Alpha ensembles and integrable systems

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Overview

1. α ensembles
2. Motivation
3. Density of states
4. Further connections

The talk is based on the following articles:

- G.M., *On the density of States of some matrices related to the beta ensembles and an application to the Toda lattice*, arXiv preprint:2008.04604.
- G.M., and P.J. Forrester, *The classical beta ensembles with beta proportional to 1/N: from loop equations to Dyson's disordered chain.* Journal of Mathematical Physics 62, 073505 (2021). DOI: 10.1063/5.0048481

α ensembles

Gaussian α ensemble	Laguerre α ensemble
$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha} & & \\ \chi_{2\alpha} & \mathcal{N}(0,2) & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{2\alpha} & \mathcal{N}(0,2) \end{pmatrix}$	$L_{\alpha,\gamma} = B_{\alpha,\gamma} B^{\intercal}_{\alpha,\gamma}$ $B_{\alpha,\gamma} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \chi_{\frac{2\alpha}{\gamma}} & & & \\ \chi_{2\alpha} & \chi_{\frac{2\alpha}{\gamma}} & & \\ & \ddots & \ddots & \\ & & \chi_{2\alpha} & \chi_{\frac{2\alpha}{\gamma}} \end{pmatrix}$

Antisymmetric Gaussian α ensemble
$$\mathrm{A}_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \chi_{2\alpha} & & \\ -\chi_{2\alpha} & 0 & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & -\chi_{2\alpha} & 0 \end{pmatrix}$$

Goal: compute the densities of states of these and related random matrix ensembles.
Why: these random matrix ensembles are related to the Gibbs ensemble of some integrable models, as first observed by H.Spohn.

The Toda Lattice in Thermal equilibrium

The classical Toda chain is the dynamical system described by the following Hamiltonian:

$$H_T(p,q) := \frac{1}{2}\sum_{j=1}^{N} p_j^2 + \sum_{j=1}^{N} V_T(q_{j+1}-q_j)\ , \quad V_T(x) = e^{-x} + x - 1\,,$$

with periodic boundary conditions $p_{j+N} = p_j\ ,\ q_{j+N} = q_j \quad \forall\, j \in \mathbb{Z}$.
Its equations of motion take the form

$$\dot{q}_j = \frac{\partial H_T}{\partial p_j} = p_j\,, \quad \dot{p}_j = -\frac{\partial H_T}{\partial q_j} = V_T'(q_{j+1}-q_j) - V_T'(q_j - q_{j-1}),\ \ j = 1,\dots,N\,.$$

One realizes that these equations are equivalent to the Lax pair

$$\dot{L} = [L, B]$$

where:

$$L(a,b) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_N \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & a_{N-1} & b_N \end{pmatrix}, \qquad \begin{cases} b_j = -p_j \\ a_j = e^{\frac{q_j - q_{j+1}}{2}} \end{cases},$$

$$B = L_+ - L_+^{\intercal}\,.$$

This implies that the eigenvalues $\lambda_1^{(N)} \leqslant \ldots \leqslant \lambda_N^{(N)}$ are constants of motion, so the system is integrable. They are also relevant to compute explicitly the solution of the Toda lattice.

The goal is to study some statistical properties of the Toda lattice in the **thermodynamic limit**, for this reason one endows the phase space

$$\mathcal{M} := \left\{ (p,q) \in \mathbb{R}^N \times \mathbb{R}^N \; : \quad \sum_{j=1}^N q_j = \sum_{j=1}^N p_j = 0 \right\} ,$$

with the Gibbs measure of the periodic Toda lattice:

$$\begin{aligned} \mathrm{d}\mu_{Toda} &:= \frac{1}{Z_{Toda}(\beta)} \; e^{-\beta H_T(p,q)} \delta_{\sum_{j=1}^N p_j} \delta_{\sum_{j=1}^N q_j} \mathrm{d}p \, \mathrm{d}q \\ &= \frac{1}{Z_{Toda}(\beta)} \; \prod_{j=1}^N a_j^{2\beta-1} e^{-\beta \sum_{j=1}^N \left(\frac{b_j^2}{2} + a_j^2 \right)} \delta_{\sum_{j=1}^N b_j} \delta_{\prod_{j=1}^N a_j - 1} \mathrm{d}a \, \mathrm{d}b \end{aligned}$$

