On some models of last passage percolation and their scaling limits

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based on joint works with D. Betea and P. Ferrari

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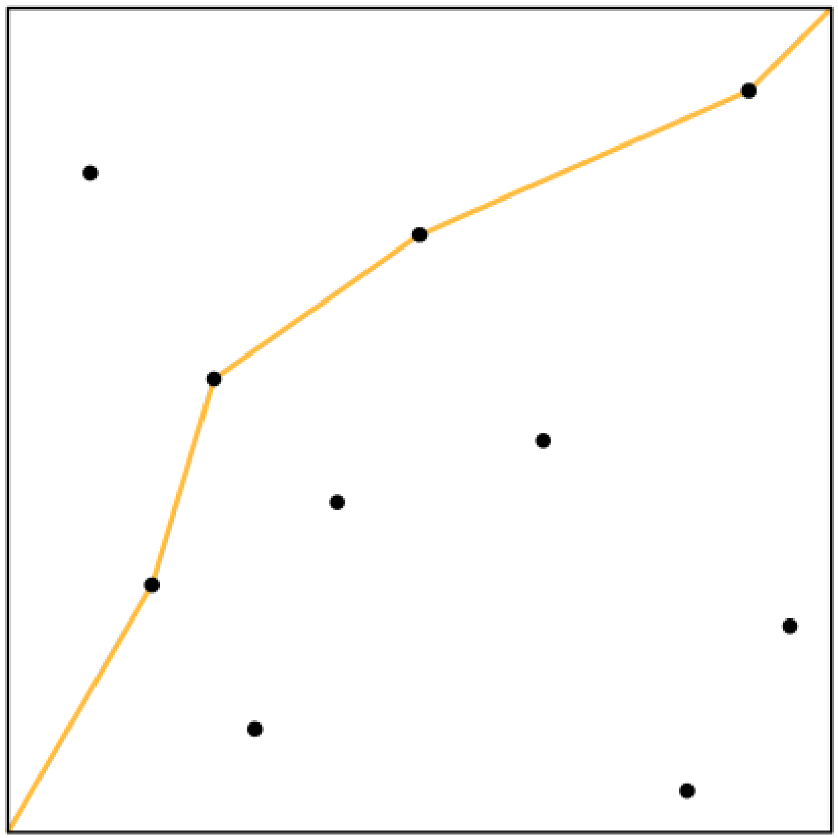
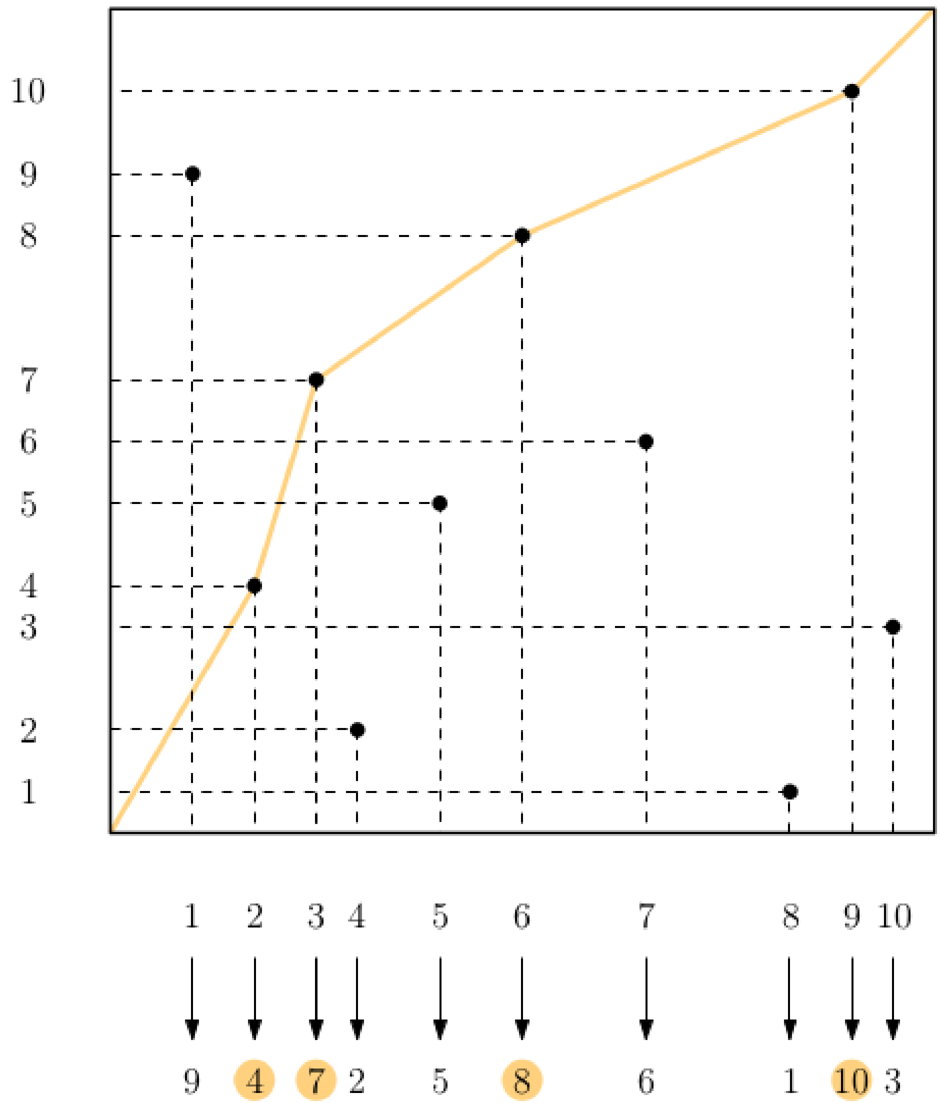
September 13th, 2021

