Vanishing of local cohomology modules

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Takeaway

Studying vanishing of local cohomology modules is a fascinating research area; there have been many remarkable theorems and there are still many interesting problems to work on.

Definition

Let R be a noetherian commutative ring and I be an ideal. Let Mod_R denote the category of R-modules.

I-torsion functor

The I-torsion functor, $\Gamma_I : Mod_R \to Mod_R$, is defined by:

- $\Gamma_I(M) = \{m \in M | I^n m = 0 \textit{ for some integer } n\} \in \mathrm{Obj}(Mod_R)$
- $\Gamma_I(M \xrightarrow{f} N) = \left(\Gamma_I(M) \xrightarrow{f_{\Gamma_I(M)}} \Gamma_I(N)\right) \in \mathrm{Mor}(Mod_R)$

Basic property

Γ_I is a covariant left-exact functor.

Local cohomology

The j-th local cohomology supported in I, denoted by $H_I^j(-)$, is the j-th derived functor of Γ_I. That is, $H_I^j(M) \cong H^j(0 \to \Gamma_I(E^\bullet))$, where $0 \to M \to E^\bullet$ is an injective resolution of M.

An example

Example

Set $R = \mathbb{Z}$ and $I = (2)$. Compute $H^j_{(2)}(\mathbb{Z})$.
An injective resolution of $\mathbb{Z}$ is given by

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

Applying $\Gamma_{(2)}$ to $0 \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$, we have

$$0 \to \Gamma_{(2)}(\mathbb{Q}) = 0 \to \Gamma_{(2)}(\mathbb{Q}/\mathbb{Z}) \to 0.$$

Hence $H^j_{(2)}(\mathbb{Z}) = 0$ for $j \neq 1$ and $H^1_{(2)}(\mathbb{Z}) \cong \Gamma_{(2)}(\mathbb{Q}/\mathbb{Z})$. Since $\mathbb{Q}/\mathbb{Z}$ consists of equivalent classes $[r]$ for rationals $0 \leq r < 1$,

$$H^1_{(2)}(\mathbb{Z}) \cong \Gamma_{(2)}(\mathbb{Q}/\mathbb{Z}) \cong \left\{[\frac{m}{2^n}] | 0 \leq m < 2^n,\ n \geq 1\right\}$$

Properties of Γ_I

- Assume $\sqrt{I} = \sqrt{J}$. Then $\Gamma_I(-) = \Gamma_J(-)$
- Since $\{x \in M | I^n x = 0\} \cong \mathrm{Hom}_R(R/I^n, M)$, we have $\Gamma_I(-) \cong \varinjlim_n \mathrm{Hom}_R(R/I^n, -)$. (This shows that Γ_I is left-exact.) And, $H^j_I(-) \cong \varinjlim_n \mathrm{Ext}^j_R(R/I^n, -)$.
- Since both $\mathrm{Hom}_R(R/I^n, -)$ and direct limits commute with flat ring homomorphisms, so does $H^j_I(-)$. I.e.

  $$H^j_I(M) \otimes_R S \cong H^j_{IS}(M \otimes_R S),$$

  for any flat $R \to S$, including localization, henselization, completion, etc.
- By the same token, Γ_I and hence $H^j_I(-)$ commutes with direct sum. (For instance, if $H^j_I(R) = 0$, then $H^j_I(F) = 0$ for all free R-modules F.)

Properties of Γ_I, continued

- Let $\mathfrak{p}$ be a prime ideal of R. Then

$$\Gamma_I(E(R/\mathfrak{p})) = \begin{cases} E(R/\mathfrak{p}) & I \subseteq \mathfrak{p} \\ 0 & \text{otherwise} \end{cases}$$

  Reason:
  1. each element in $E(R/\mathfrak{p})$ is killed by a power of $\mathfrak{p}$;
  2. every element in $R - \mathfrak{p}$ acts as an automorphism on $E(R/\mathfrak{p})$.
- $H^j_I(E) = 0$ for $j > 0$ and injective module E. (For instance, $H^j_{(2)}(\mathbb{Q}) = 0$ for all $j > 0$.)
- Short exact sequences $0 \to L \to M \to N \to 0$ induce long exact sequence on local cohomology

$$\cdots \to H^j_I(L) \to H^j_I(M) \to H^j_I(N) \to H^{j+1}_I(L) \to \cdots$$

An extension of previous example

Let $R = k[x_1, \ldots, x_d]$ and $\mathfrak{m} = (x_1, \ldots, x_d)$.

