Part 2: Introduction to SPDE

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Outline

1. What are SPDE and why are they useful/important/interesting?
2. What is space-time white noise? Construction and calculus.
3. Basic linear SPDE and and Duhamel's principle
4. Chaos expansion and multiplicative SPDE.
5. Martingale methods to identify the law of an SPDE
6. Singular SPDE: regularity computations and local subcriticality assumptions, pathwise solution theories
7. Markov property and invariant measures for SPDEs; rate of convergence.

What are SPDEs

SPDEs are to PDEs what SDEs are to ODEs. We are going to focus mainly on evolution SPDEs of the form

$$\partial_t f = L(f) + \sigma(f)\xi.$$

where L is some operator (possibly nonlinear) and σ is a linear operator. Here ξ is Gaussian space-time white noise, to be explained shortly.

Note the analogy with SDE's of the form

$$dX_t = L(X_t)dt + \sigma(X_t)dB_t.$$

Examples of L we will consider:

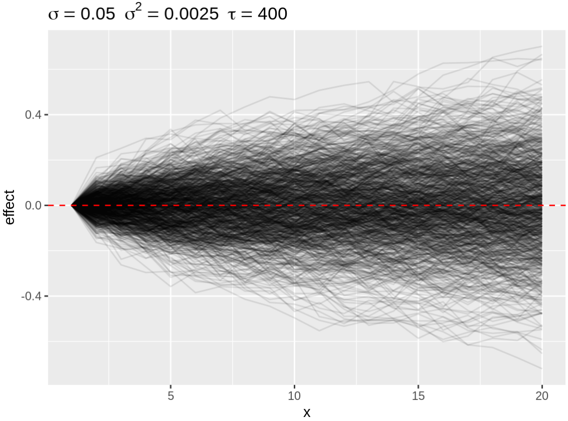
1. (SHE / Edwards-Wilkinson) $L(f) = \partial_x^2 f$ or more generally $L(f) = -(-\partial_x^2)^\alpha f$. And $\sigma(f) = I$.
2. (mSHE) $L(f) = \partial_x^2 f$ and $\sigma(f)g = fg$.
3. (KPZ) $L(f) = \partial_x^2 f + (\partial_x f)^2$ and $\sigma(f) = I$.

1. (SHE / Edwards-Wilkinson) $\partial_t f = \partial_x^2 f + \xi$, or more generally $\partial_t f = -(-\partial_x^2)^\alpha f + \xi$
2. (mSHE) $\partial_t f = \partial_x^2 f + f\xi$.
3. (KPZ) $\partial_t f = \partial_x^2 f + (\partial_x f)^2 + \xi$

Why study these?

They describe the fluctuations of various systems coming from probability, statistical mechanics.

Simple example: iid random walks or brownian motions.



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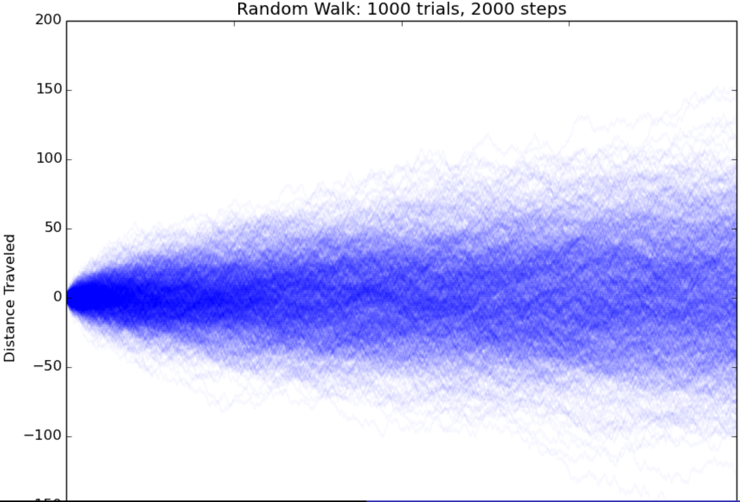