Ulam's problem and Hammersley last passage percolation



$L =$ longest up-right path from $(0, 0)$ to $(1, 1)$

Ulam's problem and Hammersley last passage percolation

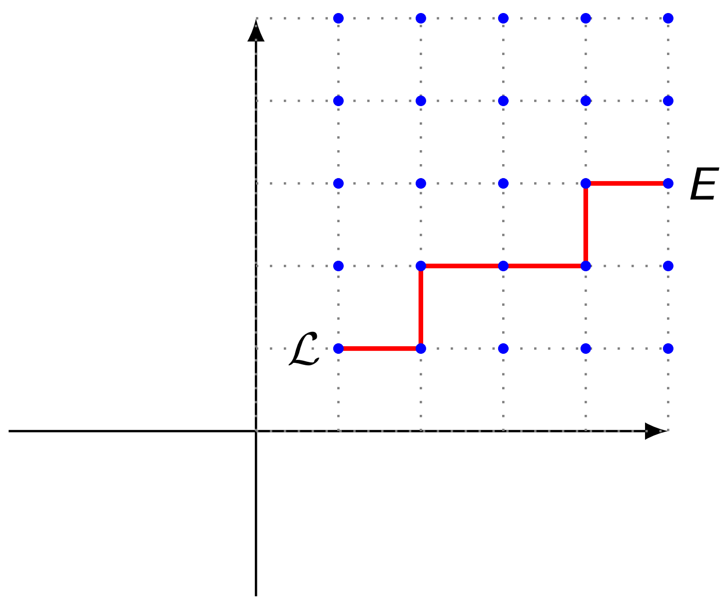


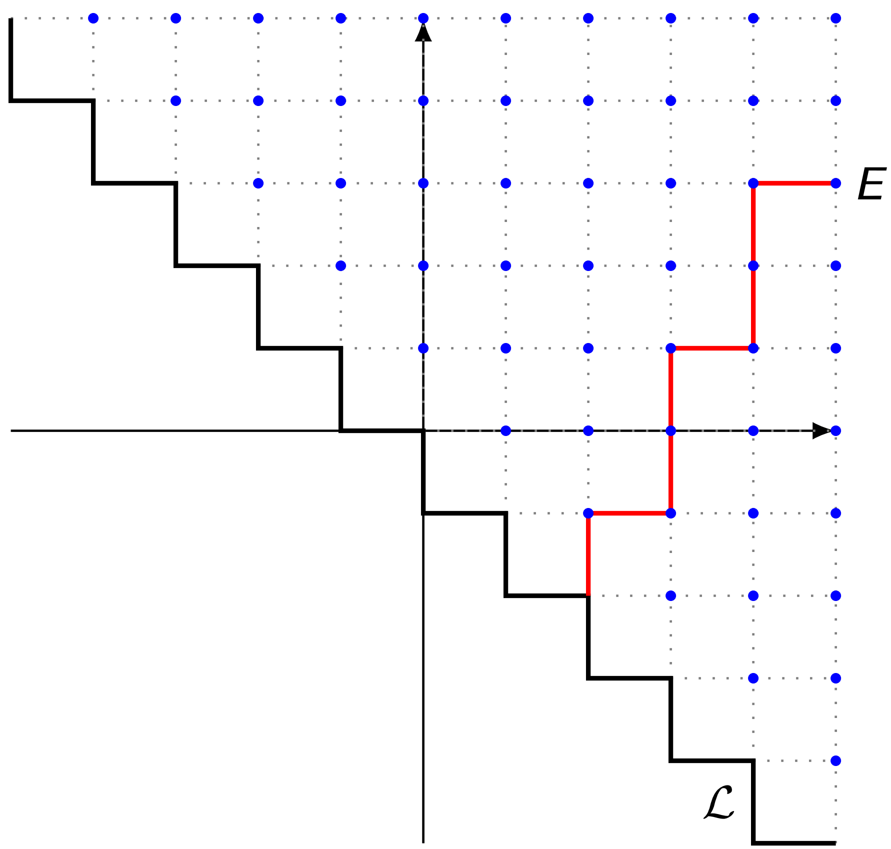
L is the length of the longest increasing subsequence in a random permutation of S_N with $N \sim Poisson(\theta^2)$

Interest: statistical properties of L when $\theta \to \infty$ (e.g. Baik–Deift–Johansson '99)

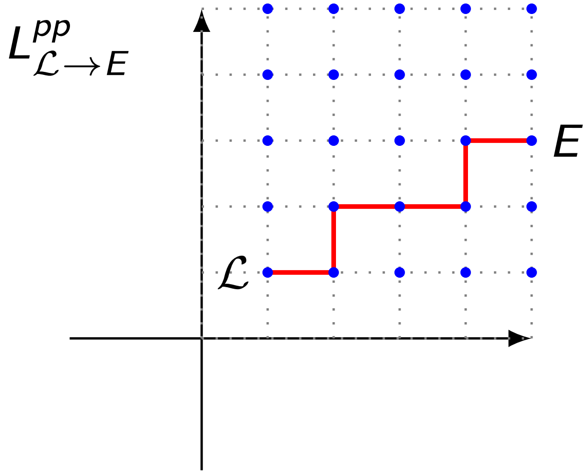
Last passage percolation on $\mathbb{Z}^2$

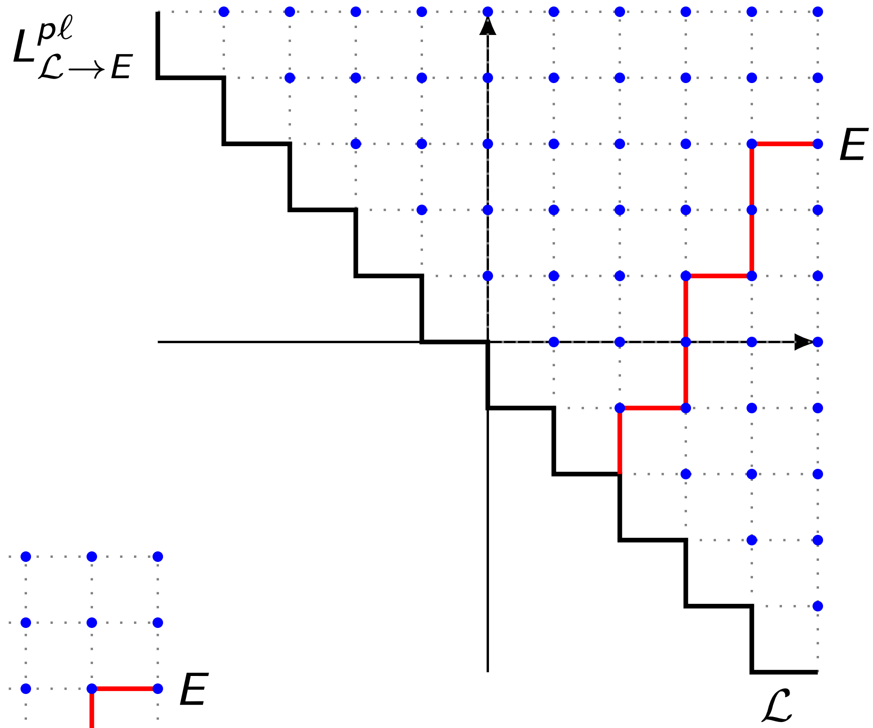
- $\mathcal{L}$ initial profile, E point in $\mathbb{Z}^2$
- $\omega_{i,j} \sim Exp(1)$, i.i.d. r.v.'s, $i,j \in \mathbb{Z}$
- Directed path π composed of $\rightarrow$ and $\uparrow$ s.t. $\pi(0) \in \mathcal{L}$ and $\pi(n) = E$
- Last passage time: $L_{\mathcal{L}\to E} = \max\limits_{\substack{\pi:A\to E\\ A\in\mathcal{L}}} \sum\limits_{1\leq k\leq n} \omega_{\pi(k)}$

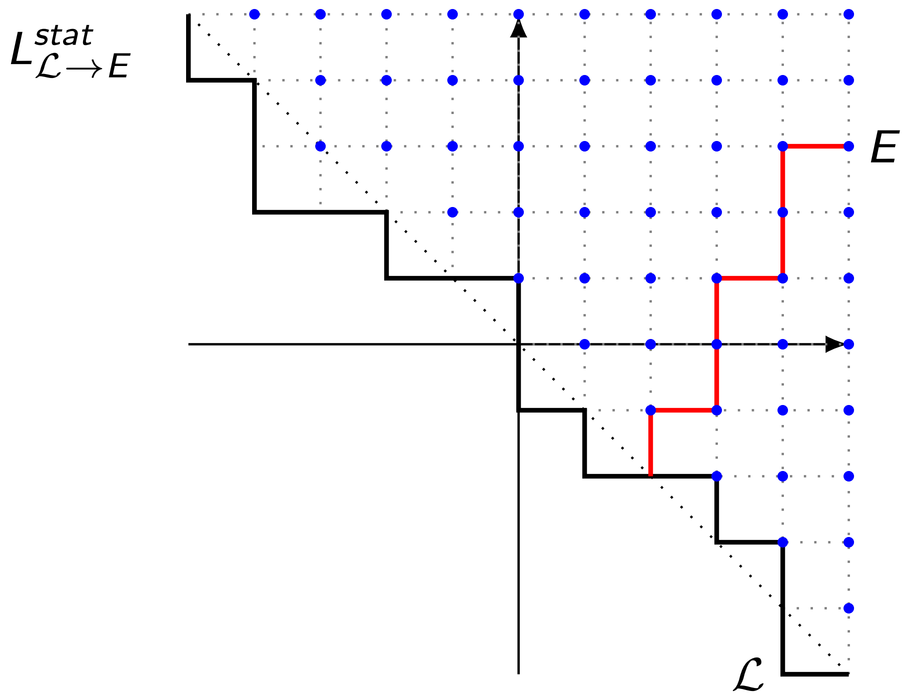




Geometries of LPP







stationary

$\mathcal{L} = \text{SRW}$

Scaling limits

We are interested in the scaling limit of last passage time $L^{\star}_{\mathcal{L}\to E_\tau(u)}$ with ending point $E_\tau(u) = \tau N(1,1) + u\,(2N)^{2/3}(1,-1)$ for $0 < \tau \le 1$ and $u \in \mathbb{R}$

$$L^{\star}_N(\tau,u) := \frac{L^{\star}_{\mathcal{L}\to E_\tau(u)} - 4\tau N}{2^{4/3}N^{1/3}}$$

$$\chi^{\star}(\tau,u) := \lim_{N\to\infty} L^{\star}_N(\tau,u)$$

for $\star \in \{pp, p\ell, stat\}$

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Airy processes

$$\chi^{pp}(1,u) = \mathcal{A}_2(u) - u^2$$

Prähofer–Spohn ‘02

$$\chi^{p\ell}(1,u) = 2^{1/3}\mathcal{A}_1(2^{-2/3}u)$$

Sasamoto ‘05

$$\chi^{stat}(1,u) = \mathcal{A}_{stat}(u)$$

Baik–Ferrari–Péché ‘09

$$\lim_{\epsilon\to 0} \epsilon^{-1/2}(\mathcal{A}_\star(\epsilon x) - \mathcal{A}_\star(0)) = \sqrt{2}B(x)$$