$$0 \to R \to \cdots \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=j} E(R/\mathfrak{p}) \to \cdots \to \bigoplus_{\mathrm{ht}(\mathfrak{p})=d} E(R/\mathfrak{p}) \to 0$$

Consequently,

$$H^j_{\mathfrak{m}}(R) = \begin{cases} E(R/\mathfrak{m}) & j = d \\ 0 & j \neq d \end{cases}$$

Remark

We need more tools to compute $H^j_I(R)$.

- Injective resolutions are not easy to construct for non-Gorenstein rings.
- Even in the injective resolution as above, differentials are somewhat mysterious.

Čech complexes

Definition

For each $f \in R$, define $\check{C}(f; R) := (0 \to R \to R_f \to 0)$. Given a sequence of elements $f_1, \ldots, f_n$, define $\check{C}(f_1, \ldots, f_n; R) := \otimes_{i=1}^n \check{C}(f_i; R)$. More explicitly:

$$0 \to R \to \bigoplus_i R_{f_i} \to \bigoplus_{j<k} R_{f_j f_k} \to \cdots \to R_{f_1 \ldots f_n} \to 0.$$

And $\check{C}(f_1, \ldots, f_n; M) := \check{C}(f_1, \ldots, f_n; R) \otimes_R M$.

Example

$\check{C}(f_1, f_2; R) = (0 \to R \xrightarrow{d^0} R_{f_1} \oplus R_{f_2} \xrightarrow{d^1} R_{f_1 f_2} \to 0)$ where

$$d^0(r) = (\frac{r}{1}, \frac{r}{1}) \quad d^1(\frac{r_1}{f_1^{n_1}}, \frac{r_2}{f_2^{n_2}}) = -\frac{r_1}{f_1^{n_1}} + \frac{r_2}{f_2^{n_2}} = \frac{r_2 f_1^{n_1} - r_1 f_2^{n_2}}{f_1^{n_1} f_2^{n_2}}$$

Čech complexes, continued

Theorem

If $I = (f_1, \ldots, f_n)$, then $H^j_I(M) \cong H^j(\check{C}(f_1, \ldots, f_n; M))$ for all j and all M.

Example

Let $R = \mathbb{Z}$ and $I = (2)$. Then

$$H^j_{(2)}(\mathbb{Z}) \cong H^j(0 \to \mathbb{Z} \to \mathbb{Z}_2 \to 0)$$

Hence $H^j_{(2)}(\mathbb{Z}) = 0$ for $j \neq 1$, and

$$H^1_{(2)}(\mathbb{Z}) \cong \mathbb{Z}_2/\mathbb{Z} \cong \left\{[\frac{m}{2^n}] | 0 \leq m < 2^n,\ n \geq 1\right\}$$

Example

Let $R = k[x_1, \ldots, x_d]$ and $\mathfrak{m} = (x_1, \ldots, x_d)$. From the Čech complex $\check{C}(x_1, \ldots, x_d; R)$:

$$0 \to R \to \bigoplus_i R_{f_i} \to \cdots \to \bigoplus_i R_{x_1 \cdots \hat{x}_i \cdots x_d} \to R_{x_1 \ldots x_d} \to 0$$

we have

$$H^d_{\mathfrak{m}}(R) = \text{Coker}\left(\bigoplus_{i=1}^{d} R_{x_1 \cdots \hat{x}_i \cdots x_d} \to R_{x_1 \cdots x_d}\right) = \bigoplus_{a_1, \ldots, a_d \geq 1} k[\frac{1}{x_1^{a_1} \cdots x_d^{a_d}}]$$

This is one way to 'visualize' $E(R/\mathfrak{m}) \cong H^d_{\mathfrak{m}}(R)$.

Similarly, one can compute ($1 \leq s \leq d$)

$$H^s_{(x_1, \ldots, x_s)}(R) \cong \bigoplus_{a_1, \ldots, a_s \geq 1} k[x_{s+1}, \ldots, x_d][\frac{1}{x_1^{a_1} \cdots x_s^{a_s}}]$$

A consequence from Čech complex

Corollary

If $\sqrt{I} = \sqrt{(f_1, \ldots, f_n)}$, *then* $H^j_I(R) = 0$ *for all* $j > n$.

