A Riemann-Hilbert approach to asymptotic analysis of Toeplitz+Hankel determinants

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The $n \times n$ Toeplitz and Hankel matrices associated respectively to the symbols ϕ and w, supported on the unit circle $\mathbb{T}$ are respectively defined as

$$T_n[\phi; r] := \{\phi_{j-k+r}\}, \qquad j, k = 0, \cdots, n-1, \qquad \phi_k = \int_{\mathbb{T}} z^{-k} \phi(z) \frac{dz}{2\pi i z},$$

and

$$H_n[w; s] := \{w_{j+k+s}\}, \qquad j, k = 0, \cdots, n-1, \qquad w_k = \int_{\mathbb{T}} z^{-k} w(z) \frac{dz}{2\pi i z},$$

for fixed *offset* values $r, s \in \mathbb{Z}$. If the Hankel symbol w is supported on a subset I of the real line, then w_k are instead given by

$$w_k = \int_I x^k w(x) dx.$$

We intended to develop a Riemann-Hilbert approach to the study of the large-n asymptotics of Toeplitz+Hankel determinants

$$\det \begin{pmatrix} \phi_r + w_s & \phi_{r-1} + w_{s+1} & \cdots & \phi_{r-n+1} + w_{s+n-1} \\ \phi_{r+1} + w_{s+1} & \phi_r + w_{s+2} & \cdots & \phi_{r-n+2} + w_{s+n} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{r+n-1} + w_{s+n-1} & \phi_{r+n-2} + w_{s+n} & \cdots & \phi_r + w_{s+2n-2} \end{pmatrix},$$

which we denote by $D_n(\phi, w; r, s)$.

Existing results

- ***P.Deift, A.Its and I.Krasovsky (2011)***

  Used the Riemann-Hilbert problem for pure Toeplitz determinants under the following assumptions

  - $\phi(z)$ of Fisher-Hartwig type.
  - $\phi(z) = w(z)$
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- ***E.Basor and T.Ehrhardt (2001,2002,2009,2013,2017)***

  Used operator theory techniques under the following assumptions
  - $\phi(z) = c(z)\phi_0(z)$
  - $w(z) = c(z)d(z)w_0(z)$
  - $\phi_0(z)$ of Fisher-Hartwig type.
  - $\phi_0(z) = \tilde{\phi}_0(z)$
  - a) $w_0(z) = \pm\phi_0(z)$, b) $w_0(z) = z\phi_0(z)$ and c) $w_0(z) = -z^{-1}\phi_0(z)$
  - $c(z), d(z)$ are not required to be even
  - $d(z)\tilde{d}(z) \equiv 1$
  - $d(\pm 1) = 1$

Perhaps one of the most important motivations behind studying T+H determinants is to study the large n asymptotics of the eigenvalues of the Hankel matrix $H_n[w]$ associated to the symbol w, simply because the characteristic polynomial

$$\det(H_n[w] - \lambda I)$$

of the Hankel matrix $H_n[w]$, is indeed a particular *Toeplitz+Hankel* determinant, with

$$\phi(z) \equiv -\lambda.$$

Clearly in the case of characteristic polynomial of a Hankel determinant, there is no relationship between ϕ and w, so to study the asymptotics of this determinant, one can not refer to the existing results mentioned above. In fact there is a methodological issue which has to be addressed at a fundamental level: formulation of a suitable Riemann-Hilbert problem.

- Hankel symbol supported on the unit circle.

The associated system of orthogonal polynomials

- Hankel symbol supported on the unit circle. If $D_n(\phi, w; r, s) \neq 0$, there exists unique monic polynomials $\mathcal{P}_n(z) = z^n + \cdots$ satisfying

$$\int_{\mathbb{T}} \mathcal{P}_n(z) z^{-k-r} \phi(z) \frac{dz}{2\pi i z} + \int_{\mathbb{T}} \mathcal{P}_n(z) z^{k+s} \tilde{w}(z) \frac{dz}{2\pi i z} = h_n \delta_{n,k},$$

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- Hankel symbol supported on the real line. If $D_n(\phi, w; r, s) \neq 0$, there exists unique monic polynomials $P_n(z) = z^n + \cdots$ satisfying

$$\int_{\mathbb{T}} P_n(z) z^{-k-r} \phi(z) \frac{dz}{2\pi i z} + \int_a^b P_n(x) x^{k+s} w(x) dx = h_n \delta_{n,k},$$

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  for $k = 0, 1, \cdots, n$. And in both cases

$$h_{n-1} \equiv \frac{D_n(\phi, w; r, s)}{D_{n-1}(\phi, w; r, s)}.$$

Assuming that $D_n(\phi, w; r, s) \neq 0$, the existence of the orthogonal polynomials follows from the explicit construction

$$\mathcal{P}_n(z) := \frac{D_{n+1}(\phi, w; r, s \big| \vec{v}(z))}{D_n(\phi, w; r, s)}, \qquad \vec{v}(z) = (1, z, \cdots, z^n),$$

where $D_n(\phi, w; r, s \big| \vec{v}(z))$ denotes the bordered Toeplitz+Hankel determinant with the last row given by the vector $\vec{v}(z)$.