Question

What is the mean eigenvalues' density (or density of States) $d\nu_{Toda}$ of the random Lax operator L, whose entries are distributed according to the Gibbs ensemble $d\mu_{Toda}$?

$$L(a,b) = \begin{pmatrix} b_1 & a_1 & 0 & \dots & a_N \\ a_1 & b_2 & a_2 & \ddots & \vdots \\ 0 & a_2 & b_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & a_{N-1} \\ a_N & \dots & 0 & a_{N-1} & b_N \end{pmatrix}, \qquad \begin{cases} b_j = -p_j \\ a_j = e^{\frac{q_j - q_{j+1}}{2}} \end{cases},$$

$$d\mu_{Toda} := \frac{1}{Z_{Toda}(\beta)} \prod_{j=1}^{N} a_j^{2\beta-1} e^{-\beta \sum_{j=1}^{N} \left(\frac{b_j^2}{2} + a_j^2 \right)} \delta_{\sum_{j=1}^N b_j} \delta_{\prod_{j=1}^N a_j - 1} da\, db, \quad a_j \geqslant 0 \ \forall j,$$

Problem: The variables are **dependent**.

Get rid of the dependence

Technical result

The density of states $\mathrm{d}\nu_{Toda}$ of the random Lax operator L with the entries distributed according to $\mathrm{d}\mu_{Toda}$ coincides to the density of states of $\frac{1}{\sqrt{\beta}}H_\alpha$ where:

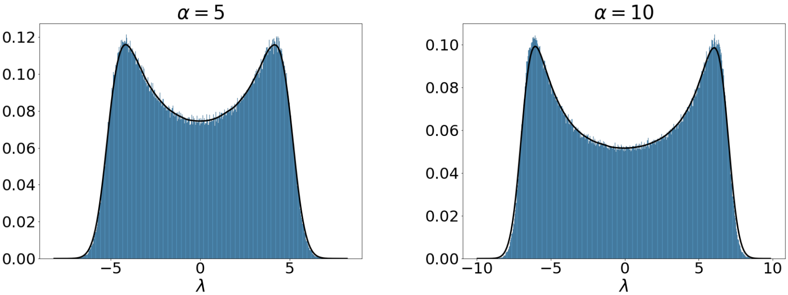
$$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha} & & \\ \chi_{2\alpha} & \mathcal{N}(0,2) & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{2\alpha} & \mathcal{N}(0,2) \end{pmatrix},$$

$\alpha = \beta + \theta$ and θ is chosen in such a way that

$$\int_{\mathbb{R}} r e^{-\beta e^{-r} - (\beta+\theta) r} \mathrm{d}r = 0\,.$$

Toda $\longleftrightarrow$ Gaussian α ensemble

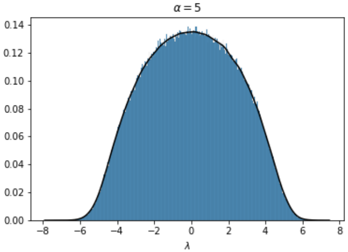
Gaussian α ensemble numerics



Gaussian β ensemble at high temperature

$$H_\beta^{h.t.} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix}, \quad \begin{cases} b_j \sim \mathcal{N}(0,2)\,, j = 1,\dots,N\,, \\ a_j \sim \chi_{2\alpha\left(1-\frac{j}{N}\right)}\,, j = 1,\dots,N-1\,. \end{cases}$$

The density of states deviate from the semicircle distribution.