Hägg ‘08, Corwin–Hammond ‘11

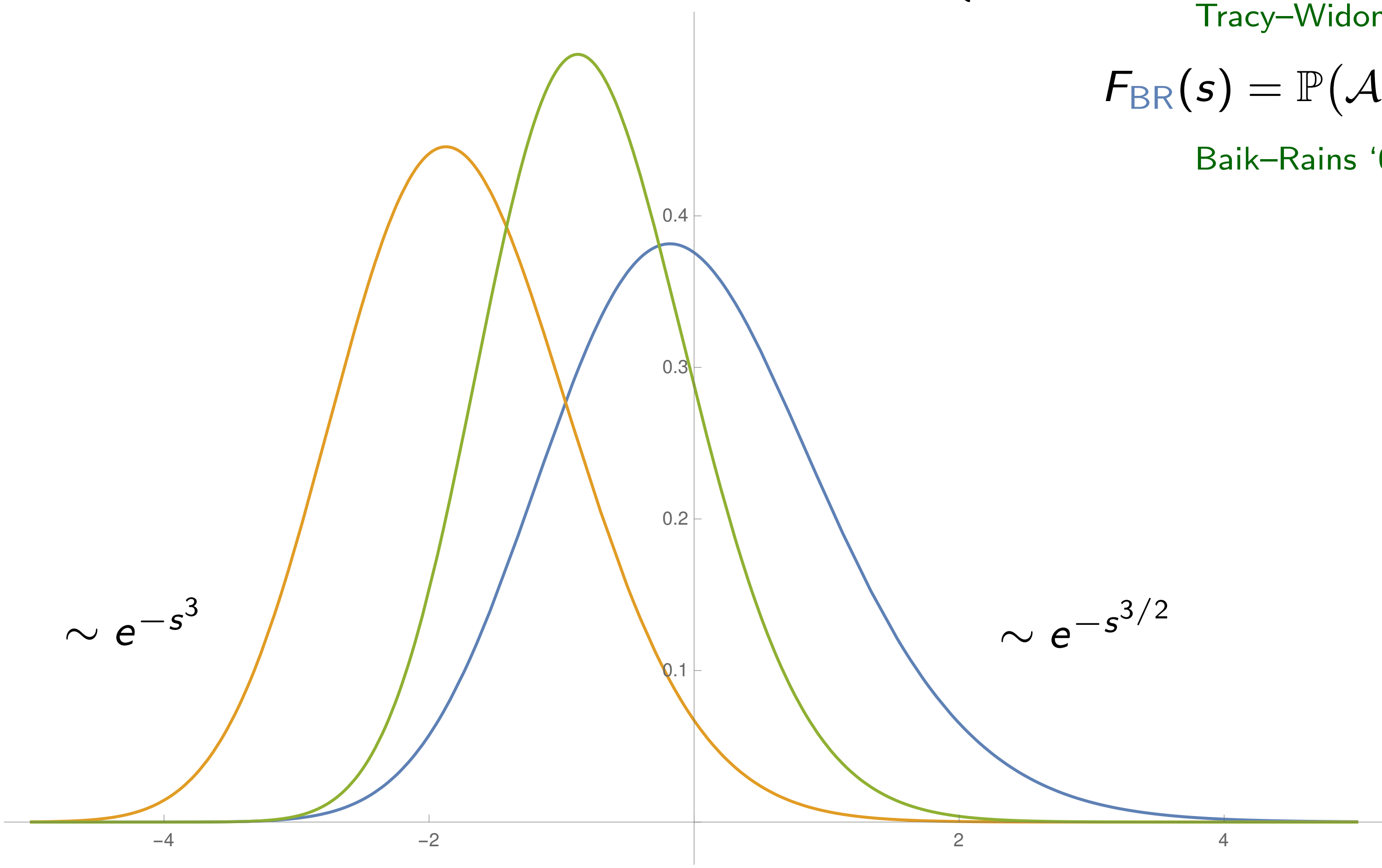
Scaling limits

Limit distributions of the largest eigenvalue in Gaussian ensembles of random matrices

$$\left\{ \begin{array}{l} F_{\mathrm{GUE}}(s) = \mathbb{P}\big(\mathcal{A}_2(0) \leq s\big) \\ \quad \text{Tracy–Widom '94} \\ F_{\mathrm{GOE}}(2s) = \mathbb{P}\big(\mathcal{A}_1(0) \leq s\big) \\ \quad \text{Tracy–Widom '96} \end{array} \right.$$

$$F_{\mathrm{BR}}(s) = \mathbb{P}\big(\mathcal{A}_{stat}(0) \leq s\big)$$

Baik–Rains '00

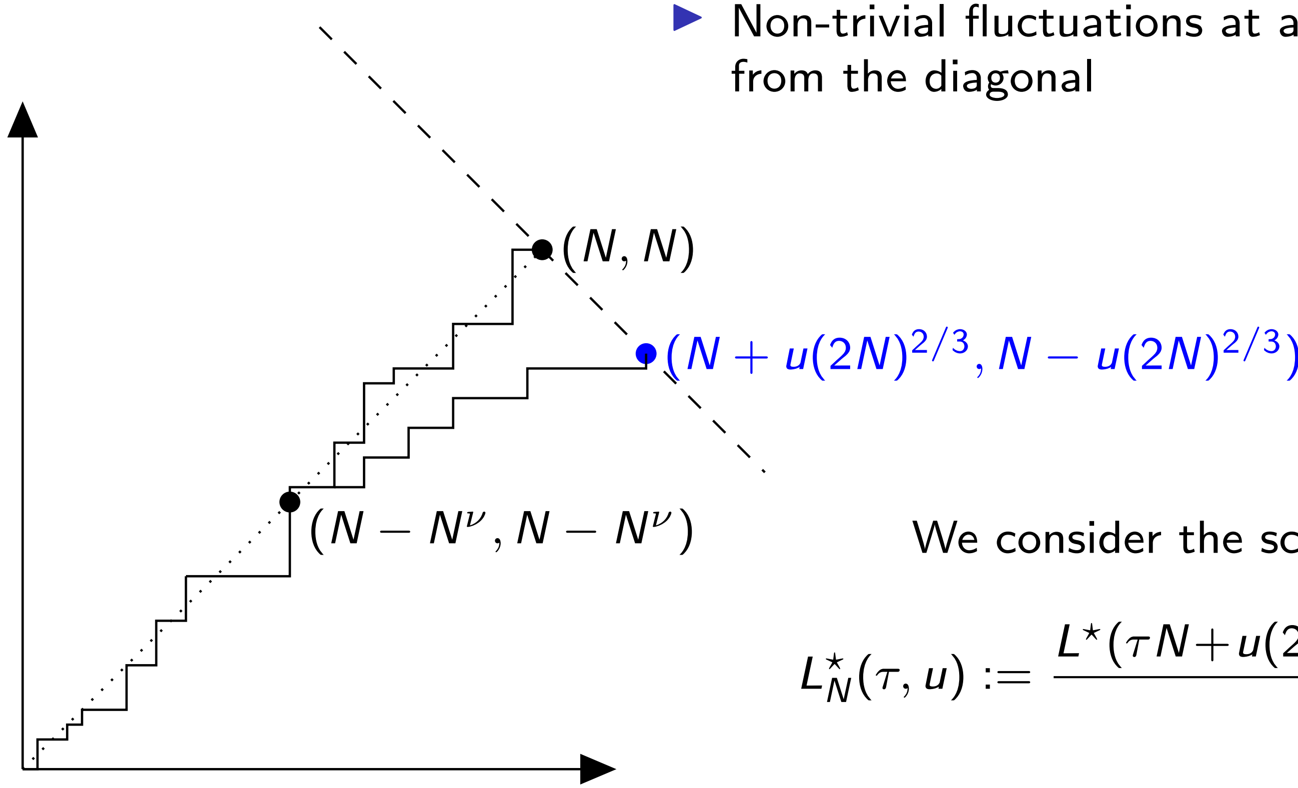


Space-time process

Slow decorrelation

Let $\nu < 1$, for any $M > 0$

$$\lim_{N\to\infty} \mathbb{P}\left(|L(N,N) - L(N-N^\nu, N-N^\nu) - 4N^\nu| \geq MN^{1/3}\right) = 0$$