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where $D_n(\phi, w; r, s \big| \vec{v}(z))$ denotes the bordered Toeplitz+Hankel determinant with the last row given by the vector $\vec{v}(z)$. The uniqueness of the polynomial $\mathcal{P}_n(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ satisfying the orthogonality conditions, simply follows from the fact that one has the following linear system for the coefficients $a_j, 0 \leq j \leq n-1$:

$$(T_n[\phi; r] + H_n[w; s]) \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{pmatrix} = \begin{pmatrix} -\phi_{-n+r} - w_{n+s} \\ -\phi_{1-n+r} - w_{1+n+s} \\ \vdots \\ -\phi_{-1+r} - w_{2n-1+s} \end{pmatrix}.$$

Let us consider the following Riemann-Hilbert problem

- **RH-$\mathcal{Y}$1** $\mathcal{Y}$ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$.
- **RH-$\mathcal{Y}$2** For $z \in \mathbb{T}$ we have

$$\mathcal{Y}_+^{(1)}(z;n) = \mathcal{Y}_-^{(1)}(z;n),$$

$$\mathcal{Y}_+^{(2)}(z;n) = \mathcal{Y}_-^{(2)}(z;n) + z^{-1+s}\tilde{w}(z)\mathcal{Y}_-^{(1)}(z;n) + z^{-1+r}\tilde{\phi}(z)\mathcal{Y}_-^{(1)}(z^{-1};n).$$

- **RH-$\mathcal{Y}$3** $\mathcal{Y}(z;n) = \big(I + \mathcal{O}(z^{-1})\big)\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$, As $z \to \infty$.

Theorem 1. The following statements are true.

1. Suppose that $D_n, D_{n-1} \neq 0$. Then, the Riemann-Hilbert problem **RH-$\mathcal{Y}$1** through **RH-$\mathcal{Y}$3** is uniquely solvable and its solution $\mathcal{Y}$ is given by

$$\begin{pmatrix} \mathcal{P}_n(z) & \displaystyle\int_{\mathbb{T}} \frac{\xi^s \tilde{w}(\xi)\mathcal{P}_n(\xi) + \xi^r \tilde{\phi}(\xi)\tilde{\mathcal{P}}_n(\xi)}{\xi - z} \frac{d\xi}{2\pi i \xi} \\ -\dfrac{1}{h_{n-1}}\mathcal{P}_{n-1}(z) & -\dfrac{1}{h_{n-1}} \displaystyle\int_{\mathbb{T}} \frac{\xi^s \tilde{w}(\xi)\mathcal{P}_{n-1}(\xi) + \xi^r \tilde{\phi}(\xi)\tilde{\mathcal{P}}_{n-1}(\xi)}{\xi - z} \frac{d\xi}{2\pi i \xi} \end{pmatrix}.$$

Moreover,

$$h_{n-1} = -\lim_{z\to\infty} z^{n-1}/\mathcal{Y}_{21}(z;n).$$

2. Suppose that the Riemann-Hilbert problem **RH-$\mathcal{Y}$1** through **RH-$\mathcal{Y}$3** has a unique solution. Then $D_n \neq 0$, and $\mathcal{P}_n(z) = \mathcal{Y}_{11}(z;n)$.

3 Suppose that the Riemann-Hilbert problem **RH-$\mathcal{Y}$1** through **RH-$\mathcal{Y}$3** has a unique solution. Suppose also that

$$\lim_{z\to\infty} \mathcal{Y}_{21}(z;n) z^{-n+1} \neq 0.$$

Then, as before, $D_n \neq 0$, $\mathcal{P}_n(z) = \mathcal{Y}_{11}(z;n)$, and, in addition,

$$D_{n-1} \neq 0, \quad h_{n-1} = -\lim_{z\to\infty} \mathcal{Y}_{21}^{-1}(z;n) z^{n-1}, \quad \mathcal{P}_{n-1}(z) = -h_{n-1}\mathcal{Y}_{21}(z).$$

Let us define the 2×4 matrix $\mathring{\mathcal{X}}$ out of the columns of $\mathcal{Y}$ as follows

$$\mathring{\mathcal{X}}(z;n) := \left(\mathcal{Y}^{(1)}(z;n), \widetilde{\mathcal{Y}}^{(1)}(z;n), \mathcal{Y}^{(2)}(z;n), \widetilde{\mathcal{Y}}^{(2)}(z;n)\right).$$