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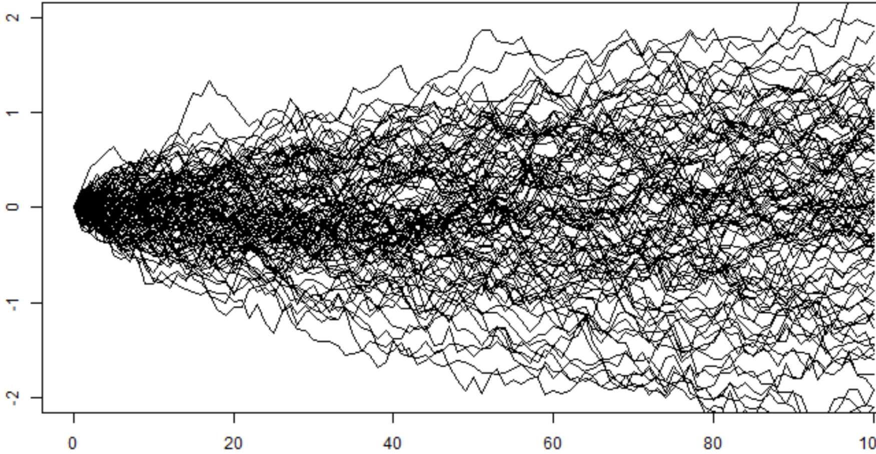

What exactly is space-time white noise?

Often space-time white noise is described as a condinuum iid Gaussian field, i.e.,

$$\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y).$$

That's not rigorous and it needs to be interpreted in an integrated sense: ξ is a random variable taking values in Schwarz distributions such that (ξ, f) is always a Gaussian and

$$\mathbb{E}\big[(\xi, f)_{L^2}(\xi, g)_{L^2}\big] = (f, g)_{L^2},$$

where $L^2 = L^2(\mathbb{R}_+ \times \mathbb{R})$.

Construction of STWN

How to prove existence of such a random variable ξ taking values in $\mathcal{S}'(\mathbb{R}^2)$? Similar to construction of Brownian motion. Several options:

1. Use Kolmogorov's extension theorem to construct a projective family of r.v. $\{(\xi, f)\}$ indexed by $f \in \mathcal{S}$ such that the convariance structure of any finite subfamily is as specified.

2. Choose an orthonormal basis $\{e_j\}$ for $L^2(\mathbb{R}_+ \times \mathbb{R})$ and let ξ_j be iid N(0,1). Then define

$$(\xi, f) := \sum_j (e_j, f)\xi_j,$$

which always converges by L^2 martingale convergence theorem.

Construction of STWN

So far this defines a family (ξ, f) indexed by $f \in \mathcal{S}$ (in fact by $f \in L^2$) such that $(\xi, f + \alpha g) = (\xi, f) + \alpha(\xi, g)$ and such that

$$\mathbb{E}[(\xi, f)^2] = \|f\|_{L^2}^2.$$

After this, one still needs to "glue together" or "modify" this family of variables so that it can actually be realized as a random element of $\mathcal{S}'(\mathbb{R}^2)$. This is possible thanks to a Kolmogorov continuity criterion together with Gaussian tail bounds:

$$\mathbb{E}[|(\xi, f)|^p] \lesssim_p \|f\|_{L^2}^p.$$

Integration against ξ

Note that $(\xi, f)_{L^2(\mathbb{R}_+ \times \mathbb{R})}$ is well-defined for all $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$. It's usually denoted suggestively as

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x)\xi(t,x)dtdx$$

or as

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x)\xi(dt\ dx),$$

though it should be remarked that ξ is a.s. neither a function nor a measure.

Linear equations

Let's return to the additive-noise stochastic heat equation:

$$\partial_t h(t,x) = \Delta_x h(t,x) + \xi(t,x),$$

with $x \in \mathbb{R}^d$ and $t \geq 0$, and $h(0,x)$ some given function. Rearrange terms and formally apply the operator $e^{-t\Delta}$ to both sides to obtain

$$\partial_t (e^{-t\Delta} h) = e^{-t\Delta}\partial_t h - e^{-t\Delta}\Delta h = e^{-t\Delta}\xi.$$