Theorem (Mayer-Vietoris sequence)

Let I and J be two ideals. Then there is a long exact sequence

$$\cdots \to H^j_{I+J}(-) \to H^j_I(-) \oplus H^j_J(-) \to H^j_{I\cap J}(-) \to H^{j+1}_{I+J}(-) \to \cdots$$

Example

Let $R = k[x, y, u, v]$ and $I = (x, y) \cap (u, v)$. Then I can not be generated by 2 elements up to radical since $H^3_I(R) \neq 0$ since

$$H^3_I(R) \to H^4_{(x,y,u,v)}(R)(\neq 0) \to H^4_{(x,y)}(R) \oplus H^4_{(u,v)}(R)(= 0).$$

Connection with sheaf cohomology

Let $I = (f_1, \dots, f_n)$ be an ideal of R. Set $U := \mathrm{Spec}(R) \backslash V(I)$. Then, in algebraic geometry, the complex computing $H^i(U, \mathcal{O}_U)$ corresponding to

$$0 \to \bigoplus_{i=1}^{n} R_{f_i} \to \bigoplus_{j<k} R_{f_j f_k} \to \cdots \to R_{f_1 \dots f_n} \to 0.$$

(R is removed and also a shift in homological degree by 1.) Consequently, $H^i(U, \mathcal{O}_U) \cong H_I^{i+1}(R)$ when $i > 0$, and

$$0 \to H_I^0(R) \to R \to H^0(U, \mathcal{O}_U) \to H_I^1(R) \to 0.$$

The graded analogue also holds. Consequently, when

$R = k[x_0, \dots, x_n]$ and $\mathfrak{m} = (x_0, \dots, x_n)$

$$H_{\mathfrak{m}}^{n+1}(R) \cong \bigoplus_{\ell \in \mathbb{Z}} H^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\ell))$$

Grothendieck's Problem

Grothendieck's Problem

Let R be a noetherian local ring, I be an ideal of R and t be an integer. Find conditions under which $H^j_I(M) = 0$ for all $j > t$ and all R-modules M.

Equivalent formulation

Equivalent to finding conditions under which $H^j_I(R) = 0$ for all $j > t$.

Proof sketch of equivalence.

Assume $H^j_I(R) = 0$ for all $j > t$ and $H^i_I(M) \neq 0$ for some M and $i > t$. Let $\ell > t$ be the greatest integer $\exists M$ such that $H^\ell_I(M) \neq 0$ (ℓ is finite, next slide). Consider $0 \to N \to F \to M \to 0$ with F free. Then $0 = H^\ell_I(F) \to H^\ell_I(M) \to H^{\ell+1}_I(N) = 0$, contradiction. Hence $H^j_I(M) = 0$ for all $j > t$ and all R-modules M. □

Grothendieck Vanishing

Theorem (Grothendieck)

Let $(R, \mathfrak{m})$ *be a noetherian local ring of dimension* d *and* I *be an ideal of* R*. Then* $H^j_I(R) = 0$ *for all* $j > d$*.*

An observation

There exist d *elements* $a_1, \ldots, a_d \in I$ *such that* $\sqrt{I} = \sqrt{(a_1, \ldots, a_d)}$*.*

Proof idea: use induction to find elements $a_1, \ldots, a_r \in I$ ($r \leq d$) such that every prime ideal of height $r - 1$ or less that contains $(a_1, \ldots, a_r)$ must also contain I.

Proof sketch.

Our observation asserts there are elements $a_1, \ldots, a_d \in I$ such that $\sqrt{I} = \sqrt{(a_1, \ldots, a_d)}$. Hence $H^j_I(R) = H^j_{(a_1,\ldots,a_d)}(R)$. Now the Čech complex with respect to $a_1, \ldots, a_d$ finishes the proof. □

This solves Grothendieck's Problem in the case when $t = \dim(R)$.