- **RH-$\mathring{\mathcal{X}}$1** $\mathring{\mathcal{X}}$ is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \{0\})$.
- **RH-$\mathring{\mathcal{X}}$2** For $z \in \mathbb{T}$, $\mathring{\mathcal{X}}$ satisfies

$$\mathring{\mathcal{X}}_+(z;n) = \mathring{\mathcal{X}}_-(z;n) \begin{pmatrix} 1 & 0 & z^{s-1}\tilde{w}(z) & -z^{-r+1}\phi(z) \\ 0 & 1 & z^{r-1}\tilde{\phi}(z) & -z^{-s+1}w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH-$\mathring{\mathcal{X}}$3** As $z \to \infty$ we have

$$\mathring{\mathcal{X}}(z;n) = \begin{pmatrix} 1+\mathcal{O}(z^{-1}) & C_1(n)+\mathcal{O}(z^{-1}) & \mathcal{O}(z^{-1}) & C_3(n)+\mathcal{O}(z^{-1}) \\ \mathcal{O}(z^{-1}) & C_2(n)+\mathcal{O}(z^{-1}) & 1+\mathcal{O}(z^{-1}) & C_4(n)+\mathcal{O}(z^{-1}) \end{pmatrix} \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

- **RH-$\mathring{\mathcal{X}}$4** As $z \to 0$ we have

$$\mathring{\mathcal{X}}(z;n) = \begin{pmatrix} C_1(n)+\mathcal{O}(z) & 1+\mathcal{O}(z) & C_3(n)+\mathcal{O}(z) & \mathcal{O}(z) \\ C_2(n)+\mathcal{O}(z) & \mathcal{O}(z) & C_4(n)+\mathcal{O}(z) & 1+\mathcal{O}(z) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix},$$

where

$$C_1(n) = \mathcal{Y}_{11}(0;n), \quad C_3(n) = \mathcal{Y}_{12}(0;n), \quad C_2(n) = \mathcal{Y}_{21}(0;n), \quad C_4(n) = \mathcal{Y}_{22}(0;n).$$

Let us assume that $r = s = 1$, the corresponding 4×4 Riemann-Hilbert problem is

- **RH-$\mathcal{X}$1** $\mathcal{X}$ is holomorphic in the complement of $\mathbb{T} \cup \{0\}$.
- **RH-$\mathcal{X}$2**
$$\mathcal{X}_+(z;n) = \mathcal{X}_-(z;n) \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T}.$$
- **RH-$\mathcal{X}$3**
$$\mathcal{X}(z;n) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad z \to \infty.$$
- **RH-$\mathcal{X}$4**
$$\mathcal{X}(z;n) = P(n)(I + \mathcal{O}(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}, \qquad z \to 0.$$

The solution of the $\mathcal{X}$-RHP is unique if it exists. Note that the matrix factor $P(n)$ is not a priori prescribed. But already we can show that

$$\operatorname{rank}(P(n)W - I_4) = 2, \qquad \text{where} \qquad W = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let us consider

$$\mathfrak{R}(z;n) := \mathring{\mathcal{X}}(z;n)\mathcal{X}^{-1}(z;n).$$

Clearly $\mathfrak{R}$ has no jumps, its behavior near zero is given by

$$\mathfrak{R}(z;n) = \begin{pmatrix} C_1(n)+\mathcal{O}(z) & 1+\mathcal{O}(z) & C_3(n)+\mathcal{O}(z) & \mathcal{O}(z) \\ C_2(n)+\mathcal{O}(z) & \mathcal{O}(z) & C_4(n)+\mathcal{O}(z) & 1+\mathcal{O}(z) \end{pmatrix} P^{-1}(n),$$

and thus $\mathfrak{R}$ is an entire function. The behaviour of $\mathfrak{R}$ near infinity is

$$\mathfrak{R}(z;n) = \begin{pmatrix} 1+\mathcal{O}(z^{-1}) & C_1(n)+\mathcal{O}(z^{-1}) & \mathcal{O}(z^{-1}) & C_3(n)+\mathcal{O}(z^{-1}) \\ \mathcal{O}(z^{-1}) & C_2(n)+\mathcal{O}(z^{-1}) & 1+\mathcal{O}(z^{-1}) & C_4(n)+\mathcal{O}(z^{-1}) \end{pmatrix}.$$

By Liouville's theorem we have

$$\mathfrak{R}(z;n) = \begin{pmatrix} 1 & C_1(n) & 0 & C_3(n) \\ 0 & C_2(n) & 1 & C_4(n) \end{pmatrix}.$$

