Part 2: Introduction to SPDE

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Outline

1. What are SPDE and why are they useful/important/interesting?
2. What is space-time white noise? Construction and calculus.
3. Basic linear SPDE and and Duhamel's principle
4. Chaos expansion and multiplicative SPDE.
5. Martingale methods to identify the law of an SPDE
6. Singular SPDE: regularity computations and local subcriticality assumptions, pathwise solution theories
7. Markov property and invariant measures for SPDEs; rate of convergence.

What are SPDEs

SPDEs are to PDEs what SDEs are to ODEs. We are going to focus mainly on evolution SPDEs of the form

$$\partial_t f = L(f) + \sigma(f)\xi.$$

where L is some operator (possibly nonlinear) and σ is a linear operator. Here ξ is Gaussian space-time white noise, to be explained shortly.

Note the analogy with SDE's of the form

$$dX_t = L(X_t)dt + \sigma(X_t)dB_t.$$

SPDEs

Examples of L we will consider:

1. (SHE / Edwards-Wilkinson) $L(f) = \partial_x^2 f$ or more generally $L(f) = -(-\partial_x^2)^\alpha f$. And $\sigma(f) = I$.
2. (mSHE) $L(f) = \partial_x^2 f$ and $\sigma(f)g = fg$.
3. (KPZ) $L(f) = \partial_x^2 f + (\partial_x f)^2$ and $\sigma(f) = I$.

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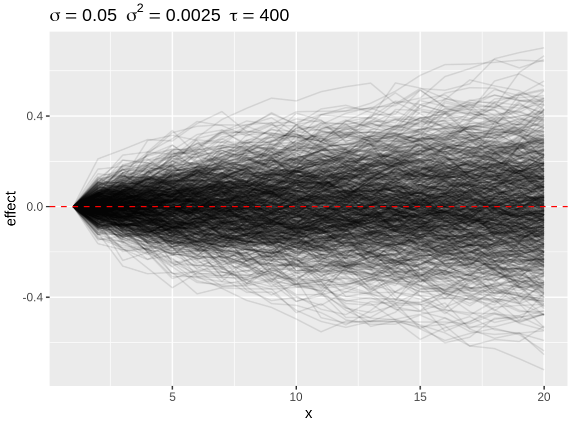
2. (mSHE) $\partial_t f = \partial_x^2 f + f\xi$.
3. (KPZ) $\partial_t f = \partial_x^2 f + (\partial_x f)^2 + \xi$

Why study these?

They describe the fluctuations of various systems coming from probability, statistical mechanics.

Simple example: iid random walks or brownian motions.



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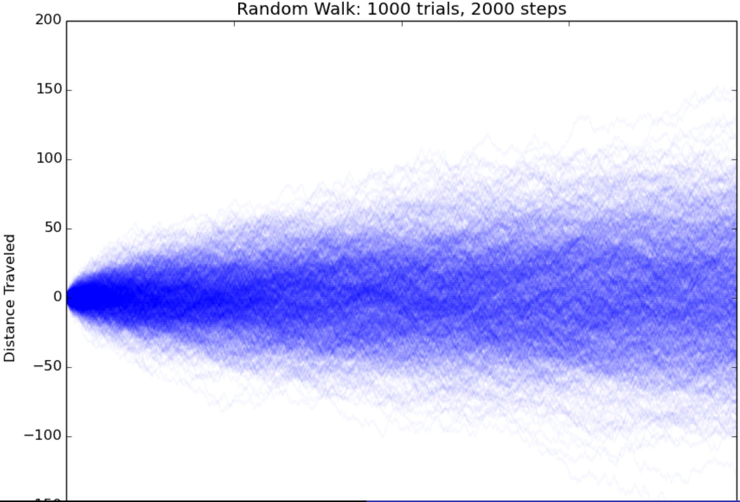

