Extrema of Branching Brownian Motion in $\mathbb{R}^d$

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Joint works with Julien Berestycki, Bastien Mallein, Eyal Lubetzky, and Ofer Zeitouni.

Connections

Paradigmatic model in the study of extrema of log-correlated fields– see Ofer's talk from 9/1.

- Random matrix theory
- Statistics of Riemann zeta function
- GFF and GMC
- Cover times of random walks
- PDEs (F-KPP equation)
- ...

Model Definition

Fix the dimension $d \geq 1$.

- Start: a single particle v at 0 performs Brownian motion in $\mathbb{R}^d$ (iid 1d BM's in each coordinate)

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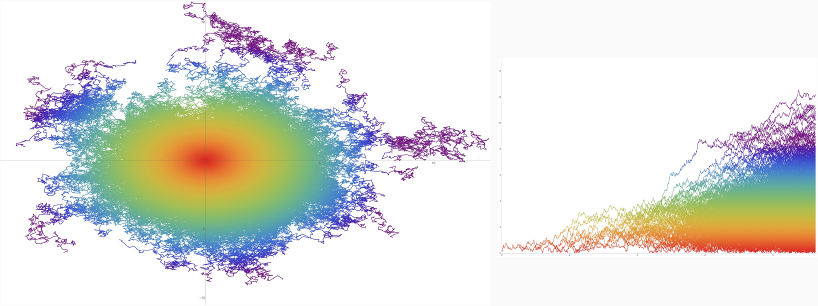

Fig: 2D BBM (left) and its modulus as a function of time (right)

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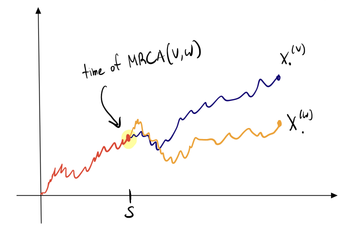
- After an $\exp(1)$ distributed time, the particle splits into two particles, which perform independent (d-dim) BM's from that point forward and carry their own exponential clocks
- Repeat.

Notation.

- Let $N_t :=$ set of particles at time t.
- $X_\cdot^{(v)} :=$ the d-dim BM path of v.

Quick facts.

- $\mathbb{E}[N_t] = e^t$
- For coordinate indices $i, j \in \{1, \ldots, d\}$, $\mathrm{Cov}(X_t^{(v)}(i), X_t^{(w)}(j)) =$ time of the most recent common ancestor of v and w



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- **Bramson**: found centering term $m_t(1) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t$, and showed $R_t^* - m_t(1)$ converges in law via connection to F-KPP equation.

Theorem (Bramson '77, '83)

$\lim_{t \to \infty} \mathbb{P}(R_t^* - m_t(1) \leq y) = w(y)$, *where* $\frac{1}{2} w'' + \sqrt{2} w' + w^2 - w = 0$.

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- **Idea from physics**: the distribution of the max. of log-correlated fields should still resemble classical (iid) extreme value distributions
- **Lalley-Selke**: characterization of the limiting law as a *randomly shifted Gumbel*, where the random shift comes from the behavior of the particles at the beginning of the process (more on this later).

Theorem (Lalley-Selke '87)

There exists a random variable Z, called the derivative martingale, and a constant $C > 0$ such that

$$\lim_{t\to\infty} \mathbb{P}(R_t^* - m_t(1) \le y) = \mathbb{E}\Big[\exp\big(-e^{-\sqrt{2}(y-CZ)}\big)\Big], \ \forall y \in \mathbb{R}.$$

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$$\mathcal{E}_t := \sum_{v \in N_t} \delta_{\|X_t^{(v)}\| - m_t(1)} .$$

- The limit was identified independently by Aidekon-Berestycki-Brunet-Shi and Arguin-Bovier-Kistler as a randomly shifted, decorated Poisson point process
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Let $\sum_{i \in \mathbb{N}} \delta_{\eta_i}$ have law $\mathrm{PPP}(C\sqrt{2}e^{-\sqrt{2}x}\mathrm{d}x)$. Let $\{\mathcal{D}^{(i)}\}_{i \in \mathbb{N}}$ be a family of iid point processes with an explicit "decoration law." Then $\mathcal{E}_t$ converges in law as $t \to \infty$ to

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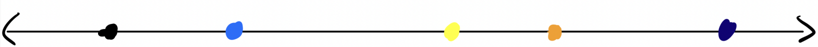

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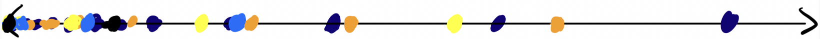

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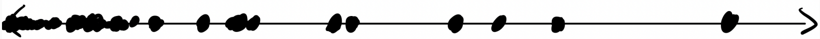

