξ, ψ for functions in one variable, the general solution is

$$\phi(z) \quad = \quad \xi(z_3) + z_2 \, \psi(z_3) + z_1 \, \psi'(z_3),$$

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$$\phi(z) = \xi(z_3) + z_2\, \psi(z_3) + z_1\, \psi'(z_3),$$

The Noetherian multipliers of Q are $B_1 = 1$ and $B_2 = z_2 + x_3 z_1$.

Their integrals in the Ehrenpreis-Palamodov Theorem are

$$\phi_1(z) = \int 1 \cdot \exp(0z_1 + 0z_2 + x_3 z_3)\, d\mu_1(x) = \xi(z_3) \qquad \text{and}$$

$$\begin{aligned} \phi_2(z) &= \int (z_2 + z_1 x_3) \cdot \exp(0z_1 + 0z_2 + x_3 z_3)\, d\mu_2(x) \\ &= z_2 \int \exp(0z_1 + 0z_2 + x_3 z_3) d\mu_2(x) + z_1 \int x_3 \exp(0z_1 + 0z_2 + x_3 z_3) d\mu_2(x) \\ &= z_2\, \psi(z_3) + z_1\, \psi'(z_3). \end{aligned}$$

Noetherian Operators

The Noetherian multipliers $B_i(x, z)$ of a primary ideal Q furnish a finite representation of the (infinite-dimensional) vector space of all solutions to the PDE. We now recycle them for ideal membership.

Switching the roles of x and z, we set $z_1 = \partial_{x_1}, \ldots, z_n = \partial_{x_n}$ in $B_i(x, z)$, with z-variables to the right of the x-variables in each monomial. This gives the *Noetherian operators* $B_i(x, \partial_x)$. These are elements in the Weyl algebra. They act on polynomials in $\mathbb{C}[x]$.

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Proposition

Noetherian operators characterize ideal membership. Namely, a polynomial $f(x)$ lies in the primary ideal Q if and only if

$$B_i(x, \partial_x) \bullet f(x) \;\; \textit{lies in} \;\; P \;\; \textit{for} \;\; i = 1, \ldots, m.$$

Example

A polynomial f lies in the primary ideal $Q = \langle x_1^2, x_2^2, x_1 - x_2 x_3 \rangle$ if and only if both f and $(x_3 \partial_{x_1} + \partial_{x_2}) \bullet f$ vanish on the x_3-axis $V(P)$.

Towards an Algorithm

Input: Generators of a (primary) ideal Q in the polynomial ring $\mathbb{C}[x]$.
Output: Noetherian multipliers (resp. Noetherial operators) for Q.

B. Mourrain: Isolated points, duality&residues, J. Pure Appl.Algebra,1997

U. Oberst: The construction of Noetherian operators, *J. Algebra*, 1999

S. Shankar: The Nullstellensatz for systems of PDE, *Advances in Applied Math*, 1999

B. Sturmfels: Solving Systems of Polynomial Equations, AMS, 2002

A. Damiano, I. Sabadini, D. Struppa: Computational methods for the construction of a class of Noetherian operators, *Experimental Math*, 2007

J. Chen, M. Härkönen, R. Krone and A. Leykin:
Noetherian operators and primary decomposition, 2006.13881

J. Chen, Y. Cid-Ruiz, M. Härkönen, R. Krone and A. Leykin:
Noetherian operators in Macaulay2, 2101.01002

R. Ait El Manssour, M. Härkönen and BSt.:
Linear PDE with constant coefficients, Glasgow Math J., 2022

Current Perspective

Fix a prime P of codimension c in $R = \mathbb{C}[x_1, \ldots, x_n]$, in Noether position. Write $\mathbb{F} = \mathbb{C}(u_1, \ldots, u_n)$ for the field of fractions of R/P.

Theorem

The following sets are in bijective correspondences:

(a) *P-primary ideals Q in R of multiplicity m,*

(b) *points in the punctual Hilbert scheme $\mathrm{Hilb}^m(\mathbb{F}[[y_1, \ldots, y_c]])$,*

(c) *m-dimensional $\mathbb{F}$-subspaces of $\mathbb{F}[z_1, \ldots, z_c]$ that are closed under differentiation,* *Inverse systems*

(d) *m-dimensional $\mathbb{F}$-subspaces of the Weyl-Noether module $\mathbb{F} \otimes_R D_{n,c}$ that are R-bi-modules, where $D_{n,c} = R\langle \partial_{x_1}, \ldots, \partial_{x_c} \rangle$.*

(c) $\rightarrow$ Noetherian multipliers (d) $\rightarrow$ Noetherian operators

Yairon Cid-Ruiz, Roser Homs Pons and BSt: Primary ideals and their differential equations, *Foundat. Computational Math*, 2021

Solving Gröbner's Problem

If $I = Q_1 \cap \cdots \cap Q_k$ is a primary decomposition then we may simply

- aggregate Noetherian operators to get a membership test for I
- aggregate Noetherian multipliers to solve the PDE given by I

Works fine if I has no embedded primes. Can do better in general.