This gives the following linear system for the constants C_j

$$\begin{pmatrix} 1 & C_1(n) & 0 & C_3(n) \\ 0 & C_2(n) & 1 & C_4(n) \end{pmatrix} = \begin{pmatrix} C_1(n) & 1 & C_3(n) & 0 \\ C_2(n) & 0 & C_4(n) & 1 \end{pmatrix} P^{-1}(n),$$

which can be rewritten as

$$\begin{pmatrix} 1 & C_1(n) & 0 & C_3(n) \\ 0 & C_2(n) & 1 & C_4(n) \end{pmatrix} (P(n)W - I_4) = 0.$$

Lemma 1. If the linear system

$$\begin{pmatrix} 1 & C_1(n) & 0 & C_3(n) \\ 0 & C_2(n) & 1 & C_4(n) \end{pmatrix} = \begin{pmatrix} C_1(n) & 1 & C_3(n) & 0 \\ C_2(n) & 0 & C_4(n) & 1 \end{pmatrix} P^{-1}(n),$$

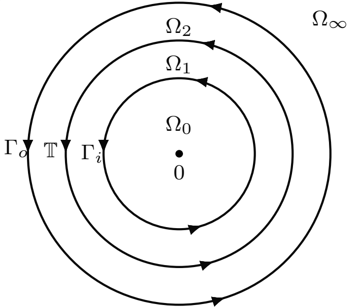
has a solution, it has to be unique.

Lemma 2. Assume that at least one of the following six inequalities is true,

$$\begin{aligned} P_{22}(n)P_{44}(n) - P_{42}(n)P_{24}(n) &\neq 0, \\ (1 - P_{21}(n))P_{42}(n) + P_{22}(n)P_{41}(n) &\neq 0, \\ (1 - P_{43}(n))P_{22}(n) + P_{23}(n)P_{42}(n) &\neq 0, \\ (1 - P_{21}(n))P_{44}(n) + P_{41}(n)P_{24}(n) &\neq 0, \\ (1 - P_{21}(n))(P_{43}(n) - 1) + P_{41}(n)P_{23}(n) &\neq 0, \\ (1 - P_{43}(n))P_{24}(n) + P_{23}(n)P_{44}(n) &\neq 0. \end{aligned}$$

Then, the linear system on C_j's is uniquely solvable.

Lemma 3. Suppose that the solution of the $\mathcal{X}$-RHP exists. Then, if at least one of the conditions of Lemma 2 holds, one can uniquely reconstruct the solution of the $\mathcal{Y}$-RHP.



$$J_{\mathcal{X}}(z) := \begin{pmatrix} 1 & 0 & \tilde{w}(z) & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -w(z) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & -\phi(z) \\ 0 & 1 & \tilde{\phi}(z) & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \tilde{w}(z) & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\equiv J_{\mathcal{X},o}(z) J_{\mathcal{X},\mathbb{T}}(z) J_{\mathcal{X},i}(z).$$

$$\mathcal{Z}(z;n) := \mathcal{X}(z;n) \begin{cases} J_{\mathcal{X},i}^{-1}(z), & z \in \Omega_1, \\ J_{\mathcal{X},o}(z), & z \in \Omega_2, \\ I, & z \in \Omega_0 \cup \Omega_\infty, \end{cases}$$

- **RH-$\mathcal{Z}$1** $\mathcal{Z}$ is holomorphic in $\mathbb{C} \setminus \Gamma$.
- **RH-$\mathcal{Z}$2** $\mathcal{Z}_+(z;n) = \mathcal{Z}_-(z;n) \begin{cases} J_{\mathcal{X},\mathbb{T}}(z), & z \in \mathbb{T}, \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o. \end{cases}$
- **RH-$\mathcal{Z}$3** $\mathcal{Z}(z;n) = (I + \mathcal{O}(z^{-1})) \begin{pmatrix} z^n & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad z \to \infty.$
- **RH-$\mathcal{Z}$4** $\mathcal{Z}(z;n) = P(n)(I + \mathcal{O}(z)) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^{-n} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^n \end{pmatrix}, \qquad z \to 0.$

$$T(z;n) := \mathcal{Z}(z;n) \begin{cases} \begin{pmatrix} z^{-n} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & z^n & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, & |z| > 1, \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & z^n & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}, & |z| < 1. \end{cases}$$

- **RH-T1** T is holomorphic in $\mathbb{C} \setminus \Gamma$.
- **RH-T2** $T_+(z;n) = T_-(z;n) J_T(z;n)$, where

$$J_T(z;n) = \begin{cases} \widehat{J}(z;n), & z \in \mathbb{T}, \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o, \end{cases} \qquad \text{where} \qquad \widehat{J}(z;n) = \begin{pmatrix} z^n & 0 & 0 & -\phi(z) \\ 0 & z^n & \tilde{\phi}(z) & 0 \\ 0 & 0 & z^{-n} & 0 \\ 0 & 0 & 0 & z^{-n} \end{pmatrix}$$

- **RH-T3** As $z \to \infty$, we have $T(z;n) = I + \mathcal{O}(z^{-1})$.