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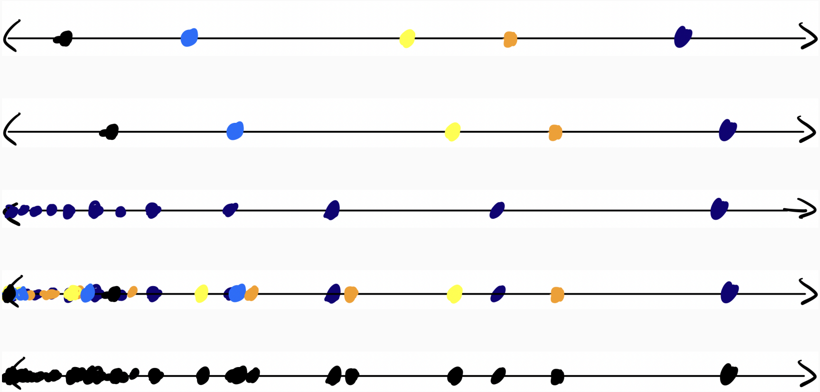

- Lots of further work: extending convergence to include genealogical info (Bovier-Hartung 2016), structure of extreme level sets (Cortines-Hartung-Louidor 2017), etc.

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Let $m_t := \sqrt{2}t + \frac{d-4}{2\sqrt{2}} \log t$*. There exists a non-degenerate, positive random variable* Z_∞ *and a constant* $C_d > 0$ *such that*

$$\lim_{t \to \infty} \mathbb{P}(R_t^* \leq m_t + y) = \mathbb{E}\Big[\exp\Big(-C_d e^{-\sqrt{2}(y - \frac{1}{\sqrt{2}}\log Z_\infty)}\Big)\Big], \ \forall y \in \mathbb{R}\,.$$

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Previous work:

- Biggins '95: $R_t^*/\sqrt{2}t \to 1$ a.s.
- Mallein 2015: $\{R_t^* - m_t\}_{t>0}$ is tight.

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Girsanov transform from Bessel to BM

$$\mathrm{d}P^R\big|_{\mathcal{F}_t} = \underbrace{\left(\frac{W_t}{W_0}\right)^{\frac{d-1}{2}}}_{\substack{\text{start/endpoint}\\ \text{dependence}}} \underbrace{\exp\left(\int_0^t \frac{c_d}{W_u^2}\mathrm{d}u\right)\mathbb{1}_{\{W_u>0,\ u\in[0,t]\}}}_{\text{pathwise dependence}} \mathrm{d}P^W\big|_{\mathcal{F}_t},$$

where $c_d > 0$ for $d \geq 3$ and $c_d < 0$ for $d = 2$.

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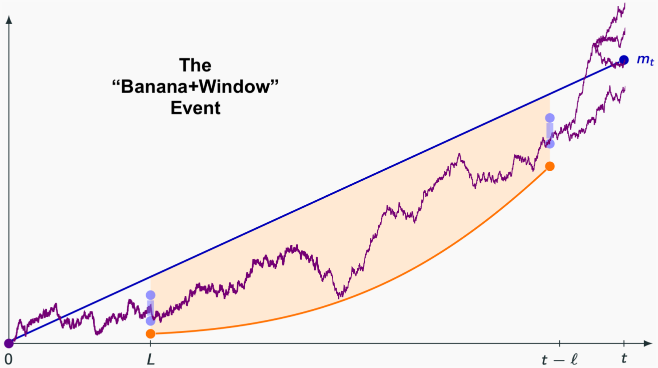

Extremal particle trajectories

W.h.p., all particles $v \in N_t$ that reach height m_t at time t did the following:

- at time L, passed through the "window" $\sqrt{2}L - [L^{1/6}, L^{2/3}]$
- at time $t-\ell$, passed through the "window" $\frac{m_t}{t}(t-\ell) - [\ell^{1/3}, \ell^{2/3}]$
- on $[L, t-\ell]$, stayed within the "banana"

Proof of Main Result

Say we have a particle $v \in N_L$ such that $R_L^{(v)} = r \in \sqrt{2}L - [L^{1/6}, L^{2/3}]$.

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We show that, on the Banana+Window event, at most one $u \in N_{t-L-\ell}$ produces a descendant that exceeds height m_t. Done via a "modified" second moment method, inspired by the work of Bramson-Ding-Zeitouni (2013) establishing convergence of the re-centered maximum of the 2d discrete GFF.

