Brownianity in KPZ

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(based on joint work with Jacob Calvert, Ivan Corwin, Alan Hammond, and Konstantin Matetski)

MSRI Seminar
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We will mainly focus on a few limiting, zero-temperature objects.

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But first: is Brownianity independently useful?

First passage percolation

Example KPZ model: first passage percolation

In FPP, each edge e in $\mathbb{Z}^2$ is given an i.i.d. non-negative weight ξ_e.

A path γ is assigned weight $\sum_{e \in \gamma} \xi_e$.

We minimize over all paths between given $x, y \in \mathbb{Z}^2$ to get the weight $T_{x,y}$.

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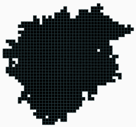

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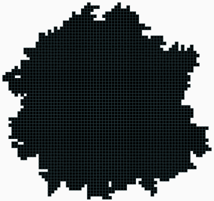

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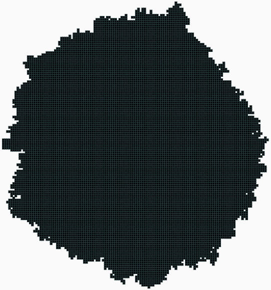

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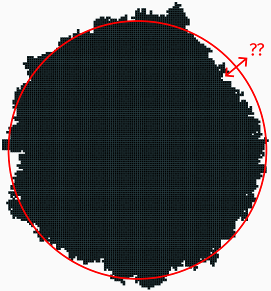
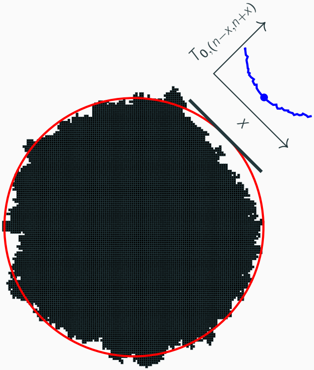


Curvature and KPZ fluctuation scales

The time constant $\mu(x) = \lim_{n\to\infty} n^{-1} T_{\mathbf{0},n(1-x,1+x)}$ is known to exist, *conjectured* to have curvature for $x \in [-1, 1]$; equivalent to curvature of limit shape.

FPP is expected to lie in the KPZ class: $T_{\mathbf{0},(n,n)}$ should have fluctuations of order $n^{1/3}$, and the natural transversal fluctuation scale for $x \mapsto T_{\mathbf{0},(n-x,n+x)}$ should be $n^{2/3}$.



Predicting KPZ exponents from Brownianity + curvature

Suppose

$$x \mapsto \tilde{T}(x) = T_{\mathbf{0},(n-x,n+x)} - \mu(x/n)$$

has "Brownian" fluctuations in x: $\tilde{T}(x) - \tilde{T}(0) \approx |x|^{1/2}$.

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Curvature says

$$T_{\mathbf{0},(n-x,n+x)} \approx \mu(x/n) \cdot n \approx \mu(0)n + n(x/n)^2 = \mu(0)n + x^2/n.$$

The gain of x^2/n competes with a Brownian fluctuation of $|x|^{1/2}$.

These are comparable when

$$|x|^{3/2} \approx n \implies |x| \approx n^{2/3}.$$

Unfortunately, we are very far from being able to prove curvature or Brownianity in *any* FPP model.

But over the past few decades, several last passage percolation models have proven tractable due to connections with integrable probability.

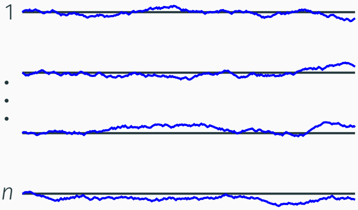
The one in which the Brownian structure is most easily seen is Brownian last passage percolation.

Brownian last passage percolation & the parabolic Airy_2 process

We consider iid Brownian motions, displayed vertically for illustration, and paths that are *directed* (unlike in FPP) to be up-right.

The weight $B[\gamma]$ of a path γ is the sum of increments of the Brownian motions.