Integrate both sides from 0 to t, then apply $e^{t\Delta}$:

$$e^{-t\Delta} h(t,\cdot) - h(0,\cdot) = \int_0^t e^{-s\Delta}\xi(ds,\cdot).$$

$$h(t,\cdot) = e^{t\Delta} h(0,\cdot) + \int_0^t e^{(t-s)\Delta}\xi(ds,\cdot).$$

Linear equations

What exactly is $e^{t\Delta}$? It's an operator that denotes the solution at time t to the solution of the equation

$$\partial_t h = \Delta h.$$

In other words $e^{t\Delta}$ is just convolution with the heat kernel:

$$e^{t\Delta} f(x) = \int_{\mathbb{R}} p(t, x-y) f(y) dy,$$

where

$$p(t,x) = \frac{1}{(2\pi t)^{d/2}} e^{-|x|^2/2t}.$$

Linear SPDE

Summarizing, we have shown formally that the "solution" of

$$\partial_t h = \Delta h + \xi$$

is given by

$$h(t,x) = \int_{\mathbb{R}} p(t, x-y)h(0,y)dy + \int_{\mathbb{R}_+ \times \mathbb{R}^d} p(t-s, x-y)\xi(ds\,dy).$$

The integral in the second term on the RHS is deterministic and in $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ when $d = 1$.

Linear SPDE

One can retroactively check that this is indeed the solution in the sense of Schwarz distributions, i.e.,

$$-(h, \partial_t \phi) = (h, \Delta\phi) + (\xi, \phi)$$

a.s. for all smooth space-time Schwarz functions ϕ.

It turns out that (the derivative of) h describes the fluctuations in the Brownian Motion picture from earlier. We will prove this later.

What about $d > 1$?

The kernel fails to be in $L^2(\mathbb{R}_+ \times \mathbb{R}^d)$ for $d > 1$.

But intuitively one expects there to be a well-defined fluctuation field for 2d noninteracting Brownian motions.

It turns out that the only issue is the singularity of the kernel at the origin.

What about $d > 1$?

In particular if $\phi \in \mathcal{S}(\mathbb{R}^{d+1})$ then one can make sense of the smoothed out field

$$h(\phi) := \int_{\mathbb{R}^d} p^\phi(t, x-y)h(0,y)dy + \int_{\mathbb{R}_+\times\mathbb{R}^2} p^\phi(t-s, x-y)\xi(ds\, dy),$$

where

$$p^\phi(t,x) = (p*\phi)(t,x) = \int_{\mathbb{R}^3} p(t-s, x-y)\phi(s,y)dsdy.$$

These random variables $h(\phi)$ as ϕ ranges through all Schwarz functions, can then be lifted to a random Schwarz distribution on $\mathbb{R}_+ \times \mathbb{R}^d$ which will solve the SHE in weak form.

The linear theory for the SHE

Summarizing, the equation

$$\partial_t h = \Delta h + \xi$$

is solved by the Duhamel formula

$$h(t,x) = \int_{\mathbb{R}} p(t, x-y)h(0,y)dy + \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y)\xi(dsdy).$$

One can show that for all d, $h(t,\cdot)$ can actually be evaluated as an element of $\mathcal{S}'(\mathbb{R}^d)$ for fixed times t and any initial data in $\mathcal{S}'(\mathbb{R}^d)$.

For fixed $t > 0$ the field $h(t,\cdot)$ is locally absolutely continuous w.r.t Brownian motion when $d = 1$ and w.r.t the Gaussian free field when $d = 2$.

Remark: What about $\sigma \neq I$?

Nothing special about space-time white noise so far. We could replace it by any (possibly correlated) noise η and the solution is still given by the Duhamel formula:

$$h(t,x) = \int_{\mathbb{R}} p(t, x-y) h(0,y) dy + \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y) \eta(s,y) ds dy,$$

provided that the integral on the right hand side makes sense (possibly in a distributional sense). This will not be the case for all Gaussian noises η but it will be true for example when $\eta = \partial_x \xi$ or $\eta = (-\Delta_x)^\alpha \xi$.

Moving onto nonlinear SPDE...