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Proof of main result:

$$\mathbb{P}(R^*_t \le m_t + y) = \mathbb{E}\Big[\prod_{v \in N_L} \mathbb{P}_{R_L^{(v)}}(R^*_{t-L} \le m_t + y)\Big] = \mathbb{E}\Big[\prod_{v \in N_L} \big(1 - \mathbb{P}_{R_L^{(v)}}(R^*_{t-L} > m_t + y)\big)\Big]$$

Proof of Main Result

Say we have a particle $v \in N_L$ such that $R_L^{(v)} = r \in \sqrt{2}L - [L^{1/6}, L^{2/3}]$.

What is the probability that v produces a descendant in N_t that exceeds height m_t?

$$\begin{aligned}
(\text{Markov}) &= \mathbb{P}_r(R^*_{t-L} > m_t,\ \text{Banana+Window event}) \\
&= \mathbb{P}_r\Big(\bigcup_{u \in N_{t-L-\ell}} BW(u) \Big) \\
&\sim e^{t-L-\ell}\mathbb{P}_r\big(BW(u)\big) \\
&\sim C_d r^{-\frac{d-1}{2}}(\sqrt{2}L - r)e^{-\sqrt{2}(\sqrt{2}L-r)} =: C_d f(r)\,.
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&\approx \mathbb{E}\Big[\prod_{v \in N_L} \big(1 - C_d f(R_L^{(v)})e^{-y\sqrt{2}}\big)\Big]
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&\approx \mathbb{E}\Big[\prod_{v \in N_L} \big(1 - C_d f(R_L^{(v)}) e^{-y\sqrt{2}}\big) \Big] \approx \mathbb{E}\Big[\exp\Big(- C_d \sum_{v \in N_L} f(R_L^{(v)}) e^{-y\sqrt{2}} \Big) \Big]
\end{aligned}$$

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Say we have a particle $v \in N_L$ such that $R_L^{(v)} = r \in \sqrt{2}L - [L^{1/6}, L^{2/3}]$.

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$$
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\mathbb{P}(R^*_t \le m_t + y) &= \mathbb{E}\Big[\prod_{v \in N_L} \mathbb{P}_{R_L^{(v)}}(R^*_{t-L} \le m_t + y) \Big] = \mathbb{E}\Big[\prod_{v \in N_L} \big(1 - \mathbb{P}_{R_L^{(v)}}(R^*_{t-L} > m_t + y)\big) \Big] \\
&\approx \mathbb{E}\Big[\prod_{v \in N_L} \big(1 - C_d f(R_L^{(v)}) e^{-y\sqrt{2}}\big) \Big] \approx \mathbb{E}\Big[\exp\Big(- C_d \sum_{v \in N_L} f(R_L^{(v)}) e^{-y\sqrt{2}} \Big) \Big] \\
&\to \mathbb{E}\Big[\exp\Big(- C_d Z_\infty e^{-y\sqrt{2}} \Big) \Big]
\end{aligned}
$$

- Note the LHS has no L dependence, while the RHS has no t dependence $\Rightarrow$ both sides converge!

Proof of Main Result

Say we have a particle $v \in N_L$ such that $R_L^{(v)} = r \in \sqrt{2}L - [L^{1/6}, L^{2/3}]$.

What is the probability that v produces a descendant in N_t that exceeds height m_t?

$$\begin{aligned}(\text{Markov}) &= \mathbb{P}_r(R^*_{t-L} > m_t,\ \text{Banana+Window event}) \\ &= \mathbb{P}_r\Big(\bigcup_{u \in N_{t-L-\ell}} BW(u)\Big) \\ &\sim e^{t-L-\ell}\mathbb{P}_r\big(BW(u)\big) \\ &\sim C_d r^{-\frac{d-1}{2}}(\sqrt{2}L - r)e^{-\sqrt{2}(\sqrt{2}L - r)} =: C_d f(r)\,.\end{aligned}$$

Proof of main result:

$$\begin{aligned}\mathbb{P}(R^*_t \le m_t + y) &= \mathbb{E}\Big[\prod_{v \in N_L} \mathbb{P}_{R_L^{(v)}}(R^*_{t-L} \le m_t + y)\Big] = \mathbb{E}\Big[\prod_{v \in N_L} \big(1 - \mathbb{P}_{R_L^{(v)}}(R^*_{t-L} > m_t + y)\big)\Big] \\ &\approx \mathbb{E}\Big[\prod_{v \in N_L} \big(1 - C_d f(R_L^{(v)})e^{-y\sqrt{2}}\big)\Big] \approx \mathbb{E}\Big[\exp\Big(-C_d \sum_{v \in N_L} f(R_L^{(v)})e^{-y\sqrt{2}}\Big)\Big] \\ &\to \mathbb{E}\Big[\exp\Big(-C_d Z_\infty e^{-y\sqrt{2}}\Big)\Big]\end{aligned}$$

- Note the LHS has no L dependence, while the RHS has no t dependence $\Rightarrow$ both sides converge!
- Z_∞ is the limit in distribution of $\sum_{v \in N_L} f(R_L^{(v)})$. Convergence in $\mathbb{P}$ forthcoming work with J. Berestycki, B. Mallein.