The last passage weight $B[(y,n) \to (x,1)]$ between points is given by maximizing over all directed paths between them.



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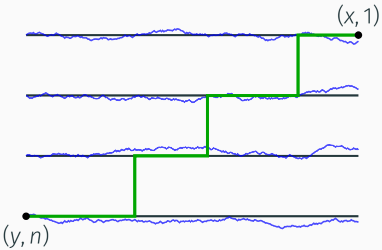
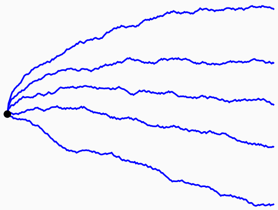


It is maybe intuitive that there should be some kind of Brownianity associated to the weight profile in Brownian LPP.

In fact, it's very explicit: $x \mapsto B[(0,n) \to (x,1)]$ is distributed as the top line of Dyson's Brownian motion!

In other words, non-intersecting Brownian motions.



Using this representation, it was shown that $B[(0,n) \to (x,1)]$ (after appropriate centering and scaling) converges to

$$\mathcal{P}(x) = \mathcal{A}(x) - x^2,$$

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Nevertheless, forms of local Brownianity hold for the *increment*:

- $\varepsilon^{-1/2}(\mathcal{P}(x_0 + \varepsilon x) - \mathcal{P}(x_0))$ converges in law to two-sided Brownian motion as $\varepsilon \to 0$ (Hägg '08).
- $\mathcal{P}$ is Hölder-$\frac{1}{2}^{-}$ (Quastel-Remenik '12).
- $\mathcal{P}$ has modulus of continuity $\delta^{1/2}(\log \delta^{-1})^{1/2}$ (Corwin-Hammond '14).

Unit-order Brownianity

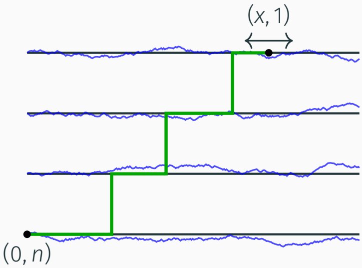
In fact, [CH14] proved a form of *qualitative* Brownianity on unit order scales:

Theorem (Corwin-Hammond)

$\mathcal{P}(\cdot) - \mathcal{P}(a)$ is absolutely continuous to BM on any interval $[a, b]$.

This implies all the properties of the previous slide, but also more.

Suppose we fix the starting point, but leave the ending point unconstrained. The geodesic will then pick an ending point $x_0(n)$ which maximizes its weight.



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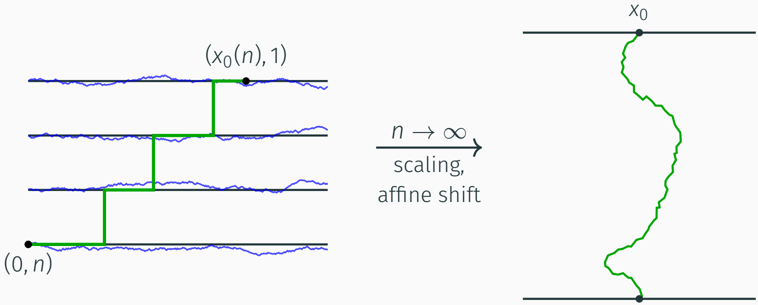

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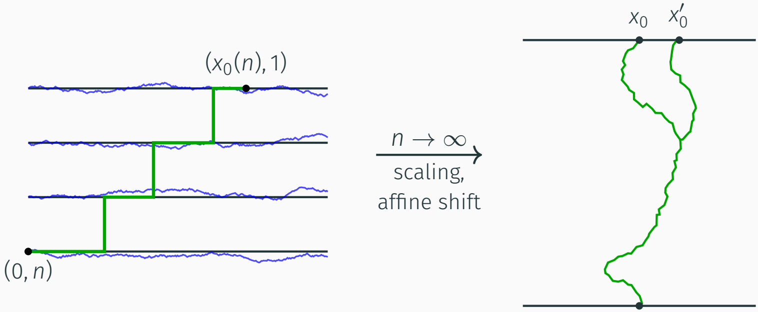

But is the limiting maximizer unique? Or can there be multiple point-to-line geodesics with different ending points?