So we can integrate deterministic integrands against ξ. What about random integrands? E.g. can we make sense of iterated integrals such as ...

$$\int \left[\int f(t,x,s,y)\xi(ds\,dy) \right] \xi(dt\,dx)$$

even with deterministic f? How about k-fold integrals such as

$$\int \cdots \int f(\mathbf{t},\mathbf{x})\xi^{\otimes k}(d\mathbf{t},d\mathbf{x})?$$

And what about things like

$$\int \sigma\left(\int f(t,x,s,y)\xi(ds\,dy) \right) \xi(dt\,dx)?$$

Motivation: why should we care about these objects?

Consider SPDE's such as the multiplicative SHE or its generalizations:

$$\partial_t f = \partial_x^2 f + f\xi.$$

$$\partial_t f = \partial_x^2 f + \sigma(f)\xi,$$

in spatial dimension $d = 1$.

We'll focus on the first one. Duhamel's principle still applies here, but as opposed to the linear case it gives an iterative relation rather than a finished solution, e.g.

$$f(t,x) = \int_{\mathbb{R}} p(t,x-y)f(0,y)dy + \int_{\mathbb{R}_+\times\mathbb{R}} p(t-s,x-y)f(s,y)\xi(ds,dy$$

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$$f(t,x) = \int_{\mathbb{R}} p(t,x-y)h(0,y)dy + \int_{\mathbb{R}_+\times\mathbb{R}} p(t-s,x-y)f(s,y)\xi(ds,dy$$

Multiplicative SHE

We can (Picard) iterate the previous relation once to obtain

$$f(t,x) = \int_{\mathbb{R}} p(t,x-y) f(0,y) dy$$

$$+ \int_{\mathbb{R}_+ \times \mathbb{R}} \left[\int_{\mathbb{R}} p(t-s,x-y) p(s,y-z) f(0,z) dz \right] \xi(ds,dz)$$

$$+ \int_{\mathbb{R}_+ \times \mathbb{R}} \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s,x-y) p(s-u,y-z) f(u,z) \xi(du,dz) \xi(ds,dy).$$

Keep iterating to obtain:

$$f(t, x) = \sum_{k=1}^{\infty} u_k(t, x)$$

where

$$u_{k+1}(t, x) = \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y) u_k(s, y) \xi(ds, dy)$$

and u_0 is just the heat flow started from $h(0, \cdot)$.

Nonrecursively we have that $u_k(t, x)$ is given by

$$\int_{(\mathbb{R}_+ \times \mathbb{R})^{k+1}} \prod_{i=1}^{k+1} p(t_i - t_{i-1}, x_i - x_{i-1}) f(0, x_{k+1}) dx_{k+1} \xi^{\otimes k}(d\mathbf{t}, d\mathbf{x}),$$

with $t_{k+1} = t$ and $x_0 = x$.

Integration of adapted processes

The filtration $\mathcal{F}_t$ of ξ is defined to be the sigma algebras generated by (f, ξ) with f supported on $[0, t] \times \mathbb{R}$.

A random space time function $f(t, x)$ is called adapted to the filtration of ξ if $f(t, x)$ is $\mathcal{F}_t$ measurable for all t, x.

A random space-time function is called predictable if it lies in the L^2 closure of the linear span of elementary functions: things of the form $f(x, t, \omega) = X(\omega)1_{(a,b]}(t)1_E(x)$ where $E \subset \mathbb{R}^d$ is Borel and X is $\mathcal{F}_a$ measurable.

Theorem: any adapted continuous function is predictable.

Integration of adapted random processes

The integral of an elementary process $f(t,x) = X \cdot 1_{(a,b]}(t) 1_E(x)$ against the noise can be defined in the obvious manner:

$$\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x) \xi(dt\,dx) = X \cdot (\xi, 1_{(a,b] \times E}).$$

One has the Ito-Walsh isometry

$$\mathbb{E}\left[\left(\int_{\mathbb{R}_+ \times \mathbb{R}} f(t,x) \xi(dt\,dx) \right)^2 \right] = \int_{\mathbb{R}_+ \times \mathbb{R}} \mathbb{E}[f(t,x)^2] dt dx,$$

which allows us to define integrals for any adapted continuous function, in particular iterated integrals as we wanted earlier (even ones which are not supported on a simplex, by symmetrization).