- Non-trivial fluctuations at a distance of order $N^{2/3}$ from the diagonal

$\Downarrow$

We consider the scaled process

$$L_N^\star(\tau, u) := \frac{L^\star(\tau N + u(2N)^{2/3}, \tau N - u(2N)^{2/3}) - 4\tau N}{2^{4/3} N^{1/3}}$$

Time-time covariance: universality for short times

Let

$$\chi^\star(\tau) = \lim_{N\to\infty} L_N^\star(\tau, u_\tau)$$

and

$$\mathrm{Cov}\left(\chi^\star(\tau), \chi^\star(1)\right) = \mathbb{E}\left[\chi^\star(\tau)\chi^\star(1)\right] - \mathbb{E}\left[\chi^\star(\tau)\right]\mathbb{E}\left[\chi^\star(1)\right]$$

Takeuchi–Sano '12, Ferrari–Spohn '16

Theorem (Ferrari–O. '19)

As $\tau \to 1$, for any $\delta > 0$

$$\mathrm{Cov}\left(\chi^\star(\tau), \chi^\star(1)\right) = \frac{1}{2}\mathrm{Var}\left(\xi^\star(u_1)\right) + \frac{\tau^{2/3}}{2}\mathrm{Var}\left(\xi^\star(\tau^{-2/3}u_\tau)\right) - \frac{(1-\tau)^{2/3}}{2}\mathrm{Var}\left(\xi_{BR}((1-\tau)^{-2/3}(u_1-u_\tau))\right) + \mathcal{O}(1-\tau)^{1-\delta}$$

Lemma (Ferrari–O. '19)

For any $\delta > 0$, as $\tau \to 1$

$$\mathrm{Var}\left[\chi^\star(1) - \chi^\star(\tau)\right] = (1-\tau)^{2/3}\mathrm{Var}\left(\xi_{BR}((1-\tau)^{-2/3}(u_1-u_\tau))\right) + \mathcal{O}(1-\tau)^{1-\delta}$$

with

$$\xi_{BR}(u) \stackrel{d}{=} \max_{v\in\mathbb{R}}\{\sqrt{2}B(v) + \mathcal{A}_2(v) - (v-u)^2\}.$$

LPP concatenation property

Consider two paths with ending points

$$E_\tau = \tau N(1,1) + u_\tau\,(2N)^{2/3}(1,-1)$$

and

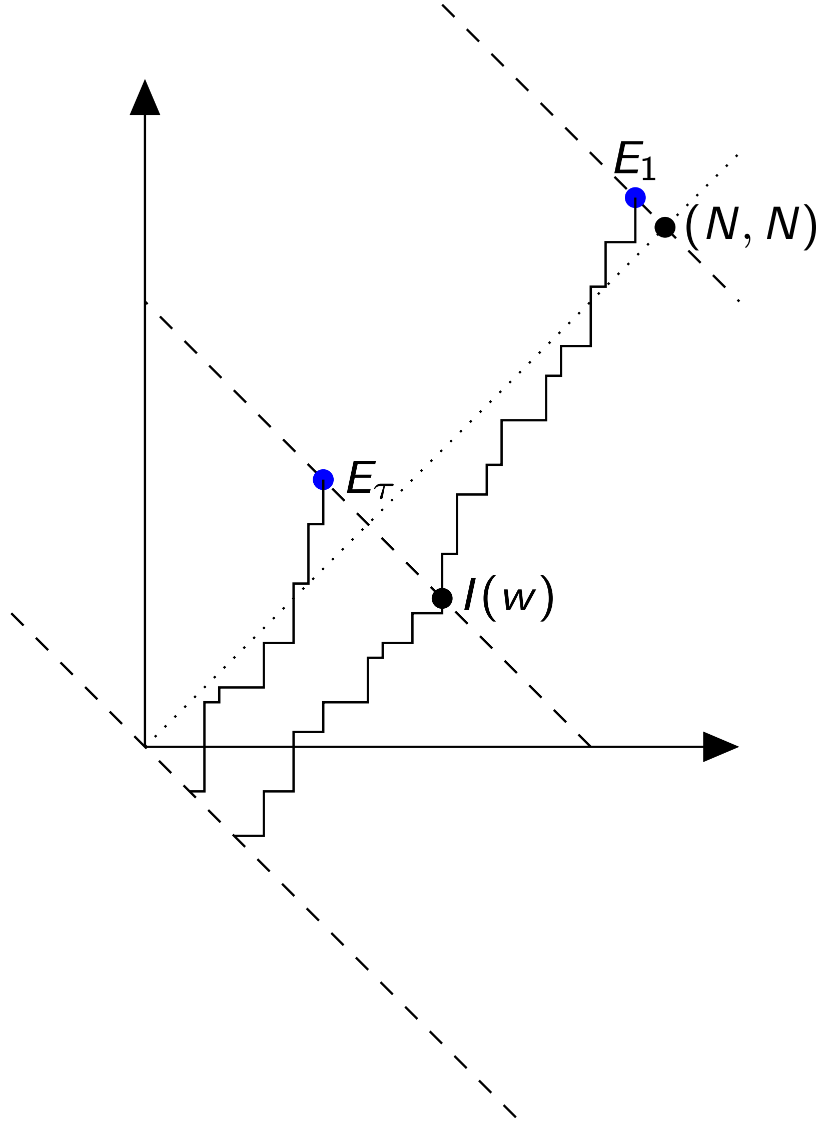
$$E_1 = N(1,1) + u_1\,(2N)^{2/3}(1,-1)$$

Define $I(w)$ as the intersection point of $L^\star_{\mathcal{L}\to E_1}$ with the antidiagonal through E_τ

$$I(w) = \tau N(1,1) + w\,(2\tau N)^{2/3}(1,-1)$$

$$\Downarrow$$

$$L^\star_{\mathcal{L}\to E_1} = \max_{w\in\mathbb{R}}\{L^\star_{\mathcal{L}\to I(w)} + L^{pp}_{I(w)\to E_1}\}$$