Interpretation of Z_∞ by Lalley-Selke

Theorem (Berestycki-K.-Lubetzky-Mallein-Zeitouni 2021)

$$\mathbb{P}\big(R_t^* - m_t \le y \,\big|\, \mathcal{F}_L\big) \xrightarrow[\substack{t\to\infty,\\ L\to\infty}]{p} \exp\big(-C_d Z_\infty e^{-\sqrt{2}y}\big)$$

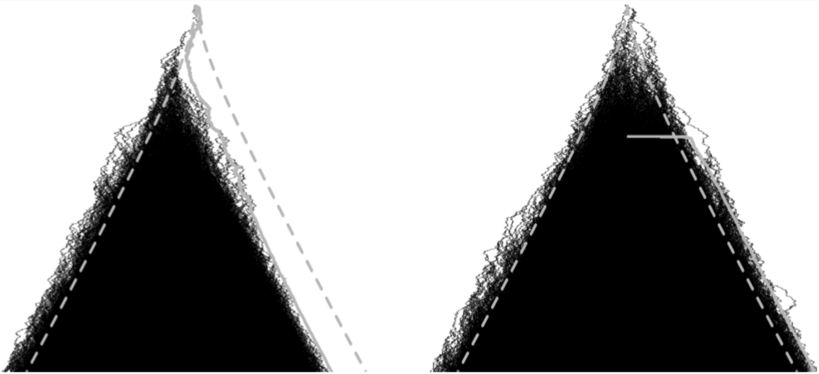


Figure 2: Left: initial particles veer to the left. Right: initial particles veer far to the right. In both pictures, we see how the initial behavior permanently shifts the maximum. (Image from Éric Brunet).

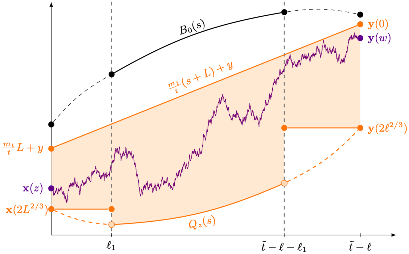
Main Results: The Extremal Point Process in $\mathbb{R}^d$

- Stasiński, Berestycki, and Mallein (2020) constructed a random measure $D_\infty(\cdot)$ on $\mathbb{S}^{d-1}$, which is the a.s., a.e. limit of an "angular" derivative martingale. So, Z_∞ and $D_\infty(\mathbb{S}^{d-1})$ have the same law.
- D_∞ is a.s. absolutely continuous wrt Leb.
- It turns out that $\sum_{v \in N_L} f(R_L^{(v)})$ converges in probability to $c_d D_\infty(\mathbb{S}^{d-1})$, for some constant $c_d > 0$.
- Let the point process $\sum_{i=1}^{\infty} \delta_{(\eta_i, \theta_i)}$ on $\mathbb{R} \times \mathbb{S}^{d-1}$ have law $\mathrm{PPP}\left(\overline{C}_d D_\infty(\mathbb{S}^{d-1}) e^{-\sqrt{2}x} \mathrm{d}x \times \frac{D_\infty(\theta)}{D_\infty(\mathbb{S}^{d-1})} \mathrm{d}\theta\right)$.
- Let $\mathcal{E}_t := \sum_{v \in N_t} \delta_{(\|X_t\| - m_t, X_t / \|X_t\|)}$ denote the extremal point process of BBM in dimension d.

Theorem

Let $\{\mathcal{D}^{(i)}\}_{i \in \mathbb{N}}$ be a collection of iid point processes with the same law as the decorations from the 1D BBM case. Then $\mathcal{E}_t$ converges in law as $t \to \infty$ to

$$\mathcal{E} := \sum_{i \in \mathbb{N}} \sum_{\xi_j \in \mathcal{D}^{(i)}} \delta_{(\eta_i + \xi_j, \theta_i)}$$


$B_0(s)$
$\mathbf{y}(0)$
$\mathbf{y}(w)$
$\frac{m_t}{t}(s+L)+y$
$\frac{m_t}{t}L+y$
$\mathbf{y}(2\ell^{2/3})$
$\mathbf{x}(z)$
$\mathbf{x}(2L^{2/3})$
$Q_z(s)$
ℓ_1
$\tilde{t}-\ell-\ell_1$
$\tilde{t}-\ell$

Thank you!