Density of states

Allez, Bouchaud, and Guionnet studied this regime, and they were able to compute explicitly the density of states for this ensemble. Specifically, they proved the following:

Theorem - Allez, Bouchaud, and Guionnet

Consider the Gaussian Beta Ensemble in the high temperature regime, i.e. $\beta N \to 2\alpha$, $\alpha \in \mathbb{R}^+$. Then the density of states $\mathrm{d}\nu_{H_\beta^{h.t.}}$ has the following density function:

$$\mu_\alpha(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left|\hat{f}_\alpha(x)\right|^{-2}, \quad \hat{f}_\alpha(x) := \sqrt{\frac{\alpha}{\Gamma(\alpha)}} \int_0^\infty t^{\alpha-1} e^{-\frac{t^2}{2}} e^{ixt} \mathrm{d}t \,.$$

This result was obtained also by Duy and Shirai with a different approach.

Almost there...

$$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha} & & & \\ \chi_{2\alpha} & \mathcal{N}(0,2) & \chi_{2\alpha} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{2\alpha} & \mathcal{N}(0,2) \end{pmatrix}$$

$$H_\beta^{h.t.} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha(1-\frac{1}{N})} & & & \\ \chi_{2\alpha(1-\frac{1}{N})} & \mathcal{N}(0,2) & \chi_{2\alpha(1-\frac{2}{N})} & & \\ & \ddots & \ddots & \ddots & \\ & & \chi_{\frac{2\alpha}{N}} & \mathcal{N}(0,2) \end{pmatrix}$$

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$$H_\beta^{h.t.} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha(1-\frac{1}{N})} & & \\ \chi_{2\alpha(1-\frac{1}{N})} & \mathcal{N}(0,2) & \chi_{2\alpha(1-\frac{2}{N})} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{\frac{2\alpha}{N}} & \mathcal{N}(0,2) \end{pmatrix}$$

Question

It is possible to recover the density of states of the matrix H_α knowing the ones of $H_\beta^{h.t.}$? How to do it?

Yes, it is possible, and we will recover the density of states for the Gaussian α ensemble applying the **moment matching technique**.

We proved that there exists a polynomial $P_m(\alpha)$ such that:

$$u_m^{H_\alpha}(\alpha) = \int_{\mathbb{R}} x^m \mathrm{d}\nu_{H_\alpha} = P_m(\alpha)\,, \qquad u_m^{H_\beta^{ht}}(\alpha) = \int_{\mathbb{R}} x^m \mathrm{d}\nu_{H_\beta}^{h.t.} = \int_0^1 P_m(\alpha x)\mathrm{d}x\,,$$

thus the following relation holds:

$$u_m^{H_\alpha} = \partial_\alpha \left(\alpha u_m^{H_\beta^{ht}}(\alpha) \right)\,,$$

We proved that there exists a polynomial $P_m(\alpha)$ such that:

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thus the following relation holds:

$$u_m^{H_\alpha} = \partial_\alpha \left(\alpha u_m^{H_\beta^{ht}}(\alpha) \right)\,,$$

which implies that:

$$\mathrm{d}\nu_{H_\alpha} = \partial_\alpha(\alpha \mathrm{d}\nu_{H_\beta^{h.t.}})\,.$$

We proved the following:

Theorem

Consider the random matrix H_α, $\alpha > 0$, then:

$$\mathrm{d}\nu_{H_\alpha} = \partial_\alpha \left(\alpha \mu_\alpha(x)\right) \mathrm{d}x$$

where

$$\mu_\alpha(x) = \frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \left|\widehat{f}_\alpha(x)\right|^{-2}, \quad \widehat{f}_\alpha(x) := \sqrt{\frac{\alpha}{\Gamma(\alpha)}} \int_0^\infty t^{\alpha-1} e^{-\frac{t^2}{2}} e^{ixt} \mathrm{d}t\,.$$

This result was first obtained by H. Spohn.