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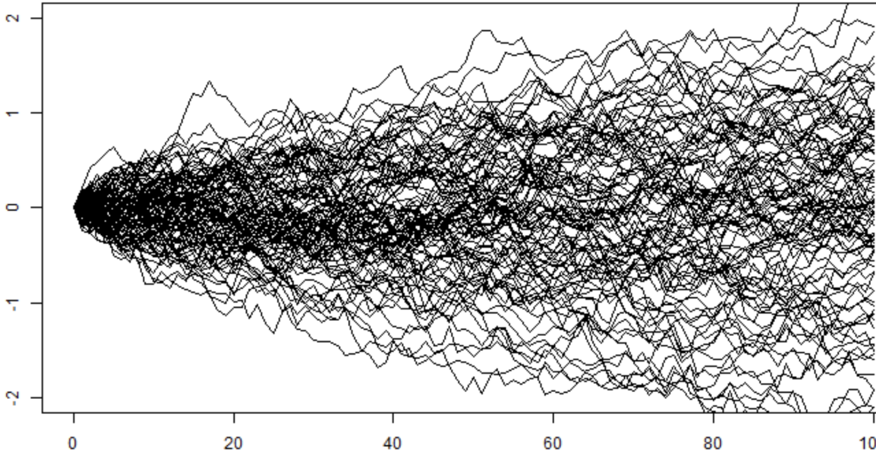

What exactly is space-time white noise?

Often space-time white noise is described as a condinuum iid Gaussian field, i.e.,

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

That's not rigorous and it needs to be interpreted in an integrated sense: ξ is a random variable taking values in Schwarz distributions such that (ξ, f) is always a Gaussian and

$$\mathbb{E}[(\xi, f)_{L^2}(\xi, g)_{L^2}] = (f, g)_{L^2},$$

where $L^2 = L^2(\mathbb{R}_+ \times \mathbb{R})$.

Construction of STWN

How to prove existence of such a random variable ξ taking values in $\mathcal{S}'(\mathbb{R}^2)$? Similar to construction of Brownian motion. Several options:

1. Use Kolmogorov's extension theorem to construct a projective family of r.v. $\{(\xi, f)\}$ indexed by $f \in \mathcal{S}$ such that the convariance structure of any finite subfamily is as specified.

2. Choose an orthonormal basis $\{e_j\}$ for $L^2(\mathbb{R}_+ \times \mathbb{R})$ and let ξ_j be iid N(0,1). Then define

$$(\xi, f) := \sum_j (e_j, f)\xi_j,$$

which always converges by L^2 martingale convergence theorem.

Construction of STWN

So far this defines a family (ξ, f) indexed by $f \in \mathcal{S}$ (in fact by $f \in L^2$) such that $(\xi, f + \alpha g) = (\xi, f) + \alpha(\xi, g)$ and such that

$$\mathbb{E}[(\xi, f)^2] = \|f\|_{L^2}^2.$$

After this, one still needs to "glue together" or "modify" this family of variables so that it can actually be realized as a random element of $\mathcal{S}'(\mathbb{R}^2)$. This is possible thanks to a Kolmogorov continuity criterion together with Gaussian tail bounds:

$$\mathbb{E}[|(\xi, f)|^p] \lesssim_p \|f\|_{L^2}^p.$$

Integration against ξ

Note that $(\xi, f)_{L^2(\mathbb{R}_+ \times \mathbb{R})}$ is well-defined for all $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$. It's usually denoted suggestively as

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x)\xi(t,x)dtdx$$

or as

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x)\xi(dt\ dx),$$

though it should be remarked that ξ is a.s. neither a function nor a measure.

Linear equations

Let's return to the additive-noise stochastic heat equation:

$$\partial_t h(t,x) = \Delta_x h(t,x) + \xi(t,x),$$

with $x \in \mathbb{R}^d$ and $t \geq 0$, and $h(0,x)$ some given function. Rearrange terms and formally apply the operator $e^{-t\Delta}$ to both sides to obtain

$$\partial_t (e^{-t\Delta} h) = e^{-t\Delta} \partial_t h - e^{-t\Delta} \Delta h = e^{-t\Delta} \xi.$$

Integrate both sides from 0 to t, then apply $e^{t\Delta}$:

$$e^{-t\Delta} h(t,\cdot) - h(0,\cdot) = \int_0^t e^{-s\Delta} \xi(ds,\cdot).$$

$$h(t,\cdot) = e^{t\Delta} h(0,\cdot) + \int_0^t e^{(t-s)\Delta} \xi(ds,\cdot).$$