Geodesic uniqueness and Johansson's conjecture

The geodesic's endpoint is just the maximizer of $\mathcal{P}$, the limiting weight profile. So Johansson conjectured that $\mathcal{P}$ a.s. has a unique maximizer—the geodesic has a unique endpoint.

Theorem (Corwin-Hammond)

$\mathcal{P}$ almost surely has a unique maximizer on $\mathbb{R}$.

Proof: $\mathcal{P}$ has a unique maximizer on $[-M, M]$ for any M by Brownian absolute continuity. Use bounds on lower tail of GUE Tracy-Widom and parabolic decay of $\mathcal{P}$ to extend to $\mathbb{R}$.

Independent proofs not using Brownianity have also been given by Moreno Flores-Quastel-Remenik and Pimentel.

A recent quantification of Brownianity

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Theorem (Calvert-Hammond-H.)

Let $d \geq 1$, $A \subseteq \mathcal{C}([-d, d])$ measurable, and ε the probability that rate two BM lies in A. Then

$$\mathbb{P}\left(\mathcal{P}(\cdot) - \mathcal{P}(-d) \in A\right) \leq \varepsilon \cdot (\textit{subpolynomial-in-}\varepsilon^{-1}).$$

In particular, the Radon-Nikodym derivative of $\mathcal{P} - \mathcal{P}(-d)$ wrt BM lies in all L^p spaces for $0 < p < \infty$.

More explicitly, the subpolynomial factor is $\exp(Gd\log(\varepsilon^{-1})^{5/6})$.

Message: Upper bounds on BM transfer to $\mathcal{P}$ (and $\mathcal{A}$) increments immediately.

An example application

Corollary

As $t \to \infty$,

$$\mathbb{P}\left(\sup_{x\in[-d,d]} |\mathcal{P}(x) - \mathcal{P}(-d)| \geq t\right) \leq \exp\left(-\frac{t^2}{8d}(1 + o(1))\right).$$

A matching lower bound of $\exp(-\frac{t^2}{8d}(1 + o(1)))$ can be obtained separately.

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Proof: By the reflection principle of Brownian motion (recall it is rate two),

$$\mathbb{P}\left(\sup_{x\in[-d,d]} |B(x)| \geq t\right) = 2 \cdot \mathbb{P}\big(|N(0, 2 \cdot 2d)| \geq t\big) \leq 4 \cdot \exp\left(-\frac{t^2}{8d}\right).$$

Other applications

The theorem has also been applied in significantly more sophisticated situations recently:

- Chaos in dynamical Brownian LPP by Hammond-Ganguly (see Alan's talk last Thursday).
- Time correlation exponents in LPP from flat initial data by Basu-Ganguly-Zhang.
- Construction of extended directed landscape by Dauvergne-Zhang.
- Three-halves variation of directed geodesics in the directed landscape by Dauvergne-Sarkar-Virág.

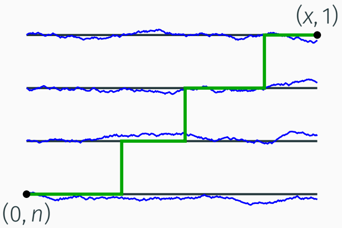
The KPZ fixed point

Going back to Brownian LPP, $\mathcal{P}$ was the scaling limit of the weight profile from a fixed starting point. What about other initial conditions?

Let

$$B_n^{\mathrm{LPP}}[y \to x] = n^{-1/3}\left(B[(2yn^{2/3}, n) \to (n + 2xn^{2/3}, 1)] - \text{centering terms}\right).$$

(So $B_n^{\mathrm{LPP}}[0 \to x] \to \mathcal{P}(x)$ as $n \to \infty$.)