We observe that for $z \in \mathbb{T}$, J_T can be factorized as follows

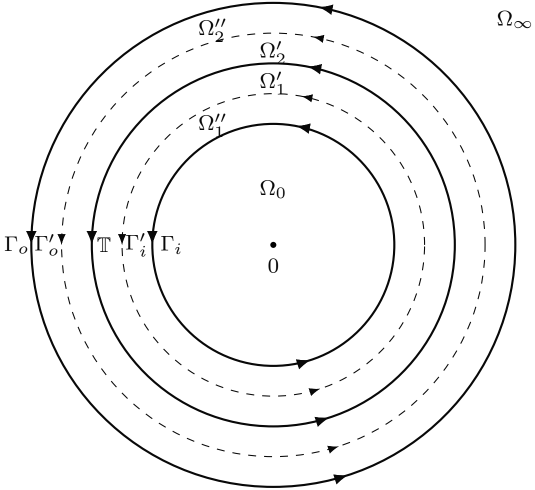
$$\widehat{J}(z;n) = \begin{pmatrix} I_2 & 0_2 \\ z^{-n}\Phi^{-1}(z) & I_2 \end{pmatrix} \begin{pmatrix} 0_2 & \Phi(z) \\ -\Phi^{-1}(z) & 0_2 \end{pmatrix} \begin{pmatrix} I_2 & 0_2 \\ z^{n}\Phi^{-1}(z) & I_2 \end{pmatrix}$$

$$\equiv J_{T,o}(z;n)\mathring{J}(z)J_{T,i}(z;n),$$

where 0_2 and I_2 are respectively 2×2 zero and identity matrices and

$$\Phi(z) = \begin{pmatrix} 0 & -\phi(z) \\ \tilde{\phi}(z) & 0 \end{pmatrix}.$$

Note that $J_{T,i}$ is exponentially close to the identity matrix for z inside of the unit circle and $J_{T,o}$ is exponentially close to the identity matrix for z outside of the unit circle.


Ω_∞
Ω_2''
Ω_2'
Ω_1'
Ω_1''
Ω_0
Γ_o
Γ_o'
$\mathbb{T}$
Γ_i'
Γ_i
0

$$S(z;n) := T(z;n) \times \begin{cases} J_{T,i}^{-1}(z;n), & z \in \Omega_1', \\ J_{T,o}(z;n), & z \in \Omega_2', \\ I, & z \in \Omega_1'' \cup \Omega_2'' \cup \Omega_0 \cup \Omega_\infty, \end{cases}$$

- **RH-S1** S is holomorphic in $\mathbb{C} \setminus \left(\mathbb{T} \cup \Gamma_i \cup \Gamma_o \cup \Gamma_i' \cup \Gamma_o'\right)$.
- **RH-S2** $S_+(z;n) = S_-(z;n) J_S(z;n)$, where

$$J_S(z;n) = \begin{cases} \mathring{J}(z), & z \in \mathbb{T}, \\ J_{T,i}(z;n), & z \in \Gamma_i', \\ J_{T,o}(z;n), & z \in \Gamma_o', \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o. \end{cases}$$

- **RH-S3** As $z \to \infty$, we have $S(z;n) = I + \mathcal{O}(z^{-1})$.

- **RH-$\mathring{S}$1** $\mathring{S}$ is holomorphic in $\mathbb{C} \setminus (\mathbb{T} \cup \Gamma_i \cup \Gamma_o)$.
- **RH-$\mathring{S}$2** $\mathring{S}_+(z) = \mathring{S}_-(z) J_{\mathring{S}}(z)$, where

$$J_{\mathring{S}}(z) = \begin{cases} \mathring{J}(z), & z \in \mathbb{T}, \\ J_{\mathcal{X},i}(z), & z \in \Gamma_i, \\ J_{\mathcal{X},o}(z), & z \in \Gamma_o. \end{cases}$$

- **RH-$\mathring{S}$3** As $z \to \infty$, we have $\mathring{S}(z) = I + \mathcal{O}(z^{-1})$.