Linear equations

What exactly is $e^{t\Delta}$? It's an operator that denotes the solution at time t to the solution of the equation

$$\partial_t h = \Delta h.$$

In other words $e^{t\Delta}$ is just convolution with the heat kernel:

$$e^{t\Delta} f(x) = \int_{\mathbb{R}} p(t, x-y) f(y) dy,$$

where

$$p(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}.$$

Linear SPDE

Summarizing, we have shown formally that the “solution” of

$$\partial_t h = \Delta h + \xi$$

is given by

$$h(t,x) = \int_{\mathbb{R}} p(t,x-y)h(0,y)dy + \int_{\mathbb{R}_+ \times \mathbb{R}^d} p(t-s,x-y)\xi(ds\ dy).$$

The integral in the second term on the RHS is deterministic and in $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ when $d = 1$.

One can retroactively check that this is indeed the solution in the sense of Schwarz distributions, i.e.,

$$-(h, \partial_t \phi) = (h, \Delta\phi) + (\xi, \phi)$$

a.s. for all smooth space-time Schwarz functions ϕ.

It turns out that (the derivative of) h describes the fluctuations in the Brownian Motion picture from earlier. We will prove this later.

What about $d > 1$?

The kernel fails to be in $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ for $d > 1$.

But intuitively one expects there to be a well-defined fluctuation field for 2d noninteracting Brownian motions.

It turns out that the only issue is the singularity of the kernel at the origin.

What about $d > 1$?

In particular if $\phi \in \mathcal{S}(\mathbb{R}^{d+1})$ then one can make sense of the smoothed out field

$$h(\phi) := \int_{\mathbb{R}^d} p^{\phi}(t, x-y)h(0,y)dy + \int_{\mathbb{R}_+ \times \mathbb{R}^2} p^{\phi}(t-s, x-y)\xi(ds\, dy),$$

where

$$p^{\phi}(t,x) = (p * \phi)(t,x) = \int_{\mathbb{R}^3} p(t-s, x-y)\phi(s,y)dsdy.$$

These random variables $h(\phi)$ as ϕ ranges through all Schwarz functions, can then be lifted to a random Schwarz distribution on $\mathbb{R}_+ \times \mathbb{R}^d$ which will solve the SHE in weak form.

Summarizing, the equation

$$\partial_t h = \Delta h + \xi$$

is solved by the Duhamel formula

$$h(t,x) = \int_{\mathbb{R}} p(t,x-y)h(0,y)dy + \int_{\mathbb{R}_+\times\mathbb{R}} p(t-s,x-y)\xi(dsdy).$$

One can show that for all d, $h(t,\cdot)$ can actually be evaluated as an element of $\mathcal{S}'(\mathbb{R}^d)$ for fixed times t and any initial data in $\mathcal{S}'(\mathbb{R}^d)$.

For fixed $t>0$ the field $h(t,\cdot)$ is locally absolutely continuous w.r.t Brownian motion when $d=1$ and w.r.t the Gaussian free field when $d=2$.

Remark: What about $\sigma \neq I$?

Nothing special about space-time white noise so far. We could replace it by any (possibly correlated) noise η and the solution is still given by the Duhamel formula:

$$h(t,x) = \int_{\mathbb{R}} p(t,x-y)h(0,y)dy + \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s,x-y)\eta(s,y)dsdy,$$

provided that the integral on the right hand side makes sense (possibly in a distributional sense). This will not be the case for all Gaussian noises η but it will be true for example when $\eta = \partial_x \xi$ or $\eta = (-\Delta_x)^\alpha \xi$.

Moving onto nonlinear SPDE...

So we can integrate deterministic integrands against ξ. What about random integrands? E.g. can we make sense of iterated integrals such as ...

$$\int \left[\int f(t,x,s,y)\xi(ds\,dy) \right] \xi(dt\,dx)$$

even with deterministic f? How about k-fold integrals such as

$$\int \cdots \int f(\mathbf{t},\mathbf{x})\xi^{\otimes k}(d\mathbf{t},d\mathbf{x})?$$

And what about things like

$$\int \sigma\left(\int f(t,x,s,y)\xi(ds\,dy) \right) \xi(dt\,dx)?$$

Motivation: why should we care about these objects?

Consider SPDE's such as the multiplicative SHE or its generalizations:

$$\partial_t f = \partial_x^2 f + f\xi.$$

$$\partial_t f = \partial_x^2 f + \sigma(f)\xi,$$

in spatial dimension $d = 1$.