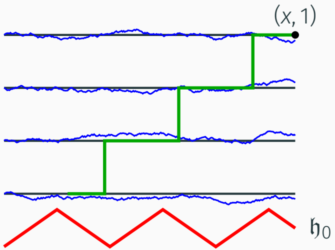


General initial conditions: the KPZ fixed point

For a general initial condition $\mathfrak{h}_0 : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$, we *augment* the weight of a path starting at y by $\mathfrak{h}_0(y)$. Maximizing the augmented weight defines

$$B_n^{\mathrm{LPP}}[\mathfrak{h}_0 \to x] = \sup_y \left\{ \mathfrak{h}_0(y) + B_n^{\mathrm{LPP}}[y \to x] \right\}.$$

Taking $n \to \infty$ gives the KPZ fixed point $\mathfrak{h}_1$ (i.e., at time 1), first constructed by Matetski-Quastel-Remenik.



History of Brownianity in KPZ fixed point

Various formulations of qualitative local Brownianity are known for the fixed point for a fairly wide class of initial data.

- $\varepsilon^{-1/2}(\mathfrak{h}_1(x_0 + \varepsilon x) - \mathfrak{h}_1(x))$ converges to two-sided Brownian motion (Matetski-Quastel-Remenik '17)
- $\mathfrak{h}_1$ is Hölder-$\frac{1}{2}^-$ almost surely (MQR17).
- The modulus of continuity of $\mathfrak{h}_1$ is at most of order $\delta^{1/2}(\log \delta^{-1})^{2/3}$ (Hammond '17)
- $\mathfrak{h}_1 - \mathfrak{h}_1(a)$ is absolutely continuous to Brownian motion on any interval $[a, b]$ (Sarkar-Virág '20)

Quantified Brownianity for the fixed point?

Currently, quantitative general Brownian comparisons are not available.

Techniques to transfer Brownian regularity from $\mathcal{P}$ to $\mathfrak{h}_1$ can give some quantified information in some cases.

For example, combining work of Hammond with the Brownian comparison for $\mathcal{P}$ gives the following.

Theorem (Hammond, Calvert-Hammond-H.)

Fix $\varepsilon > 0$. For a wide class of initial data $\mathfrak{h}_0$, there exist G and x_0 such that, for $|x| < x_0$,

$$\mathbb{E}\left[|\mathfrak{h}_1(x) - \mathfrak{h}_1(0)|^{2-\varepsilon}\right] \leq G|x|^{\frac{1}{2}(2-\varepsilon)}.$$

(For BM, of course, $\mathbb{E}[|B(x)|^{2-\varepsilon}]$ is of order $|x|^{\frac{1}{2}(2-\varepsilon)}$ for all $\varepsilon \geq 0$.)

Returning to Johansson's conjecture

Johansson's conjecture concerned the uniqueness of the geodesic endpoint from fixed starting point. The same question can be asked for general initial condition, i.e., is the maximizer of $\mathfrak{h}_1$ unique a.s.?

Theorem (Corwin-Hammond-H.-Matetski)

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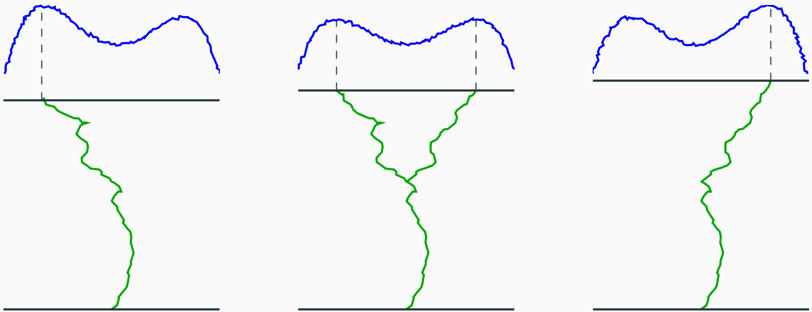
The KPZ fixed point is actually defined as a function-valued process in time, $t \mapsto \mathfrak{h}_t$ [MQR17]. We've focused on the $t = 1$ marginal till now.