A nonvanishing theorem

Theorem (Grothendieck)

Let $(R, \mathfrak{m})$ *be a noetherian local ring of dimension* d*. Then* $H^d_{\mathfrak{m}}(R) \neq 0$*.*

Remark

- If $d > 0$, then $H^d_{\mathfrak{m}}(R)$ is *not* finitely generated.
- Set $g := \mathrm{grade}(I)$. Then $H^j_I(R) = 0$ for all $j < g$ and $H^g_I(R) \neq 0$.

Corollary

If $f_1 \ldots, f_c$ *is a regular sequence in* R*, then* $H^j_{(f_1,\ldots,f_c)}(R) \neq 0 \Leftrightarrow j = c$*.*

Example

Let $R = k[x_1, \ldots, x_d]$ and $I = (x_1, \ldots, x_s)$. Then $H^j_I(R) = 0$ when $j \neq s$. We have calculated $H^s_I(R)$.

Hartshorne-Lichtenbaum Vanishing

Theorem (Hartshorne-Lichtenbaum, 1968)

Let $(R, \mathfrak{m})$ be a complete local domain of dimension d and I be an ideal of R. The following are equivalent:

- $H^j_I(R) = 0$ *for all* $j > d - 1$*;*
- $\sqrt{I} \neq \mathfrak{m}$.

This solves Grothendieck's Problem in the case when $t = \dim(R) - 1$.

Remark

This is a highly non-trivial result, and has found numerous applications.

Hartshorne's Theorem

So far we have seen solutions to Grothendieck's Problem in the cases when $t = \dim(R)$ and when $t = \dim(R) - 1$. What about the case when $t = \dim(R) - 2$?

Theorem (Hartshorne's Second Vanishing Theorem, 1968)

Let X be a (geometrically) connected closed subscheme of $\mathbb{P}^d_k$ over a field k, of dimension ≥ 1. Then

$$H^{d-1}(\mathbb{P}^d - X, \mathcal{F}) = 0$$

for every coherent sheaf $\mathcal{F}$.
Or equivalently, let $R = k[x_0, \ldots, x_d]$ where k is separably closed and I be a homogeneous ideal. Assume that $\dim(R/I) \geq 2$ and $\operatorname{Spec}(R/I) - \{\mathfrak{m}\}$ is connected. Then $H^j_I(R) = 0$ for all $j > \dim(R) - 2$.

Why separably closed?

Example

Let $R = \mathbb{Q}[[x, y, u, v]]$ and $I = (u^2 - 3x^2, v^2 - 3y^2, uv - 3xy, vx - uy)$. Then I is a prime ideal ($R/I \cong \mathbb{Q}[x, x\sqrt{3}, y, y\sqrt{3}]$) and hence $\mathrm{Spec}(R/I) - \{\mathfrak{m}\}$ is connected. However, in $\overline{R} = \overline{\mathbb{Q}}[[x, y, u, v]]$, we have

$$I\overline{R} = (u - x\sqrt{3}, v - y\sqrt{3}) \cap (u + x\sqrt{3}, v + y\sqrt{3})$$

and hence $\mathrm{Spec}(\overline{R}/I) - \{\mathfrak{m}\}$ is disconnected.
Or similarly, let $R = \mathbb{Q}[x, y, u, v]$ and $\overline{R} = \overline{\mathbb{Q}}[x, y, u, v]$ and let I be the same. Then $\mathrm{Proj}(R/I)$ is connected, but $\mathrm{Proj}(\overline{R}/I\overline{R})$ is disconnected.

Remark

If not separably closed, then apply strict henselization (faithfully flat). In local case, after a sequence of strict henselization and completion, one may assume the local ring is complete with separably closed residue field.

Hartshorne's Problem

Definition

Let $(R, \mathfrak{m})$ be a complete local ring whose residue field is separably closed. We say that the Second Vanishing Theorem holds for R, if the following are equivalent for every ideal I of R:

- $H^j_I(R) = 0$ for all $j > \dim(R) - 2$;
- $\dim(R/I) \geq 2$ and $\mathrm{Spec}(R/I) - \{\mathfrak{m}\}$ is connected.

Hartshorne's Problem (1968)

Prove that the Second Vanishing Theorem holds for all complete regular local rings whose residue field is separably closed.