Now define

$$\Lambda(z) := \mathring{S}(z) \times \begin{cases} J_{\mathcal{X},i}(z), & z \in \Omega_1, \\ J_{\mathcal{X},o}^{-1}(z), & z \in \Omega_2, \\ I, & z \in \Omega_0 \cup \Omega_\infty, \end{cases}$$

Λ satisfies what we refer to as *the model Riemann-Hilbert problem for the pair* (ϕ, w):

- **RH-Λ1** Λ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$.
- **RH-Λ2** $\Lambda_+(z) = \Lambda_-(z) J_\Lambda(z)$, for $z \in \mathbb{T}$, where

$$J_\Lambda(z) = \begin{pmatrix} 0 & 0 & 0 & -\phi(z) \\ -\dfrac{w(z)}{\phi(z)} & 0 & \tilde{\phi}(z) - \dfrac{w(z)\tilde{w}(z)}{\phi(z)} & 0 \\ 0 & -\dfrac{1}{\tilde{\phi}(z)} & 0 & 0 \\ \dfrac{1}{\phi(z)} & 0 & \dfrac{\tilde{w}(z)}{\phi(z)} & 0 \end{pmatrix}.$$

- **RH-Λ3** As $z \to \infty$, we have $\Lambda(z) = I + \mathcal{O}(z^{-1})$.

What if w is supported on the line?

Let w be supported on the interval $[a, b]$, $0 < a < b < 1$. Consider

$$Y(z) := \begin{pmatrix} P_n(z) & \displaystyle\int_a^b \frac{P_n(x)x^s w(x)}{x-z}dx + \int_{\mathbb{T}} \frac{\tilde{\phi}(\xi)\xi^r \tilde{P}_n(\xi)}{\xi - z}\frac{d\xi}{2\pi i\xi} \\ -\dfrac{1}{h_{n-1}}P_{n-1}(z) & -\dfrac{1}{h_{n-1}}\left\{\displaystyle\int_a^b \frac{P_{n-1}(x)x^s w(x)}{x-z}dx + \int_{\mathbb{T}} \frac{\tilde{\phi}(\xi)\xi^r \tilde{P}_{n-1}(\xi)}{\xi - z}\frac{d\xi}{2\pi i\xi}\right\} \end{pmatrix},$$

where we recall

$$\int_{\mathbb{T}} P_n(z)z^{-k-r}\phi(z)\frac{dz}{2\pi iz} + \int_a^b P_n(x)x^{k+s}w(x)dx = h_n\delta_{n,k}, \quad k = 0, \dots, n,$$

and

$$h_{n-1} \equiv \frac{D_n(\phi, w; r, s)}{D_{n-1}(\phi, w; r, s)}.$$

The RHP for OPs, w supported on $[a,b]$

- **RH-Y1** Y is holomorphic in $\mathbb{C}\setminus(\mathbb{T}\cup[a,b])$.
- **RH-Y2** For $z\in\mathbb{T}$ we have

$$Y_+^{(1)}(z;n)=Y_-^{(1)}(z;n),$$

and

$$Y_+^{(2)}(z;n)=Y_-^{(2)}(z;n)+z^{r-1}\tilde{\phi}(z)Y_-^{(1)}(z^{-1};n).$$

- **RH-Y3** For $x\in[a,b]$ we have

$$Y_+^{(1)}(x;n)=Y_-^{(1)}(x;n)$$

and

$$Y_+^{(2)}(x;n)=Y_-^{(2)}(x;n)+2\pi i x^s w(x)Y_-^{(1)}(x;n).$$

- **RH-Y4** As $z\to\infty$

$$Y(z)=\left(I+O(\frac{1}{z})\right)z^{n\sigma_3}.$$

Similar to the previous case, we can associate 2×4 and 4×4 RHPs to this Riemann-Hilbert problem and analyse it for $r=1$ and arbitrary $s\in\mathbb{Z}$. It is remarkable that we arrive to the *same* model problem, now for the pair $(\phi,-\tilde{u})$, where

$$u(z) := z\int_a^b \frac{t^{s-1}w(t)}{t-z}\,dt.$$

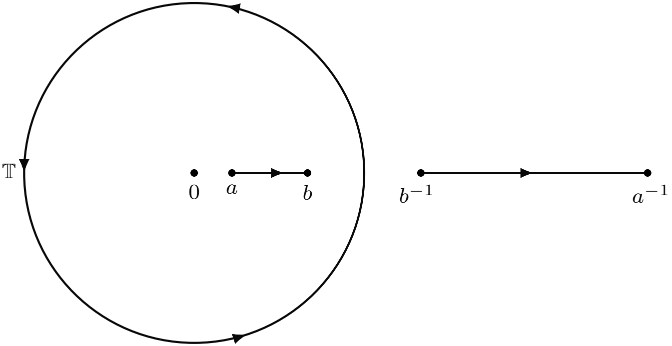


Figure: Jump contour for the X and T-RHPs.