In the geodesic picture, varying t varies the height difference between the starting and ending points.

It is possible that there exist random exceptional times t such that $\mathfrak{h}_t$ has multiple maximizers.

Geodesic instability at exceptional times

In addition to endpoint uniqueness, these exceptional times are interesting from the point of view of the dynamic: they are times of geodesic instability.



exceptional time

The set of exceptional times

For $T > 0$, let the "twin peaks" set be

$$\mathcal{T} = \Big\{ t \in [0, T] : \mathfrak{h}_t \text{ has multiple maximizers} \Big\}.$$

For fixed t, we know from before that $t \notin \mathcal{T}$ almost surely, so $\mathrm{Leb}(\mathcal{T}) = 0$ a.s.

But we can understand its sparsity by its fractal (Hausdorff) dimension:

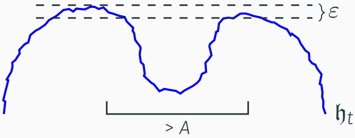
Theorem (Corwin-Hammond-H.-Matetski)

For a wide class of initial data, $\mathcal{T} \neq \emptyset$ with positive probability. Conditioned on this event, $\dim(\mathcal{T}) = \frac{2}{3}$ almost surely.

(The value $\frac{2}{3}$ is related to Hölder continuity properties of $t \mapsto \mathfrak{h}_t$, i.e. in time.)

The role of Brownianity

As might be intuitive, an important ingredient of the proof is to estimate the probability of having "ε-twin peaks" in $\mathfrak{h}_t$ for fixed t:



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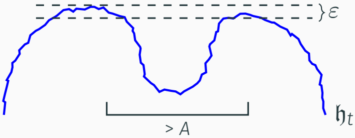

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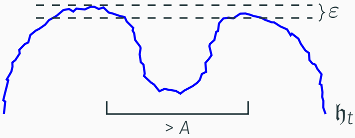

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Proposition

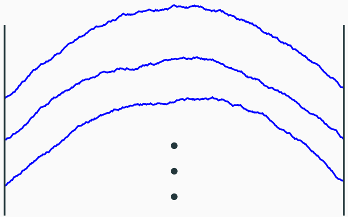
There exist t-dependent constants $c_1(t)$ and $c_2(t)$ such that

$$c_1\varepsilon \leq \mathbb{P}\left(\mathfrak{h}_t \text{ has } \varepsilon\text{-twin peaks}\right) \leq c_2\varepsilon.$$

The Brownian Gibbs property

Brownian Gibbs property

To understand the source of Brownianity in the theorems presented, we need the parabolic Airy line ensemble: an infinite collection of random non-intersecting continuous curves whose top curve is $\mathcal{P}$.



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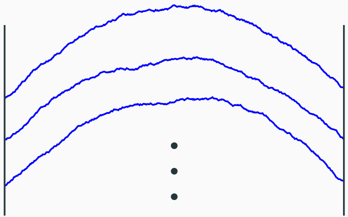

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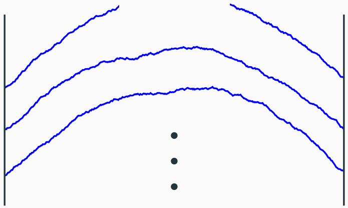

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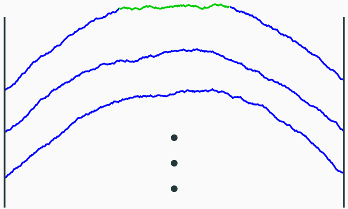

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Summary

- Many limiting zero-temperature KPZ objects have local Brownian structure, for various meanings of "local" and "Brownian".
- The strongest results are available for the parabolic Airy process $\mathcal{P}$ (narrow-wedge initial data), because of the Brownian Gibbs property.
- With work and in some cases, techniques or control of $\mathcal{P}$ can be adapted to the general initial data object, the KPZ fixed point.
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Thank you!