Some positive results

Theorem (Peskine-Szpiro, Ogus, 1973)

The Second Vanishing Theorem holds

- *for regular local ring of equi-characteristic $p > 0$ (due to Peskine-Szpiro, 1973), and*
- *for regular local ring of equi-characteristic 0 (due to Ogus, 1973).*

Remark

Huneke-Lyubeznik (1990) discovered a proof that works for all regular local ring of equi-characteristic. (A refinement of a theorem of Faltings (1978); more on this later.)

Question

What about regular local rings of mixed characteristic?

Second Vanishing Theorem, Unramified Case

Theorem (Zhang, 2021)

The Second Vanishing Theorem holds for all **unramified** *regular local rings of mixed characteristic.*

Proof Sketch.

We sketch the proof of one implication. Assume $\dim(R/I) \geq 2$ and $\mathrm{Spec}(R/I) - \{\mathfrak{m}\}$ is connected. Hartshorne-Lichtenbaum Vanishing implies that $H^d_I(R) = 0$ and $\mathrm{Supp}(H^{d-1}_I(R)) \subseteq \{\mathfrak{m}\}$. A result of Lyubeznik (2000) implies $\dim_k \mathrm{Soc}(H^{d-1}_I(R)) < \infty$. Combining these two shows $H^{d-1}_I(R)$ is artinian. Now invoke a remarkable theorem of Peskine-Szpiro (1973): if $H^{d-1}_I(R)$ is artinian then $H^{d-1}_I(R) = 0$. □

Local cohomological invariant of local rings

Let $(A, \mathfrak{m})$ be an equi-characteristic local ring. Assume that A is a homomorphic image of an equi-characteristic regular local ring $(R, \mathfrak{n})$ of dimension n. Write $A \cong R/I$. Consider

$$\lambda_{i,j}(A) := \dim_{R/\mathfrak{n}} \operatorname{Ext}^i_R(R/\mathfrak{n}, H^{n-j}_I(R)).$$

Theorem (Lyubeznik, 1993)

With $A, R, I, \lambda_{i,j}(A)$ as above, we have

- *$\lambda_{i,j}(A)$ is independent of the choice of R (or the surjection $R \twoheadrightarrow A$).*
- *$\lambda_{i,j}(A) = \lambda_{i,j}(\widehat{A})$.*

Remark

$\lambda_{i,j}(A)$ are called *Lyubeznik numbers* (of A) in the literature.

A topological characterization

Definition

Let A be a local ring. Define a graph G_A as follows. The vertices of G_A consists of top-dimensional minimal primes of A. Two vertices P, Q are joined by an edge iff $\mathrm{ht}(P+Q) = 1$.

Conjecture (Lyubeznik, 1999)

Let $(A, \mathfrak{m})$ be a equi-characteristic complete local ring whose residue field is separably closed. Set $d := \dim(A)$. Then $\lambda_{d,d}(A)$ agrees with the number of connected components of a graph G_A.

Remark

- If A is not complete or the residue field is not separably closed, then consider $\widetilde{A} = \widehat{\hat{A}^{sh}}$ ($\widehat{()}$=completion; $()^{sh}$=strict henselization).
- This conjecture was proved in characteristic p by Lyubeznik (2006) and in full generality (equi-characteristic) by Zhang (2007).

An example

Example

Let $R = \mathbb{Q}[[x, y, u, v]]$ and $I = (u^2 - 3x^2, v^2 - 3y^2, uv - 3xy, vx - uy)$. Set $A = R/I$ (integral domain, only one minimal prime).

Set $\overline{R} = \overline{\mathbb{Q}}[[x, y, u, v]]$ and $\overline{A} = \overline{R}/I\overline{R}$ (now $\overline{A}$ is complete with a separably closed residue field).

Recall: $I\overline{R} = (u - x\sqrt{3}, v - y\sqrt{3}) \cap (u + x\sqrt{3}, v + y\sqrt{3})$ (hence $\overline{A}$ has two top-dim minimal primes).

Then the graph G_A consists of a vertex, while the graph $G_{\overline{A}}$ consists of two vertices with no edge.

It follows that

$$\lambda_{2,2}(A) = \lambda_{2,2}(\overline{A}) = 2.$$

A similar invariant of local rings, mixed char.