We'll focus on the first one. Duhamel's principle still applies here, but as opposed to the linear case it gives an iterative relation rather than a finished solution, e.g.

$$f(t,x) = \int_{\mathbb{R}} p(t,x-y)f(0,y)dy + \int_{\mathbb{R}_+\times\mathbb{R}} p(t-s,x-y)f(s,y)\xi(ds,dy$$

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Multiplicative SHE

We can (Picard) iterate the previous relation once to obtain

$$f(t,x) = \int_{\mathbb{R}} p(t, x-y) f(0,y) dy$$

$$+ \int_{\mathbb{R}_+ \times \mathbb{R}} \left[\int_{\mathbb{R}} p(t-s, x-y) p(s, y-z) f(0,z) dz \right] \xi(ds, dz)$$

$$+ \int_{\mathbb{R}_+ \times \mathbb{R}} \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y) p(s-u, y-z) f(u,z) \xi(du, dz) \xi(ds, dy).$$

Keep iterating to obtain:

$$f(t,x) = \sum_{k=1}^{\infty} u_k(t,x)$$

where

$$u_{k+1}(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y) u_k(s,y) \xi(ds, dy)$$

and u_0 is just the heat flow started from $h(0,\cdot)$.

The explicit form of u_k :

Nonrecursively we have that $u_k(t, x)$ is given by

$$\int_{(\mathbb{R}_+ \times \mathbb{R})^{k+1}} \prod_{i=1}^{k+1} p(t_i - t_{i-1}, x_i - x_{i-1}) f(0, x_{k+1}) dx_{k+1} \xi^{\otimes k}(d\mathbf{t}, d\mathbf{x}),$$

with $t_{k+1} = t$ and $x_0 = x$.

Integration of adapted processes

The filtration $\mathcal{F}_t$ of ξ is defined to be the sigma algebras generated by (f, ξ) with f supported on $[0, t] \times \mathbb{R}$.

A random space time function $f(t, x)$ is called adapted to the filtration of ξ if $f(t, x)$ is $\mathcal{F}_t$ measurable for all t, x.

A random space-time function is called predictable if it lies in the L^2 closure of the linear span of elementary functions: things of the form $f(x, t, \omega) = X(\omega)1_{(a,b]}(t)1_E(x)$ where $E \subset \mathbb{R}^d$ is Borel and X is $\mathcal{F}_a$ measurable.

Theorem: any adapted continuous function is predictable.

Integration of adapted random processes

The integral of an elementary process $f(t,x) = X \cdot 1_{(a,b]}(t)1_E(x)$ against the noise can be defined in the obvious manner:

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x)\xi(dt\ dx) = X \cdot (\xi, 1_{(a,b]\times E}).$$

One has the Ito-Walsh isometry

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x)\xi(dt\ dx)\right)^2\right] = \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{E}[f(t,x)^2]dtdx,$$

which allows us to define integrals for any adapted continuous function, in particular iterated integrals as we wanted earlier.

Note that if $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$ then it is not true that

$$\int_{(\mathbb{R}_+ \times \mathbb{R})^k} \prod_1^k f(t_i, x_i) \xi^{\otimes k}(d\mathbf{t}, d\mathbf{x}) = (f, \xi)^k.$$

Rather the right hand side equals $H_k((\xi, f))$ when $\|f\|_{L^2} = 1$, where H_k is the k^{th} Hermite polynomial. Again, ξ is **not** a measure or a function.

In particular all k-fold iterated integrals are orthogonal to all n-fold iterated integrals for $k \neq n$. The set of all k-fold iterated integrals is called the k^{th} homogeneous chaos of ξ, denoted $\mathcal{H}^k(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem: $L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{k \in \mathbb{N}} \mathcal{H}^k(\Omega, \mathcal{F}, \mathbb{P})$.

Why does this happen?

Think about the simple case $k = 2$ with just a Brownian motion instead of white noise. Recall the computation of how

$$\int_0^1 B_t dB_t = \frac{1}{2}(B_t^2 - t).$$

There's a law of large numbers averaging happening at the second order. This is referred to as renormalization and tends to become relevant in all SPDE's with a nonlinear term such as a product.