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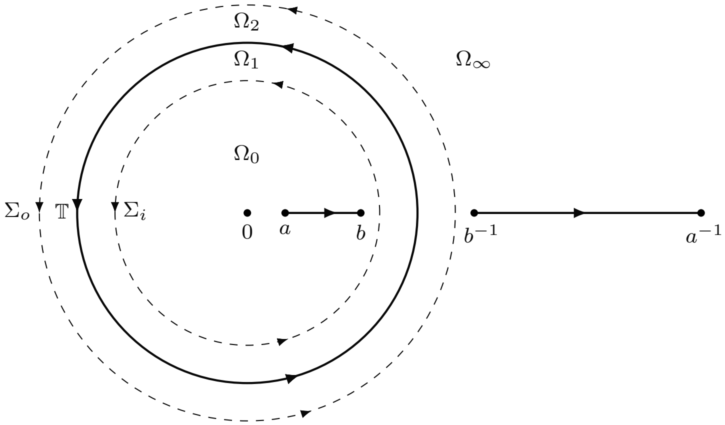

Figure: Jump contour for the S-RHP.

It is remarkable that we arrive to the *same* model problem, now for the pair $(\phi, -\tilde{u})$, where

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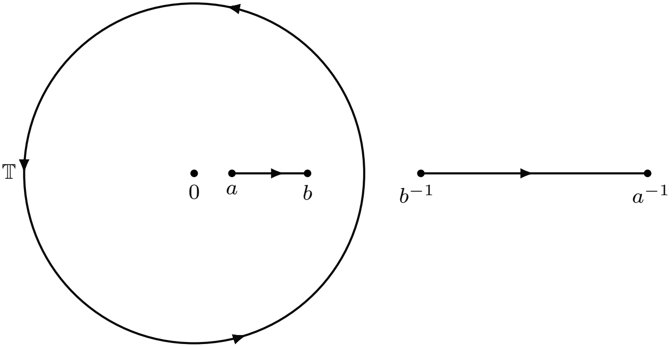

Figure: Jump contour for the global parametrix $\mathring{S}$.

It is remarkable that we arrive to the *same* model problem, now for the pair $(\phi, -\tilde{u})$, where

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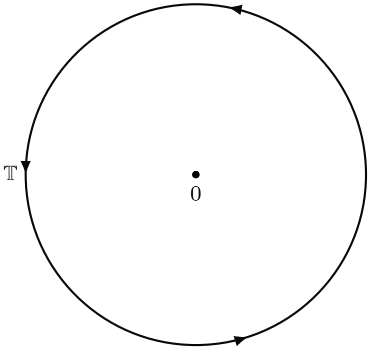

Figure: Jump contour for the model RHP for the pair $(\phi, -\tilde{u})$.

- **RH-Λ1** Λ is holomorphic in $\mathbb{C} \setminus \mathbb{T}$.
- **RH-Λ2** $\Lambda_+(z) = \Lambda_-(z) J_\Lambda(z)$, for $z \in \mathbb{T}$, where

$$J_\Lambda(z) = \begin{pmatrix} 0 & 0 & 0 & -\phi(z) \\ -\dfrac{w(z)}{\phi(z)} & 0 & \tilde{\phi}(z) - \dfrac{w(z)\tilde{w}(z)}{\phi(z)} & 0 \\ 0 & -\dfrac{1}{\tilde{\phi}(z)} & 0 & 0 \\ \dfrac{1}{\phi(z)} & 0 & \dfrac{\tilde{w}(z)}{\phi(z)} & 0 \end{pmatrix}.$$

- **RH-Λ3** As $z \to \infty$, we have $\Lambda(z) = I + \mathcal{O}(z^{-1})$.

The answer is yes! at least for the pair $(\phi, d\phi)$, where ϕ and d are smooth and nonzero on the unit circle, with zero winding number, and we further require d to satisfy $d(e^{i\theta})d(e^{-i\theta}) = 1$, $\theta \in [0, 2\pi)$. For instance, a class of functions satisfying these conditions is given by

$$d(z) = \prod_{j=1}^{m} d_j(z), \qquad d_j(z) = \pm \left(\frac{z - b_j}{z - a_j}\right)^{\alpha_j} \left(\frac{a_j z - 1}{b_j z - 1}\right)^{\alpha_j}, \tag{1}$$

where $\alpha_j \in \mathbb{C}$, all factors are defined by their principal branch, and

$$0 < a_1 < b_1 < a_2 < b_2 < \cdots < a_m < b_m < 1.$$

The last condition on d makes $J_{\Lambda,23} = 0$, while the first three conditions on ϕ and d allow us to factorize them on the unit circle using the Szegő functions:

$$\alpha(z) = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\ln(\phi(\tau))}{\tau - z} d\tau\right], \qquad \beta(z) = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\ln(d(\tau))}{\tau - z} d\tau\right].$$

By Plemelj-Sokhotskii formula α, β, $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy the following jump conditions on the unit circle:

$$\alpha_+(z) = \alpha_-(z)\phi(z), \qquad \beta_+(z) = \beta_-(z)d(z),$$
$$\tilde{\alpha}_-(z) = \tilde{\alpha}_+(z)\tilde{\phi}(z), \qquad \tilde{\beta}_-(z) = \tilde{\beta}_+(z)\tilde{d}(z).$$