Further generalizations

$$H_\alpha = \frac{1}{\sqrt{2}} \begin{pmatrix} b_1 & a_1 & & & \\ a_1 & b_2 & a_2 & & \\ & a_2 & b_3 & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ & & & a_{N-1} & b_N \end{pmatrix}$$

$$\mathrm{d}\mu \propto \prod_{j=1}^{N-1} a_j^{2\alpha-1} e^{-\mathrm{Tr}(V(T))} \mathrm{d}a \mathrm{d}b, \quad V(x) = \frac{x^2}{2}$$

H. Spohn , in a series of papers, proved this result for polynomial potential, and computed the correlation functions for the Toda lattice.

A. Guionnet, and R. Memin generalized this result for general potential, and obtained large deviations principles for the empirical measure.

What have we done?

$$H_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha} & & \\ \chi_{2\alpha} & \mathcal{N}(0,2) & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{2\alpha} & \mathcal{N}(0,2) \end{pmatrix}$$

$$H_\beta^{h.t.} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{N}(0,2) & \chi_{2\alpha(1-\frac{1}{N})} & & \\ \chi_{2\alpha(1-\frac{1}{N})} & \mathcal{N}(0,2) & \chi_{2\alpha(1-\frac{2}{N})} & \\ & \ddots & \ddots & \ddots \\ & & \chi_{\frac{2\alpha}{N}} & \mathcal{N}(0,2) \end{pmatrix}$$

$$u_m^{H_\alpha} = \partial_\alpha \left(\alpha u_m^{H_\beta^{ht}}(\alpha) \right)$$

What about the other β, and α ensembles? Is it possible to apply the same ideas?

Antisymmetric α ensemble

$$A_\alpha \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \chi_{2\alpha} & & \\ -\chi_{2\alpha} & 0 & \chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \\ & & -\chi_{2\alpha} & 0 \end{pmatrix}$$

$$A_\beta^{h.t.} \sim \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \chi_{2\alpha(1-\frac{1}{N})} & & \\ -\chi_{2\alpha(1-\frac{1}{N})} & 0 & \chi_{2\alpha(1-\frac{2}{N})} & \\ & \ddots & \ddots & \ddots \\ & & -\chi_{\frac{2\alpha}{N}} & 0 \end{pmatrix}$$

$$u_m^{A_\alpha} = \partial_\alpha \left(\alpha u_m^{A_\beta^{ht}}(\alpha) \right)$$

$$\mathrm{d}\nu_{A_\beta^{h.t.}} = \phi_\alpha(x)\mathrm{d}x \quad \mathrm{d}\nu_\alpha = \partial_\alpha(\alpha\phi_\alpha(x))\mathrm{d}x$$

It is true any time that you can change a stair into a slide!

Any relation to Integrable Systems?

Random Matrix	**Integrable System**
GαE	Toda lattice
CαE	Ablowitz-Ladik lattice
JαE	Discrete mKdV
AGαE	Discrete KdV

The relation between Toda and the GαE, and the one between JαE and the mKdV were first realized by H. Spohn. The work on the AGαE is in preparation with T. Grava, and M. Gisonni.

Summarizing

- We defined some new random matrix models called α ensembles;
- We computed their density of states, relying on the classical β ensembles;
- We connected the α ensembles to some integrable models.

Thank you for the attention!

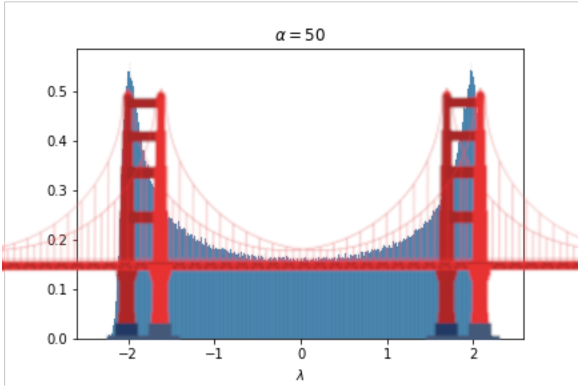

Figura: Wait what!?!?! That's the Golden Gate!