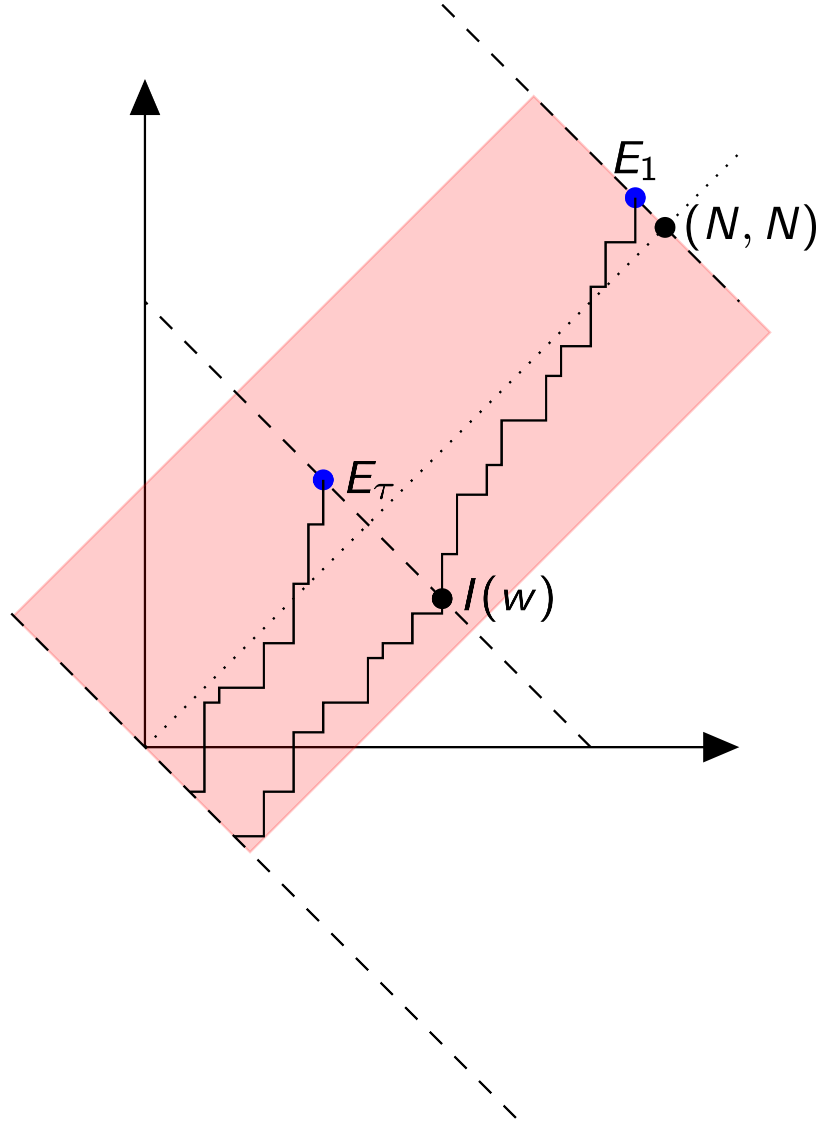
Path localization

$$L^{\star}_{\mathcal{L}\to E_1} = \max_{w\in\mathbb{R}} \{L^{\star}_{\mathcal{L}\to I(w)} + L^{pp}_{I(w)\to E_1}\}$$

1. Localization: the probability that the maximizing path passes through $I(w)$ with $|w| > M$ is bounded by Ce^{-cM^2} uniformly in N

   (obtained via comparison with the stationary model)

2. Local convergence: convergence of the covariance on the region $|w| \leq M$



Path localization

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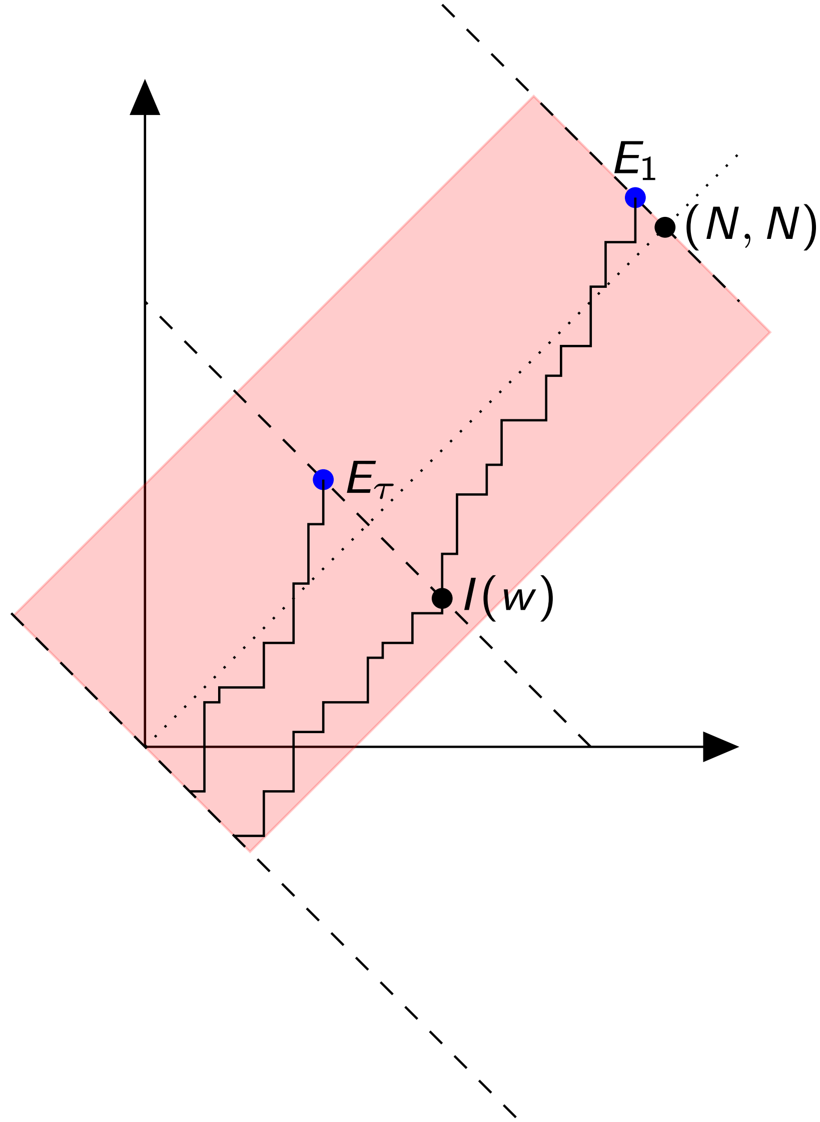

Covariance behavior as $\tau \to 1$

Case $u_\tau = u_1 = 0$

As $\tau \to 1$ $\quad \mathrm{Var}\left[\chi^\star(1) - \chi^\star(\tau)\right] = (1-\tau)^{2/3}\,\mathrm{Var}\left(\xi_{BR}\right) + \mathcal{O}(1-\tau)^{1-\delta}$ with

$$\xi_{BR} \stackrel{d}{=} \max_{v \in \mathbb{R}} \{\sqrt{2}B(v) + \mathcal{A}_2(v) - v^2\}$$

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Sketch of the proof

$$\chi^\star(1) - \chi^\star(\tau) = \max_{w\in\mathbb{R}}\Big\{\tau^{1/3}\left[\mathcal{A}^\star(\tau^{-2/3}w) - \mathcal{A}^\star(0)\right] + (1-\tau)^{1/3}\left[\mathcal{A}_2\left((1-\tau)^{-2/3}w\right) - (1-\tau)^{-4/3}w^2\right]\Big\}$$

$w = (1-\tau)^{2/3}v$

$$= (1-\tau)^{1/3}\max_{v\in\mathbb{R}}\Big\{\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\left[\mathcal{A}^\star\left(\left(\tfrac{1-\tau}{\tau}\right)^{2/3}v\right) - \mathcal{A}^\star(0)\right] + \mathcal{A}_2(v) - v^2\Big\}$$

Covariance behavior as $\tau \to 1$

Case $w = 0$

As $\tau \to 1$ $\quad \mathrm{Var}\left[\chi^\star(1) - \chi^\star(\tau)\right] = (1-\tau)^{2/3}\,\mathrm{Var}\left(\xi_{BR}\right) + \mathcal{O}(1-\tau)^{1-\delta}$ with

$$\xi_{BR} \stackrel{d}{=} \max_{v\in\mathbb{R}}\{\sqrt{2}B(v) + \mathcal{A}_2(v) - v^2\}$$

Sketch of the proof

$$\begin{aligned}\chi^\star(1) - \chi^\star(\tau) = &\max_{w\in\mathbb{R}}\Big\{\tau^{1/3}\left[\mathcal{A}^\star(\tau^{-2/3}w) - \mathcal{A}^\star(0)\right] \\ &+ (1-\tau)^{1/3}\left[\mathcal{A}_2\left((1-\tau)^{-2/3}w\right) - (1-\tau)^{-4/3}w^2\right]\Big\}\end{aligned}$$

$$\begin{aligned}=&(1-\tau)^{1/3}\max_{v\in\mathbb{R}}\Big\{\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\left[\mathcal{A}^\star\left(\left(\tfrac{1-\tau}{\tau}\right)^{2/3}v\right) - \mathcal{A}^\star(0)\right] \\ &+ \mathcal{A}_2(v) - v^2\Big\}\end{aligned}$$

As $\tau \to 1$,

$$\left(\tfrac{\tau}{1-\tau}\right)^{1/3}\left[\mathcal{A}^\star\left(\left(\tfrac{1-\tau}{\tau}\right)^{2/3}v\right) - \mathcal{A}^\star(0)\right] \simeq \sqrt{2}B(v)$$

with an error of order $\mathcal{O}(1-\tau)^{1-\delta}$, for any $\delta > 0$.