Properties of stochastic integrals

Note that if $f \in L^2(\mathbb{R}_+ \times \mathbb{R})$ then it is not true that

$$\int_{(\mathbb{R}_+ \times \mathbb{R})^k} \prod_1^k f(t_i, x_i) \xi^{\otimes k}(d\mathbf{t}, d\mathbf{x}) = (f, \xi)^k.$$

Rather the right hand side equals $H_k((\xi, f))$ when $\|f\|_{L^2} = 1$, where H_k is the k^{th} Hermite polynomial. Again, ξ is **not** a measure or a function.

In particular all k-fold iterated integrals are orthogonal to all n-fold iterated integrals for $k \neq n$. The set of all k-fold iterated integrals is called the k^{th} homogeneous chaos of ξ, denoted $\mathcal{H}^k(\Omega, \mathcal{F}, \mathbb{P})$.

Theorem: $L^2(\Omega, \mathcal{F}, \mathbb{P}) = \bigoplus_{k \in \mathbb{N}} \mathcal{H}^k(\Omega, \mathcal{F}, \mathbb{P})$.

Why does this happen?

Think about the simple case $k = 2$ with just a Brownian motion instead of white noise. Recall the computation of how

$$\int_0^1 B_t dB_t = \frac{1}{2}(B_t^2 - t).$$

There's a law of large numbers averaging happening at the second order. This is referred to as renormalization and tends to become relevant in all SPDE's with a nonlinear term such as a product.

For elements $X \in \mathcal{H}^k$ one has that $\|X\|_p \leq C_{k,p}\|X\|_2$ where optimally one may take $C_{k,p} = (2p-1)^{k/2}$. This can be proved using Burkholder or hypercontractivity of the OU semigroup.

Returning to the multiplicative SHE

Recall our formula for the solution of

$$\partial_t f = \partial_x^2 f + f\xi$$

was given by

$$f(t,x) = \sum_{k=1}^{\infty} u_k(t,x)$$

where

$$u_{k+1}(t,x) = \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y) u_k(s,y) \xi(ds, dy).$$

The iteration

So by the Ito isometry we have

$$\mathbb{E}[u_{k+1}(t,x)^2] = \int_{\mathbb{R}_+ \times \mathbb{R}} p(t-s, x-y)^2 \mathbb{E}[u_k(s,y)^2] dsdy.$$

One can thus obtain inductive bounds that will show that

$$\sum_k \mathbb{E}[u_k(s,y)^2] < \infty.$$

To show this, one however needs fairly stringent assumptions on initial conditions, e.g.

$$\sup_{x \in \mathbb{R}} e^{-a|x|} \mathbb{E}[f(0,x)^2] < \infty.$$

Also d **cannot** be larger than 1. And this is not just a purely technical restriction of functions vs distributions.

Given a space-time process defined on some probability space, how can one identify it as the solution of some SPDE?

First consider SDE. Suppose X_t is a continuous process defined on some space with the property that

$$M_t := X_t - \int_0^t b(X_s)ds$$

is a martingale such that

$$\langle M \rangle_t = \int_0^t \sigma(X_s)^2 ds,$$

where b, σ are smooth with $\sigma > 0$.

Change of topic: Martingale methods

Then X_t must have the same law as the diffusion

$$dX = b(X)dt + \sigma(X)dB.$$

Proof: Let $B_t = \int_0^t \sigma(X_s)^{-1} dM_s$. Note that B is a martingale with quadratic variation t and therefore is a Brownian motion. Moreover by construction

$$X_t - \int_0^t b(X_s)ds = M_t = \int_0^t \sigma(X_s)dB_s.$$

Similar theory for white-noise driven SPDEs

Theorem [Konno-Shiga, '88]: suppose that $(f(t,x))_{t\geq 0, x\in\mathbb{R}}$ is a continuous process with the property that the processes

$$M_t(\phi) = (f(t,\cdot),\phi)_{L^2(\mathbb{R})} - \int_0^t (f(s,\cdot),\phi'')_{L^2(\mathbb{R})}ds$$