Now let ρ be defined on the unit circle by

$$\rho(z) = -\left(\tilde{\beta}_-(z)\beta_+(z)\tilde{\alpha}_-(z)\alpha_+(z)\right)^{-1}.$$

Can we solve the model problem?

$$\Lambda(z) = \Lambda_\infty^{-1} \begin{pmatrix} 1 & 0 & 0 & 0 \\ \mathcal{C}_\rho(z) & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \times \begin{cases} \begin{pmatrix} -\beta(z) & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\tilde{\alpha}(z)\tilde{\beta}(z)\alpha(z)} & 0 \\ 0 & -\tilde{\alpha}(z) & 0 & 0 \\ 0 & 0 & 0 & -\alpha(z) \end{pmatrix}, & |z| < 1, \\ \begin{pmatrix} 0 & \beta(z) & 0 & 0 \\ 0 & 0 & 0 & \dfrac{1}{\tilde{\beta}(z)\tilde{\alpha}(z)\alpha(z)} \\ 0 & 0 & \tilde{\alpha}(z) & 0 \\ \alpha(z) & 0 & 0 & 0 \end{pmatrix}, & |z| > 1, \end{cases}$$

where

$$\Lambda_\infty^{-1} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \dfrac{1}{\alpha(0)} & 0 \\ 0 & \alpha(0) & 0 & 0 \end{pmatrix}, \qquad \text{and} \qquad \mathcal{C}_\rho(z) = \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\rho(\tau)}{\tau - z} d\tau.$$

Recall

$$\alpha(z) = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\ln(\phi(\tau))}{\tau - z} d\tau\right], \qquad \beta(z) = \exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\ln(d(\tau))}{\tau - z} d\tau\right],$$

and

$$\rho(z) = -\left(\tilde{\beta}_-(z)\beta_+(z)\tilde{\alpha}_-(z)\alpha_+(z)\right)^{-1}.$$

Define

$$g_{23}(z) = -\frac{\alpha(0)\tilde{d}(z)\beta(z)}{\tilde{\alpha}(z)}, \qquad g_{43}(z) = -\alpha^2(0)\beta(z)\left(\frac{\alpha(z)}{\tilde{\phi}(z)} + \frac{\tilde{d}(z)C_\rho(z)}{\tilde{\alpha}(z)}\right),$$

$$R_{1,23}(z;n) = \frac{1}{2\pi i}\int_{\Gamma_i'} \frac{\mu^n g_{23}(\mu)}{\mu - z} d\mu, \qquad R_{1,43}(z;n) = \frac{1}{2\pi i}\int_{\Gamma_i'} \frac{\mu^n g_{43}(\mu)}{\mu - z} d\mu,$$

where the contour Γ_i' is a circle, oriented counter-clockwise, with radius $r_0 < 1$ so that the functions ϕ and d are analytic in the annulus $\{z : r_0 \le |z| < 1\}$. Finally denote

$$\mathcal{E}(n) := \frac{2}{\alpha(0)} R_{1,43}(0;n) - C_\rho(0)R_{1,23}(0;n),$$

which is the leading order term in the asymptotic expansion of

$$(1 - P_{21}(n))P_{42}(n) + P_{22}(n)P_{41}(n).$$

Theorem. Suppose that $\phi(e^{i\theta})$ is smooth and nonzero on the unit circle with zero winding number which admits an analytic continuation in a neighborhood of the unit circle. Let $w = d\phi$, where d satisfies all the properties of ϕ in addition to $d(e^{i\theta})d(e^{-i\theta}) = 1$, for all $\theta \in [0, 2\pi)$. Also assume that the exists a $C > 0$, such that for all n sufficiently large

$$|\mathcal{E}(n)| \geq Cr^n, \quad \text{for some } r: \quad r_0 \leq r < 1,$$

where r_0 is the inner radius of the largest symmetric annulus (outer radius is the reciprocal of the inner radius) in which both functions d and ϕ are analytic. Then, for sufficiently large n the determinant $D_n(\phi, w; 1, 1) \neq 0$ and the asymptotics of

$$h_{n-1} \equiv \frac{D_n(\phi, w; 1, 1)}{D_{n-1}(\phi, w; 1, 1)},$$

is given by

$$h_{n-1} = -\alpha(0) \frac{\mathcal{E}(n)}{\mathcal{E}(n-1)}(1 + \mathcal{O}(e^{-c_1 n})), \qquad n \to \infty,$$

where $c_1 = -\log\left(\frac{r_1^2}{r}\right) > 0$, and r_1 is any number satisfying the conditions:

$$r < r_1 < 1 \qquad \text{and} \qquad r_1^2 < r.$$

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7. Characteristic polynomial of a Hankel matrix.

Thank you!