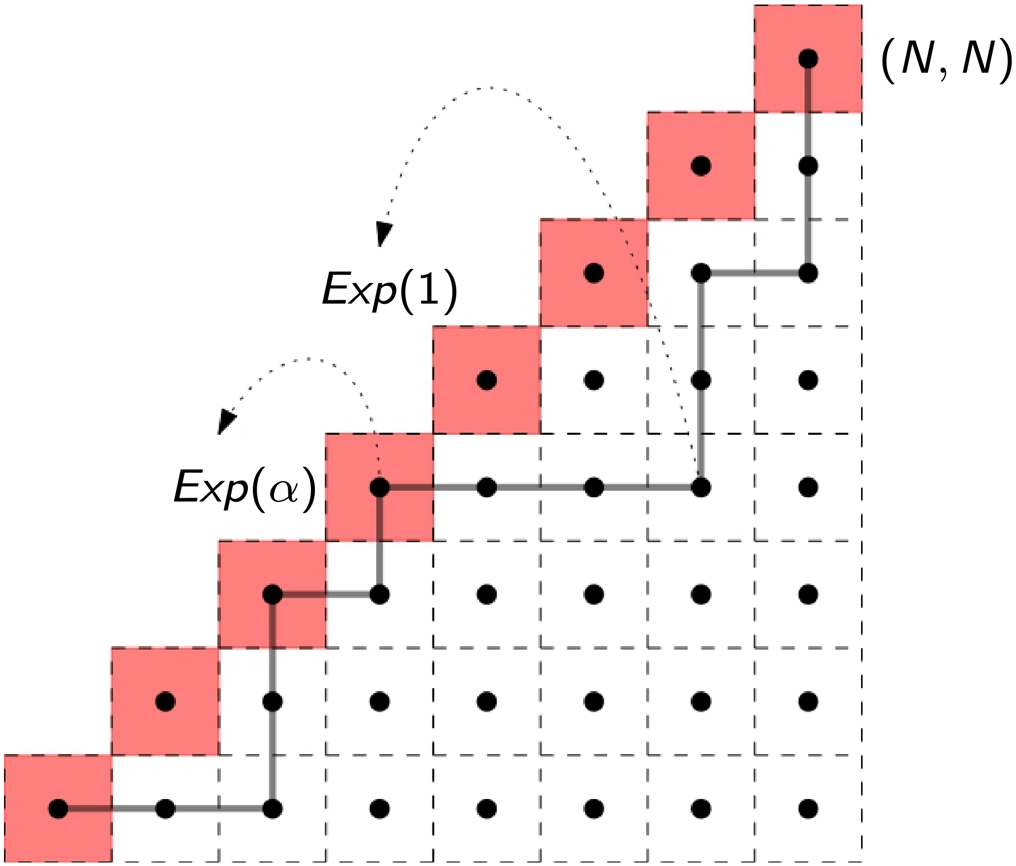
Half-space last passage percolation

- Model in half-space: TASEP on half-line with reservoir in the origin
- Equivalent to LPP on the full space with weights symmetric w. r. t. the diagonal

$$\omega_{i,j} \sim \begin{cases} Exp(1), & i \geq j+1 \\ Exp(\alpha), & i = j \end{cases}$$

Symmetrized LPP with geometric weights
Baik–Rains ’01
Sasamoto–Imamura ‘04

and exponential weights
Baik–Barraquand–Corwin–Suidan ‘18



Half-space last passage percolation

Theorem (BBCS '18)

a) *For* $\alpha > 1/2$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{N,N} - 4N}{2^{4/3} N^{1/3}} < s\right) = F_{GSE}(s).$$

b) *For* $\alpha = 1/2$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{N,N} - 4N}{2^{4/3} N^{1/3}} < s\right) = F_{GOE}(s).$$

c) *For* $\alpha < 1/2$ *and* $\sigma = \frac{(1-2\alpha)^{1/2}}{\alpha(1-\alpha)}$

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{N,N} - \frac{N}{\alpha(1-\alpha)}}{\sigma N^{1/2}} < s\right) = G(s).$$

d) *For any* $\kappa \in (0,1)$ *and* $\alpha > \frac{\sqrt{\kappa}}{1+\sqrt{\kappa}}$,

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{N,\kappa N} - (1+\sqrt{\kappa})^2 N}{\sigma N^{1/3}} < s\right) = F_{GUE}(s),$$

where $\sigma = \frac{(1+\sqrt{\kappa})^{4/3}}{\kappa^{1/6}}$.

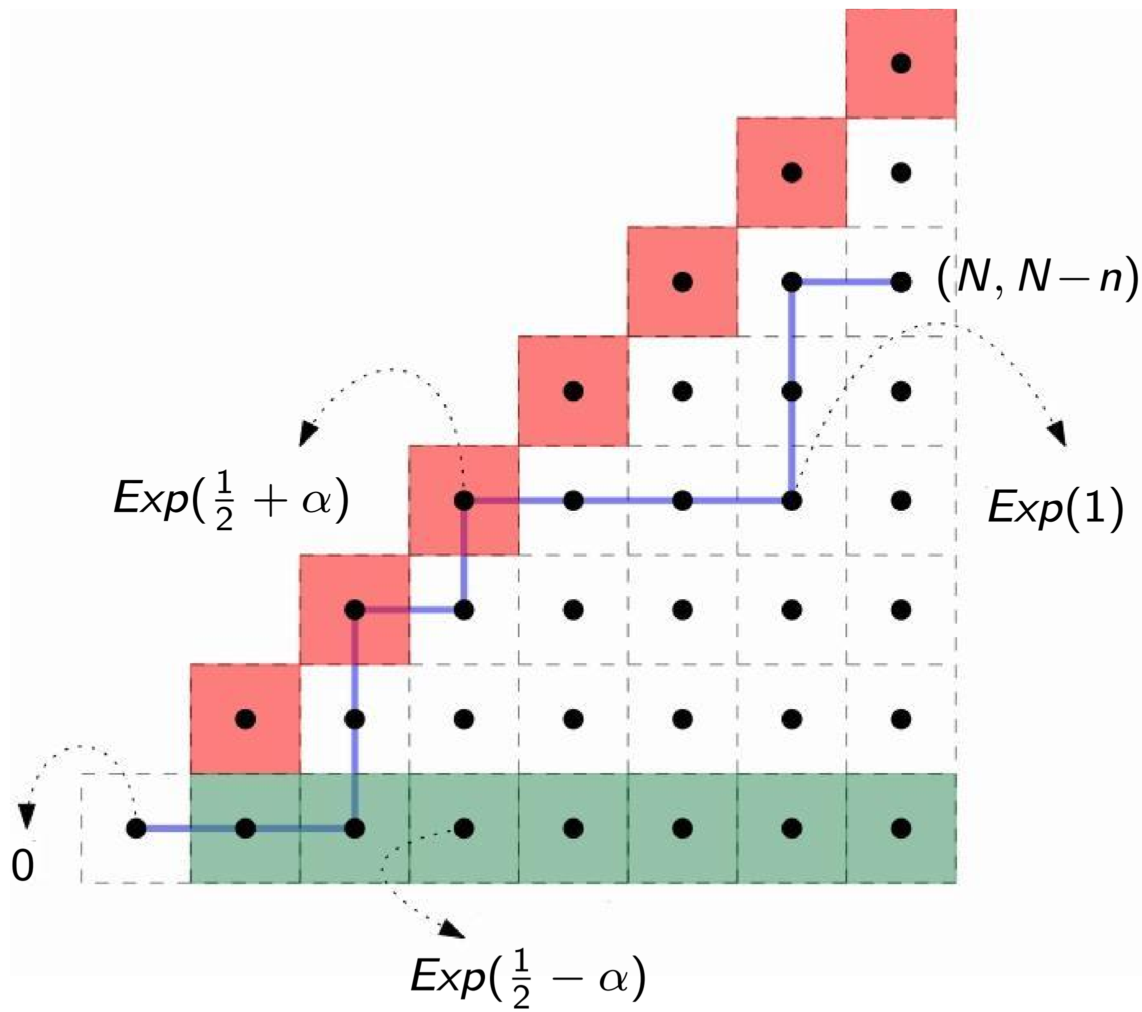
Stationary half-space LPP

We consider the half-space LPP from the origin to $(N, N-n)$ with the following weights

$$\omega_{i,j} \sim \begin{cases} Exp(\frac{1}{2}+\alpha) & i=j>1 \\ Exp(\frac{1}{2}-\alpha) & j=1, i>1 \\ 0 & i=j=1 \\ Exp(1) & \text{otherwise} \end{cases}$$