are martingales with respect to the filtration of f and that

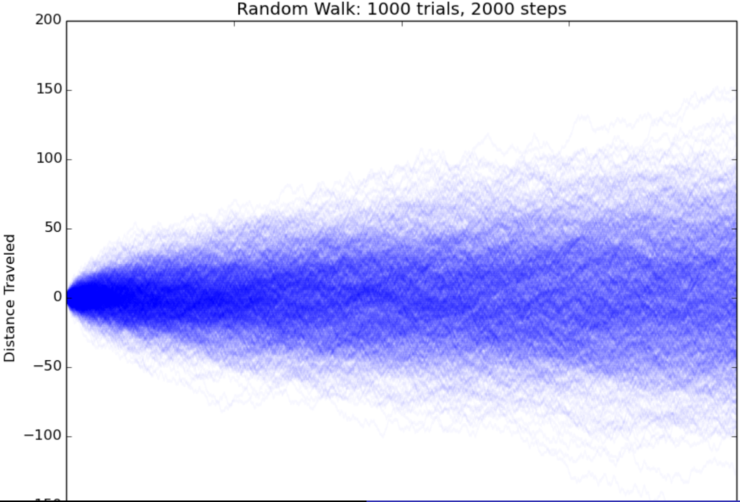
$$\langle M(\phi)\rangle_t = \int_0^t (\sigma(f(\cdot,s))^2\phi,\phi)_{L^2(\mathbb{R})}ds,$$

for all Schwartz functions ϕ on $\mathbb{R}$. Then f is distributed as the solution of

$$\partial_t f = \partial_x^2 f + \sigma(f)\xi.$$

Returning to the Brownian motions picture

Let's do a computation with the empirical measures.



The global limit

The global limit is the solution of the SPDE

$$\partial_t u = \partial_x^2 u + \partial_x\left(\sqrt{p(t,x)} \cdot \xi\right)$$

which happens to be the spatial derivative of

$$\partial_t h = \partial_x^2 h + \sqrt{p(t,x)} \cdot \xi,$$

which looks like SHE (Edwards-Wilkinson) if one zooms in locally around any deterministic space-time point.

For linear equations it was clear that the Duhamel formula gave sense to a bona fide weak solution in a pathwise sense.

For the multiplicative SHE, it is questionable what is so intrinsic about the Duhamel notion of solution to the equation. That is because $f\xi$ is not defined pathwise.

In recent years a tremendous amount of progress has been made in terms of giving a intrinsic pathwise notion of solution to equations with nonlinear terms such as these.

We'll talk a little bit about KPZ and Φ^4_d.

KPZ and Φ^4_d

$$KPZ: \qquad \partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi, \qquad t \geq 0, x \in \mathbb{R}$$

Hopf-cole transform of multiplicative SHE, canonical model of interface growth.

$$\Phi^4_d: \qquad \partial_t \Phi = \Delta\Phi - \Phi^3 + \xi, \qquad t \geq 0, x \in \mathbb{R}^d.$$

Important in physics, invariant measure was constructed in dimensions 2 and 3 as a breakthrough in constructive QFT.

Note that the KPZ equation and mSHE

$$\partial_t h = \partial_x^2 h + (\partial_x h)^2 + \xi$$

$$\partial_t f = \partial_x^2 f + f\xi$$

are formally related by the Hopf-Cole transform $h = \log(f)$.

Thus one may define a **Hopf-cole solution** of h as being equal to $\log(f)$ where f solves the mSHE in the Duhamel sense defined earlier. But the same question of what is so natural or intrinsic of this notion of solution comes up.

This Hopf-Cole notion of solution is sufficient for many purposes but one often wants to study finer properties of the solutions, e.g.

1. Prove that space-time mollifications of the noise lead to classical solutions that converge uniformly on compacts (after renormalization) to the Hopf-Cole solution.
2. Explore properties of the **solution map**.
3. Prove that the difference of two solutions (with different initial data) has better regularity than individual ones.

None of this is possible with the Hopf-Cole notion of solution.

Regularity and subcriticality

Both KPZ and Φ^4_d are nonlinear perturbations of the additive SHE.

In spatial dimension d the additive SHE has "parabolic regularity" $1-\frac{d}{2}$ as it is invariant under the scaling $\epsilon^{-1+\frac{d}{2}}h(\epsilon^2 t, \epsilon x)$. Parabolic regularity in this context means space-time Holder regularity with respect to the parabolic metric

$$d\big((t,x),(s,y)\big) = |t-s|^{1/2} + |x-y|,$$

where "time counts as two spatial dimensions." This can all be made rigorous using parabolic Schauder estimates (convolution with heat kernel improves parabolic regularity by two) which follow by studying the singularity of the heat kernel.