Example (Fat point on a double line)

$$I \;=\; \langle x_1^2, x_2^2, x_1x_3 - x_2x_3^2 \rangle \;=\; \langle x_1^2, x_2^2, x_1 - x_2x_3 \rangle \;\cap\; \langle x_1^2, x_2^2, x_3 \rangle$$

The naive method gives six Noetherian multipliers, namely two for the line and four for the point. But we need only four of them:

prime	$\langle x_1, x_2 \rangle$	$\langle x_1, x_2, x_3 \rangle$
multipliers	$1\,,\; z_2 + x_3 z_1$	$z_1\,,\; z_1 z_2$
operators	$1, \partial_{x_2} + x_3 \partial_{x_1}$	$\partial_{x_1}, \partial_{x_1}\partial_{x_2}$

Commutative Algebra

Fix $R = \mathbb{C}[x]$. Consider **any** ideal $I \subset R$. Associated primes $P_1, \ldots, P_k$. A *differential primary decomposition* of I is a list $(P_1, \mathcal{A}_1), \ldots, (P_k, \mathcal{A}_k)$ where $\mathcal{A}_i$ is a finite subset of $D_{n,n}$ with

$$I \;=\; \big\{ f \in R \mid \delta \bullet f \in P_i \text{ for all } \delta \in \mathcal{A}_i \text{ and } i = 1, \ldots, k \big\}.$$

Its *arithmetic multiplicity* is $\mathrm{amult}(I) \;=\; \sum_{j=1}^{k} \mathrm{mult}_I(P_j)$, where

$$\mathrm{mult}_I(P) \;=\; \frac{\texttt{degree(saturate(I,P)/I)}}{\texttt{degree(P)}}$$

is the length of the largest ideal of finite length in R_P / IR_P.

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This makes sense for any algebra R that is *essentially of finite type over a perfect field*. The Weyl algebra $D_{n,n}$ gets replaced by the ring of differential operators on R, which is usually not noetherian.

Yairon Cid-Ruiz and BSt:
Primary decomposition with differential operators, 2101.03643

Main Result

Theorem

Let I be an ideal in $R = \mathbb{C}[x]$, or an algebra R as above. The size of a differential primary decomposition of I is at least $\mathrm{amult}(I)$. This lower bound is tight. More precisely:

(i) *The ideal I has a differential primary decomposition $(P_1, \mathcal{A}_1), \ldots, (P_k, \mathcal{A}_k)$ such that $|\mathcal{A}_i| = \mathrm{mult}_I(P_i)$.*

(ii) *If $(P_1, \mathcal{A}_1), \ldots, (P_k, \mathcal{A}_k)$ is any differential primary decomposition for I, then $|\mathcal{A}_i| \geq \mathrm{mult}_I(P_i)$.*

This theoretical result yields a practical algorithm for computing a minimal DPD for I in $R = \mathbb{C}[x]$. The output translates into Noetherian multipliers, and hence into a general solution to the PDE given by I. Try the command `solvePDE` in `Macaulay2`.

Slogan: Primary decompositions are unique (up to change of basis).

Macaulay 2

Computing a minimal differential primary decomposition:

```
i1 : needsPackage "NoetherianOperators";

i2 : R = QQ[x,y,z];

i3 : I = ideal(x^2,y^2,x*z-y*z^2);

o3 : Ideal of R

i4 : amult(I)

o4 = 4

i5 : solvePDE(I)

o5 = {{ideal (y, x),     {| 1 |, | zdx+dy |}},
       {ideal (z, y, x),  {| dx |, | dxdy | }}}
```

This is double line with a fat point:

$P_1 = \langle x, y \rangle\,, \;\; \mathcal{A}_1 = \{1, z\partial_x + \partial_y\}$

$P_2 = \langle x, y, z \rangle\,, \; \mathcal{A}_2 = \{\partial_x,\, \partial_x \partial_y\}$

Modules

The treatment of Ehrenpreis-Palamodov in books on analysis emphasizes PDE for vector-valued functions $\psi : \mathbb{C}^n \to \mathbb{C}^k$.

[J.-E. Björk: Rings of Differential Operators], [L. Hörmander: An Introduction to Complex Analysis in Several Variables]

In calculus we learn how to rewrite one higher-order ODE as a system of first order ODE, and in algebraic geometry we learn how to appreciate matrix representations of geometric objects:

$$\begin{array}{ccc} \text{Ideals} & \longrightarrow & \text{Schemes} \\ \text{Modules} & \longrightarrow & \text{Coherent Sheaves} \end{array}$$

A system of ℓ linear PDE for ψ is represented by a $k \times \ell$ matrix with entries in $R = \mathbb{C}[x_1, \ldots, x_n]$. The image of this matrix is a submodule M of R^k. Primary decomposition makes sense here:

$$M \;=\; M_1 \cap \cdots \cap M_k.$$

... and so does **differential primary decomposition** aka as `solvePDE`

Coherent Sheaves

```
needsPackage "NoetherianOperators";
R = QQ[x1,x2,x3,x4];
M = image matrix {
{x1*x3, x1*x2, x1^2*x2},
{ x1^2,  x2^2, x1^2*x4}};
amult(M)
solvePDE(M)
```