$\alpha \in (-1/2, 1/2)$

$L_{N,N-n}$ is stationary in the sense of Balász–Cator–Seppäläinen '16,
i.e. it has stationary increments along the vertical and the horizontal directions



Stationary full-space LPP: Baik–Rains '00

Limit distribution

Theorem (Betea–Ferrari–O. '19)
Let $\delta \in \mathbb{R}$, $u > 0$. *Let*

$$\alpha = 2^{-4/3}\delta N^{-1/3}, \quad n = u 2^{5/3} N^{2/3}.$$

Then

$$\lim_{N\to\infty} \mathbb{P}\left(\frac{L_{N,N-n} - 4N + 4u(2N)^{2/3}}{2^{4/3}N^{1/3}} \leq S \right) = F_{0,\,half}^{(\delta,u)}(S)$$

where

$$F_{0,\,half}^{(\delta,u)}(S) = \partial_S \left\{ pf(J - \overline{\mathcal{A}}) G_{\delta,u}(S) \right\}$$

with $J = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ *and*

$$G_{\delta,u}(S) = \mathcal{e}^{\delta,u}(S) - \left\langle -\mathcal{g}_1^{\delta,u} \quad \mathcal{g}_2^{\delta,u} \middle| (\mathbb{1} - J^{-1}\overline{\mathcal{A}})^{-1} \begin{pmatrix} -\mathcal{h}_1^{\delta,u} \\ \mathcal{h}_2^{\delta,u} \end{pmatrix} \right\rangle$$

- $\overline{\mathcal{A}} = \lim_{N\to\infty} \overline{K}$ is the limit kernel of Sasamoto–Imamura '04 and Baik–Barraquand–Corwin–Suidan '18 interpolating between the GOE, GSE, GUE and Gaussian distributions

A Pfaffian model

Consider the half-space LPP $\tilde{L}_{N,N-n}$ with weights

$$\tilde{\omega}_{i,j} \sim \begin{cases} Exp(\frac{1}{2}+\alpha) & i=j>1 \\ Exp(\frac{1}{2}+\beta) & j=1, i>1 \\ Exp(\alpha+\beta) & i=j=1 \\ Exp(1) & \text{otherwise} \end{cases}$$

where $\alpha \in (-1/2, 1/2)$, $\beta \in (0, 1/2)$ and $\alpha+\beta>0$

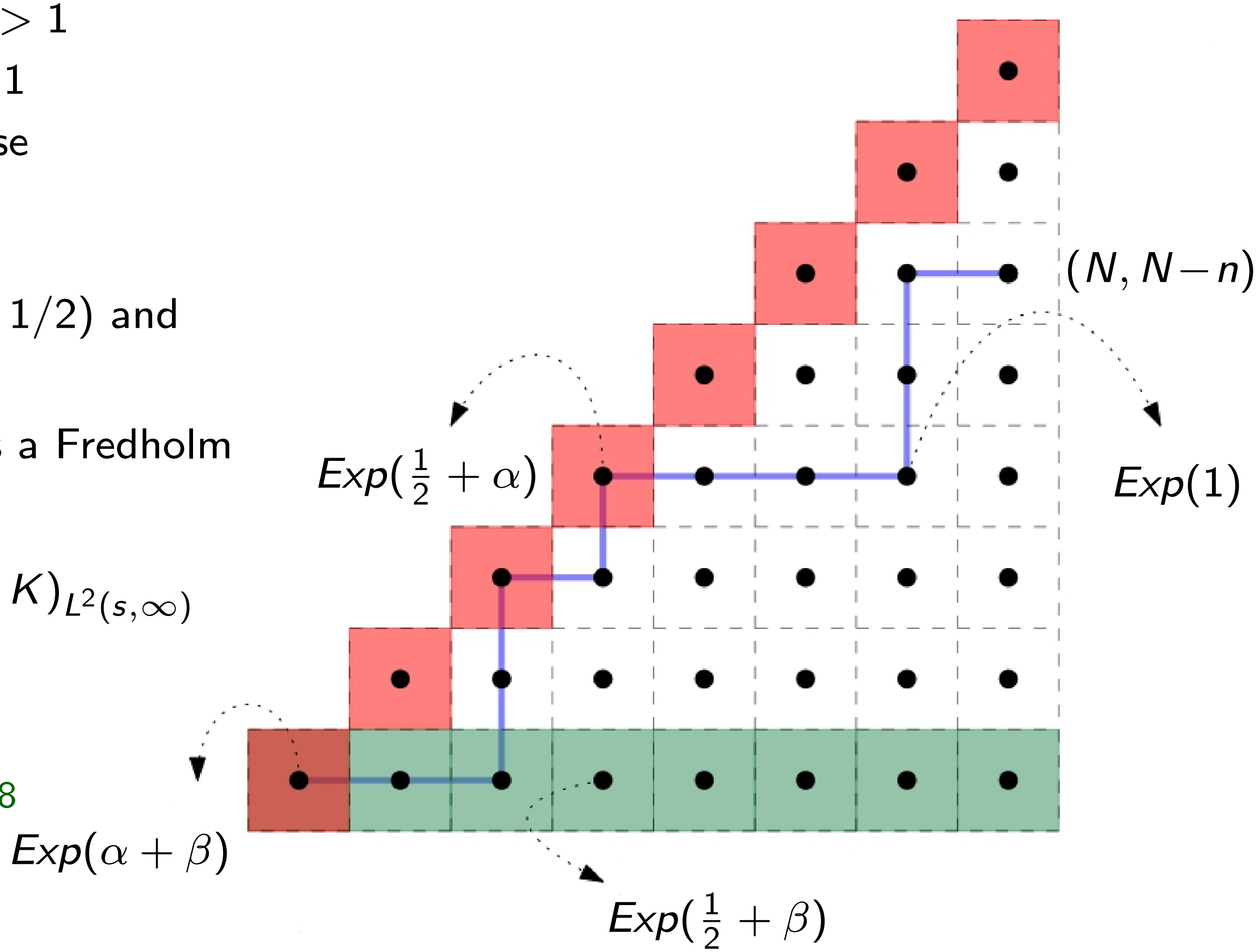
$\Rightarrow$ the distribution of $\tilde{L}_{N,N-n}$ is a Fredholm pfaffian

$$\mathbb{P}(\tilde{L}_{N,N-n} \le s) = pf(J-K)_{L^2(s,\infty)}$$

where K is a 2×2 matrix kernel

Rains '00

Baik–Barraquand–Corwin–Suidan '18



From integrable to stationary

1. **Shift argument**: Let $L^0_{N,N-n} = \tilde{L}_{N,N-n} - \tilde{\omega}_{1,1}$. For $\alpha + \beta > 0$,

$$\mathbb{P}(L^0_{N,N-n} \leq s) = \left(\mathbb{1} + \frac{1}{\alpha+\beta}\partial_s\right)\mathbb{P}(\tilde{L}_{N,N-n} \leq s)$$

   GOAL: obtain $L_{N,N-n} = \lim_{\alpha+\beta\to 0} L^0_{N,N-n}$

2. **Kernel decomposition**: The kernel K of $\tilde{L}_{N,N-n}$ splits as

$$K = \overline{K} + (\alpha+\beta)R$$

   where

$$R = \begin{pmatrix} |g_1\rangle\left\langle f_+^\beta\right| - \left|f_+^\beta\right\rangle\langle g_1| & \left|f_+^\beta\right\rangle\langle g_2| \\ -|g_2\rangle\left\langle f_+^\beta\right| & 0 \end{pmatrix}$$

$$\Rightarrow \mathbb{P}(L_{N,N-n} \leq s) = \lim_{\alpha+\beta\to 0} \partial_S \left\{\mathrm{pf}(J - \overline{K})\left(\frac{1}{\alpha+\beta} - \langle Y, (\mathbb{1} - \overline{G})^{-1}X\rangle\right)\right\}$$

   with $X = \left|\begin{matrix} 0 \\ f_+^\beta \end{matrix}\right\rangle$ and $Y = \langle -g_1 \quad g_2|$ and $\overline{G} = J^{-1}\overline{K}$