Let $(A, \mathfrak{m}, k)$ be a local ring of mixed characteristic that is a homomorphic image of an unramified regular local ring $(R, \mathfrak{n})$. Set $n = \dim(R)$, $d = \dim(A)$, and write $A \cong R/I$. Consider

$$\lambda_{i,j}(A) := \dim_k \operatorname{Soc}(H^i_{\mathfrak{n}} H^{n-j}_I(R)).$$

Remark

- $\lambda_{i,j}(A) = \lambda_{i,j}(\hat{A})$
- If $(R', \mathfrak{n}')$ is another unramified regular local ring of dimension n' such that $A \cong R'/I'$. Then

$$\dim_k \operatorname{Soc}(H^i_{\mathfrak{n}} H^{n-j}_I(R)) = \dim_k \operatorname{Soc}(H^i_{\mathfrak{n}'} H^{n'-j}_{I'}(R))$$

That is, $\lambda_{i,j}(A)$ is independent of the choice of R (or the surjection $R \twoheadrightarrow A$).

A topological characterization

Theorem (Zhang, 2021)

Let $(A, \mathfrak{m}, k)$ be a d-dimensional complete local ring of mixed characteristic whose residue field is separably closed. Then $\lambda_{d,d}(A)$ agrees with the number of connected components of a graph G_A.

Proof idea

Use 2nd Vanishing to handle dim-2 case, then induction on dimension.

Remark

- In equi-characteristic,

$$\dim_k \operatorname{Ext}^i(R/\mathfrak{n}, H_I^{n-j}(R)) = \dim_k \operatorname{Soc}(H_\mathfrak{n}^i H_I^{n-j}(R))$$

  for all i, j.
- In mixed characteristic, they may be different.

Questions

Question 1

Does the second vanishing theorem hold for **ramified** regular local rings?

Huneke Conjecture (1990)

Let R be a regular local ring and I an ideal. Then

$$\dim_{\kappa(\mathfrak{p})} \operatorname{Ext}^i_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), H^j_I(R)_\mathfrak{p}) < \infty$$

for all i, j, where $\mathfrak{p}$ is a prime and $\kappa(\mathfrak{p})$ is the residue field at $\mathfrak{p}$.

Remark

Huneke Conjecture remains open for **ramified** regular local rings. To answer Question 1 in the affirmative, it suffices to show that $\operatorname{Soc}(H^{d-1}_\mathfrak{p}(R))$ is finite for all primes $\mathfrak{p}$ of height $d-2$ (Zhang, 2021).

An alternate approach to Question 1

Theorem (Faltings, 1978)

Let A be a complete local ring containing its residue field. Let I be an ideal of A and set

$$t := \text{emb.}\dim(A) - \min\{\dim(A/P) | P \text{ is minimal prime of } I\}$$

Let $m > t$ be an integer and M be a finitely generated A-module. Assume that, for every integer s with $0 < s < t$ and every prime ideal $\mathfrak{p} \subset A$ with $\dim(A/\mathfrak{p}) > s$, $H^q_{IA_\mathfrak{p}}(M_\mathfrak{p}) = 0$ for all $q \geq m - s$. Then

$$H^q_I(M) = 0, \quad \forall q \geq m.$$

Question 2

Does Faltings Theorem or Huneke-Lyubeznik's refinement hold in mixed characteristic?

Lyubeznik's Problem

Lyubeznik's Problem (1999)

Let $(R, \mathfrak{m})$ be a complete local domain of dimension d whose residue field is separably closed

1. Find necessary and sufficient condition on I under which $H^j_I(R) = 0$ for all $j > d - 2$.
2. Let I be a prime ideal. Assume that $\mathrm{ht}(I + \mathfrak{p}) < d$ for every height-1 prime ideal $\mathfrak{p}$. Is it true that $H^j_I(R) = 0$ for all $j > d - 2$?

An example

Example (Hochster-Zhang, 2018)

Let $R = \mathbb{C}[[x, y, z, u, v]]/(x^3 + y^3 + z^3, z^2 - ux - vy)$ and $I = (x, y, z)$. Then

- $\dim(R) = 3$ and I is a prime ideal of height 1;
- $\mathrm{ht}(I + \mathfrak{p}) < 3$ for every height-1 prime ideal $\mathfrak{p}$;
- $H^2_I(R) \neq 0$.

This example answers 2nd part of Lyubeznik's question in the negative.

Remark

1st part of Lyubeznik's question is wide open.

Thank you!

Stay safe and healthy!