Returning to the multiplicative SHE

Recall our formula for the solution of

$$\partial_t f = \partial_x^2 f + f\xi$$

was given by

$$f(t,x) = \sum_{k=1}^{\infty} u_k(t,x)$$

where

$$u_{k+1}(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y) u_k(s,y) \xi(ds, dy).$$

The iteration

So by the Ito isometry we have

$$\mathbb{E}[u_{k+1}(t,x)^2] = \int_{\mathbb{R}_+\times\mathbb{R}} p(t-s,x-y)^2 \mathbb{E}[u_k(s,y)^2] dsdy.$$

One can thus obtain inductive bounds that will show that

$$\sum_k \mathbb{E}[u_k(s,y)^2] < \infty.$$

To show this, one however needs fairly stringent assumptions on initial conditions, e.g.

$$\sup_{x\in\mathbb{R}} e^{-a|x|}\mathbb{E}[f(0,x)^2] < \infty.$$

Also d cannot be larger than 1.

Martingale methods

Given a space-time process defined on some probability space, how can one identify it as the solution of some SPDE?

First consider SDE. Suppose X_t is a continuous process defined on some space with the property that

$$M_t := X_t - \int_0^t b(X_s)ds$$

is a martingale such that

$$\langle M \rangle_t = \int_0^t \sigma(X_s)^2 ds,$$

where b, σ are smooth with $\sigma > 0$.

Then X_t must have the same law as the diffusion

$$dX = b(X)dt + \sigma(X)dB.$$

Proof: Let $B_t = \int_0^t \sigma(X_s)^{-1} dM_s$. Note that B is a martingale with quadratic variation t and therefore is a Brownian motion. Moreover by construction

$$X_t - \int_0^t b(X_s)ds = M_t = \int_0^t \sigma(X_s)dB_s.$$

Similar theory for white-noise driven SPDEs

Theorem [Konno-Shiga, '88]: suppose that $(f(t,x))_{t\geq 0, x\in\mathbb{R}}$ is a continuous process with the property that the processes

$$M_t(\phi) = (f(t,\cdot),\phi)_{L^2(\mathbb{R})} - \int_0^t (f(s,\cdot),\phi'')_{L^2(\mathbb{R})} ds$$

are martingales with respect to the filtration of f and that

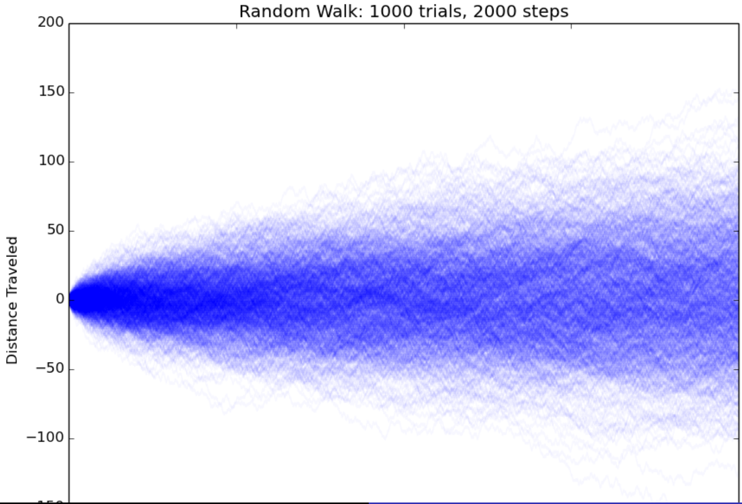
$$\langle M(\phi)\rangle_t = \int_0^t (\sigma(f(\cdot,t))^2\phi,\phi)_{L^2(\mathbb{R})} ds,$$

for all Schwartz functions ϕ on $\mathbb{R}$. Then f is distributed as the solution of

$$\partial_t f = \partial_x^2 f + \sigma(f)\xi.$$

Returning to the Brownian motions picture

Let's do a computation with the empirical measures.



The global limit

The global limit is the solution of the SPDE

$$\partial_t u = \partial_x^2 u + \partial_x\left(\sqrt{p(t,x)} \cdot \xi\right)$$

which happens to be the spatial derivative of

$$\partial_t h = \partial_x^2 h + \sqrt{p(t,x)} \cdot \xi,$$

which looks like SHE (Edwards-Wilkinson) if one zooms in locally around any deterministic space-time point.