Regularity and subcriticality

The KPZ and Φ^4_d $(d < 4)$ equations have **subcritical** nonlinearities and thus we expect them to have the same regularity as the linearized equation. In particular the nonlinearities are ill-posed.

Subcritical nonlinearity: Formally we say that a nonlinearity F is subcritical if under the scaling which sends $u(t,x)$ to $u^\epsilon(t,x) := \epsilon^{-1+\frac{d}{2}} u(\epsilon^2 t, \epsilon x)$, the equation

$$\partial_t u = \Delta u + F(u, \nabla u) + \xi$$

gets transformed into

$$\partial_t u = \Delta u + F_\epsilon(u, \nabla u) + \xi$$

where $F_\epsilon \to 0$ as $\epsilon \to 0$.

Subcriticality computation for KPZ and Φ^4_d

For the KPZ equation one has $F(u, \nabla u) = (\nabla u)^2$ and then one gets $F_\epsilon(u, \nabla u) = \epsilon^{1/2}(\nabla u)^2$.

For Φ^4_d one has $F(u, \nabla u) = -u^3$ and then $F_\epsilon(u, \nabla u) = -\epsilon^{4-d} u^3$. Thus we see that the subcriticality assumption fails unless $d < 4$.

Pathwise Solution theories

Theorem [Hairer, and independently Gubinelli-Imkeller-Perkowski]: For the Hopf-Cole solution of KPZ one has a factorization of the solution map

$$\xi \mapsto \psi(\xi) \mapsto h(\xi)$$

where the first map ψ consists of of a finite number of Gaussian chaoses and the second map is continuous (locally Lipchitz even) on some closed subset which supports ψ. All topologies are those of Holder spaces of the optimal exponents.

The theory can also be used to make sense of Φ^4_d as well as essentially any subcritical nonlinearity.

The continuous map is where the most difficult theory lies. The first map is easier and will be explained shortly.

Note that the analogous statement even for SDEs in $\mathbb{R}^n$ is not trivial.

That is, how would one study (even for smooth and bounded b, σ) structural properties of the solution map that sends $B \mapsto X$ where

$$dX = b(X)dt + \sigma(X)dB?$$

The corresponding result for SDEs says that one has a factorization

$$B \mapsto \left(B, \int_0^{\bullet} B_i(s)dB_j(s)\right)_{1\le i<j\le n} \mapsto X$$

where the first map ic called the "rough path lift of B" and the second map is continuous (the **Ito-Lyons map**). See the book on rough paths by Friz and Hairer.

Brief idea for Φ^4_2

We consider here the dynamical Φ^4_2 model $(d = 2)$ with zero i.c. for simplicity. To make sense of the equation let Φ formally solve

$$\partial_t \Phi = \Delta \Phi - \phi^3 + \xi$$

and let f solve

$$\partial_t h = \Delta h + \xi$$

with the same realization of ξ. Define $v = \Phi - h$ and note that v formally solves

$$\begin{aligned}\partial_t v &= \Delta v - \Phi^3 \\ &= \Delta v - (v + h)^3 \\ &= \Delta v - v^3 - 3v^2 h - 3vh^2 - h^3.\end{aligned}$$

Brief idea for Φ^4_2

Recall h solves the linearized equation and thus by the Duhamel formula

$$h(t,x) = \int_{\mathbb{R}_+\times\mathbb{R}^2} p(t-s, x-y)\xi(ds\,dy).$$

In $d=2$ needs to be interpreted by testing both sides against a Schwarz function. So h does not really exist as a function.

So what exactly do h^2 and h^3 mean? One needs to interpret these as iterated integrals:

$$h^{:2:}(t,x) := \int_{\mathbb{R}_+\times\mathbb{R}^2}\int_{\mathbb{R}_+\times\mathbb{R}^2} p(t-s,x-y)p(t-u,x-z)\xi(ds\,dy)\xi(du\,dz).$$

Again this needs to be interpreted by integration against Schwarz functions. Likewise $h^{:3:}$ may be defined as a threefold iterated integral. Note the **renormalization** happening here.