Let $M \subset R^2$ be the module spanned by the columns of

$$\begin{bmatrix} \partial_1\partial_3 & \partial_1\partial_2 & \partial_1^2\partial_2 \\ \partial_1^2 & \partial_2^2 & \partial_1^2\partial_4 \end{bmatrix}.$$

This represents PDE for functions $\psi : \mathbb{C}^4 \to \mathbb{C}^2$. We seek $\psi(z) = \big(\psi_1(z_1, z_2, z_3, z_4), \psi_2(z_1, z_2, z_3, z_4)\big)$ such that

$$\frac{\partial^2\psi_1}{\partial z_1 \partial z_3} + \frac{\partial^2\psi_2}{\partial z_1^2} = \frac{\partial^2\psi_1}{\partial z_1 \partial z_2} + \frac{\partial^2\psi_2}{\partial z_2^2} = \frac{\partial^3\psi_1}{\partial z_1^2 \partial z_2} + \frac{\partial^3\psi_2}{\partial z_1^2 \partial z_4} = 0.$$

The module M has six associated primes, namely $P_1 = \langle \partial_1 \rangle$, $P_2 = \langle \partial_2, \partial_4 \rangle$, $P_3 = \langle \partial_2, \partial_3 \rangle$, $P_4 = \langle \partial_1, \partial_3 \rangle$, $P_5 = \langle \partial_1, \partial_2 \rangle$, $P_6 = \langle \partial_1^2 - \partial_2\partial_3, \partial_1\partial_2 - \partial_3\partial_4, \partial_2^2 - \partial_1\partial_4 \rangle$. Primes P_4, P_5 are embedded. Arithmetic multiplicity: $1{+}1{+}1{+}1{+}4{+}1 = 9 = \mathrm{amult}(M)$.

To solve the PDE, we compute a differential primary decomposition.

Making Waves

Analysts are interested in **wave solutions**. Collaboration with Jonas Hirsch and Bogdan Raita. Here is one example motivated by

A. Arroyo-Rabasa, G. De Philippis, J. Hirsch and F. Rindler: Dimensional estimates and rectifiability for measures satisfying linear PDE constraints, *Geometric and Functional Analysis*, 2019.

Fix $n = \ell = 4, k = 7$ and let M be the image in R^7 of

$$\begin{bmatrix} x_1 & 0 & 0 & 0 \\ x_2 & x_1 & 0 & 0 \\ x_3 & x_2 & x_1 & 0 \\ x_4 & x_3 & x_2 & x_1 \\ 0 & x_4 & x_3 & x_2 \\ 0 & 0 & x_4 & x_3 \\ 0 & 0 & 0 & x_4 \end{bmatrix}$$

This module is primary with $P = \{0\}$ and $\text{amult}(M) = 3$. It represents a first-order PDE for functions $\phi : \mathbb{R}^4 \to \mathbb{R}^7$. Here `solvePDE` outputs three Noetherian multipliers; these are left syzygies. They span all syzygies as a vector space over $\mathbb{R}(x)$.

Hankel Matrix

We get solutions ϕ to the PDE from any syzygy by applying that differential operator to any function $f(z_1, z_2, z_3, z_4)$.

For instance, one Noetherian multiplier gives

$$\phi = \; (f_{2222} - 3f_{1223} + f_{1133} + 2f_{1124},\, 2f_{1123} - f_{1222} - f_{1114},\, f_{1122} - f_{1113},\, -f_{1112},\, f_{1111},\, 0,\, 0\,).$$

Consider the Hankel matrix

$$H(u) \;=\; \begin{bmatrix} u_1 & u_2 & u_3 & u_4 \\ u_2 & u_3 & u_4 & u_5 \\ u_3 & u_4 & u_5 & u_6 \\ u_4 & u_5 & u_6 & u_7 \end{bmatrix}.$$

Wave cones of [ADHR] are the varieties $\{u \in \mathbb{P}^6 : \text{rank}(H(u)) \leq r\}$.
$r = 1$: rational normal curve in $\mathbb{P}^6$; $r = 3$: variety of secant planes.
Parametrization as the span of our three Noetherian multipliers.

Any $u \in \mathbb{P}^6$ with $H(u)$ of low rank yields wave solutions to M.

Distributions

Example

The Hankel matrix $H(u)$ has rank one for

$$u = (1, 2, 4, 8, 16, 32, 64).$$

Kernel of $H(u)$ is spanned by $2e_1 - e_2, 2e_2 - e_3, 2e_3 - e_4$.
Any function $f : \mathbb{R}^3 \to \mathbb{R}$ yields a solution

$$\phi(z) = f(2z_1 - z_2, 2z_2 - z_3, 2z_3 - z_4) \cdot u.$$

This vector is a wave solution as in 1747. If f is the Dirac distribution at the origin in $\mathbb{R}^3$ then ϕ is a distributional solution supported on a line in $\mathbb{R}^4$. Characterizing such supports is the point of [ADHR].

THE END

Many thanks for your attention!!