3. **Analytic continuation** $f_\beta(x) \sim e^{-\beta x}$ is diverging for $\beta < 0 \Rightarrow$ determine an expression of the kernel analytic in $(\alpha, \beta) \in (-1/2, 1/2)^2$

Limit to the Baik–Rains distribution

- Two-parameters family of distributions:

  u = distance of the end point from the diagonal

  δ = limit strength of the diagonal weights

Theorem (Betea–Ferrari–O. '19)
Let $S = s + \delta(2u + \delta)$ and $u + \delta = w$ fixed. Then

$$\lim_{u \to \infty} F^{(\delta,u)}_{0,\, half}(S) = F_{BR,w}(s)$$

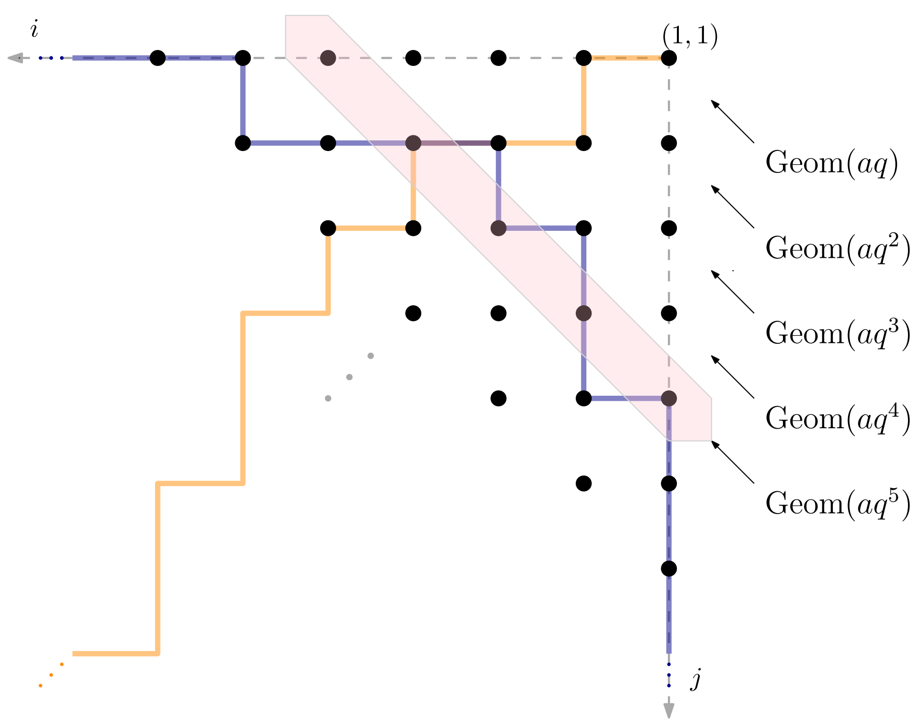
where $F_{BR,w}(s)$ is the extended Baik–Rains distribution

$$F_{BR,w}(s) = \partial_s \left[F_{GUE}(s + w^2) \left(\mathcal{R}_w - \left\langle \Psi_w \left| (\mathbb{1} - \mathcal{K}_{Ai,w})^{-1} \Phi_w \right\rangle \right) \right]$$

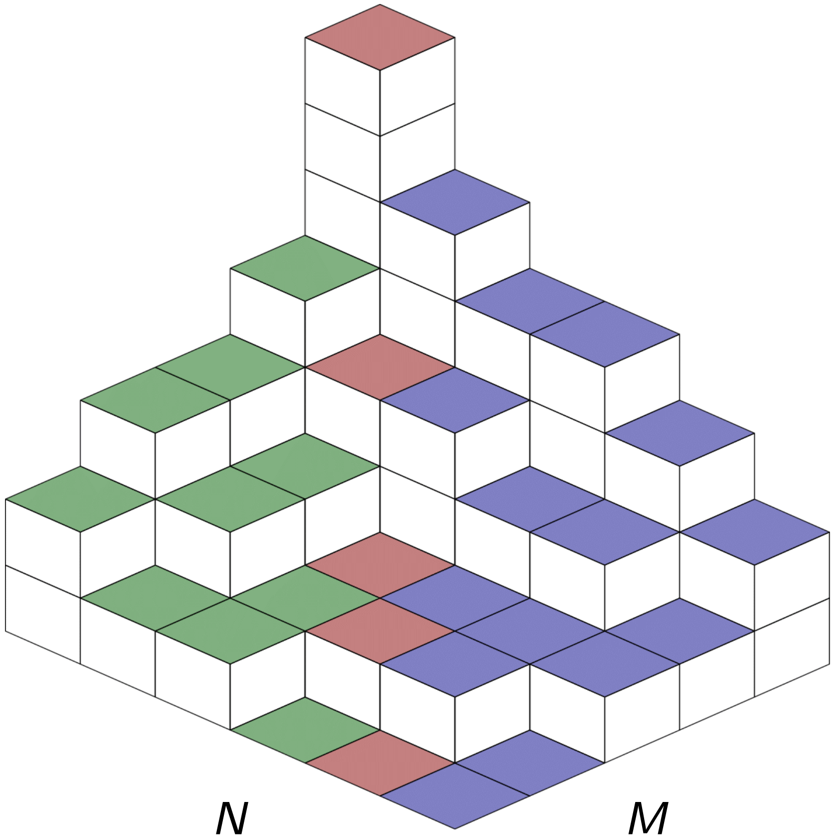
with $\mathcal{K}_{Ai,w}$ the (shifted) Airy kernel.

LPP with infinite geometry

- equi-distributed-by-diagonal full quarter plane: on $i+j=k+1$ $(k \geq 1)$ each variable is iid $Geom(aq^{i+j-1}) = Geom(aq^k)$
- $\mathbb{P}(Geom(u) = n) = u^n(1-u)$, $n \geq 0$
- L_1 = maximal path from $(1,1) \to (\infty,\infty)$ using down-left steps (orange)
- L_2 = maximal path from $(\infty,1) \to (1,\infty)$ using down-right steps (blue)
- by RSK correspondences L_1 and L_2 have the same distribution



Discrete Muttalib–Borodin distribution



$$\mathbb{P}(\Lambda) \propto q^{\eta \text{ left vol}} \left(aq^{\frac{\eta+\theta}{2}}\right)^{\text{central vol}} q^{\theta \text{ right vol}}$$

- We consider $\Lambda_{1,1}$ the corner (largest) part of a Muttalib–Borodin-distributed plane partition Λ when $\eta = \theta = 1$ and $M = N = \infty$

Main result

Theorem (Betea–O. '21)

Fix $\alpha \geq 0$. Let $q = e^{-\epsilon}$, $a = e^{-\alpha\epsilon}$ and $L \in \{L_1, L_2, \Lambda_{1,1}\}$. We have:

$$\lim_{\epsilon \to 0+} \mathbb{P}\left(\epsilon L + 2\log(\epsilon) < s\right) = \det(1 - O_\alpha)_{L^2(s,\infty)} := G_\alpha(s)$$

where $O_\alpha(x,y) = e^{-\frac{x}{2} - \frac{y}{2}} Bess_\alpha(e^{-x}, e^{-y})$ (Johansson '08) and $Bess_\alpha(x,y) = \int_0^1 J_\alpha(2\sqrt{tx}) J_\alpha(2\sqrt{ty}) dt$ is the RMT Bessel kernel (hard edge of Laguerre/Jacobi ensembles) with J the Bessel function.

Remark

G_α has interpolating properties (Johansson '08):

- $G_0(s) = e^{-e^{-s}}$ is the Gumbel distribution
- $\lim_{\alpha \to \infty} G_\alpha(-2\log(2(\alpha - 1)) + (\alpha - 1)^{-2/3} s) = F_{GUE}(s)$

Theorem (Betea–O. '21)

As $q \to 1-$, L has

- *Gumbel fluctuations, if $a = 1$*
- *transitional (exponential) hard-edge Bessel fluctuations, if $a \to 1$ critically*
- *Tracy–Widom fluctuations, if $0 < a < 1$ fixed ($N^{1/3}$ scaling)*

Thank you
for your attention!