Brief idea for Φ^4_2

Going back to our remainder equation we replace the formal powers of h by the Wick powers:

$$\partial_t v = \Delta v - v^3 - 3v^2 h - 3vh^{:2:} - h^{:3:}.$$

This gives us a fixed point problem for v:

$$v(t,x) = -\int_{\mathbb{R}_+ \times \mathbb{R}^2} p(t-s, x-y)\left(v^3 + 3v^2 h + 3vh^{:2:} + h^{:3:}\right)(s,y)\,dsdy.$$

It turns out that this fixed point problem can be sensibly and globally solved...

Brief idea for Φ^4_2

...in the sense that the map $\mathcal{S}$ which sends a triple $(f, g, h) \in C_w^{-\alpha}(\mathbb{R}_+ \times \mathbb{R})^3$ to the fixed point of the map

$$v \mapsto -p * (v^3 + 3v^2 f + 3vg + h)$$

on $C_w^{2-\alpha}(\mathbb{R}_+ \times \mathbb{R})$, is locally Lipchitz.

In particular we have our factorization

$$\xi \mapsto (h, h^{:2:}, h^{:3:}) \overset{\mathcal{S}}{\mapsto} \Phi$$

because $\Phi = v + h$.

See [Mourrat & Weber, Global well-posedness...] as well as the original paper [Da Prato & Debussche, Strong solutions...] for more details.

Regularity Structures and Paracontrolled Products

The theories of Hairer and Gubinelli-Imkeller-Perkowski build on these ideas but are more complicated in that the continuous part of the solution map cannot be built on the entire space but must rather be built on some nonlinear part of the space in which the chaoses live.

This requires heavy analytic machinery such as the use of Daubechies wavelets to patch together local Taylor expansions (regularity structures) or the use of Littlewood Paley blocks, Bony paraproducts, and paralinearization formulae for nonlinearities.

Consider the solution of the equation

$$\partial_t f = \Delta f + \sigma(f)\xi,$$

where σ can be some smooth nonlinearity, possibly even operator-valued.

This process is Markov: by Duhamel formula we have (in abbreviated form) that

$$f_t = p_t * f_0 + \int_0^t p_{t-a} * (f_a \cdot \xi(da)) = p_{t-s} * f_s + \int_s^t p_{t-a} * (f_a \cdot \xi(da)).$$

Thus one may ask about invariant measures and convergence to equilibrium.

Markov property and equilibrium

Even for linear equations $\sigma(f) \equiv Q$ the question is nontrivial. Note that one can view the equation as a gradient flow in $L^2(\mathbb{R})$ with respect to the potential

$$V(f) = \int |\nabla f|^2.$$

An equilibrium probability measure doesn't exist but if we replace Δ by (say) $\Delta - \alpha I$ for $\alpha > 0$ or any strictly contractive semigroup $S(t)$ with generator L, then the equilibrium measure is a Gaussian measure with Cameron martin norm $\langle f, Af \rangle$ where

$$A = \int_0^\infty S(t) Q Q^* S^*(t) dt.$$

In finite dimensions one can hope to use entropy methods to control the total variation distance to equilibrium measure.

Convergence to equilibrium

Hairer and Mattingly found a proof that doesn't use this (arXiv 0810.2777).

Theorem (Harris)

Let P_t be a Markov semigroup on a Polish space E such that there exists $V : E \to [0, \infty)$ and $T > 0$ such that

1. $P_t V(x) \le \gamma V(x) + K$ *for some* $\gamma \in (0, 1)$ *and* $K > 0$.
2. *Given* $C > 0$ *there exists* $\delta > 0$ *such that* $\|P_T^* \delta_x - P_T^* \delta_y\|_{TV} < 2 - \delta$ *for all* x, y *with* $V(x) + V(y) \le C$.

Then there exists $S > 0$ such that P_S^* is a strict contraction in TV norm (in particular, there is a unique invariant measure and exponentially fast convergence to it).

One can leverage this idea to prove exponentially fast convergence to the invariant measure in case of a linear equation with contractive drift, with $V(x) = \|x\|_B$. It can even be adapted to the case of slight nonlinearity.

Thanks.

Thank You!