Conformal blocks on a torus via Fredholm determinants

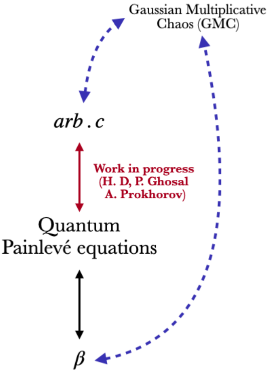
Harini Desiraju

joint work with F. Del Monte, P. Gavrylenko (arXiv: 2011.06292v3)

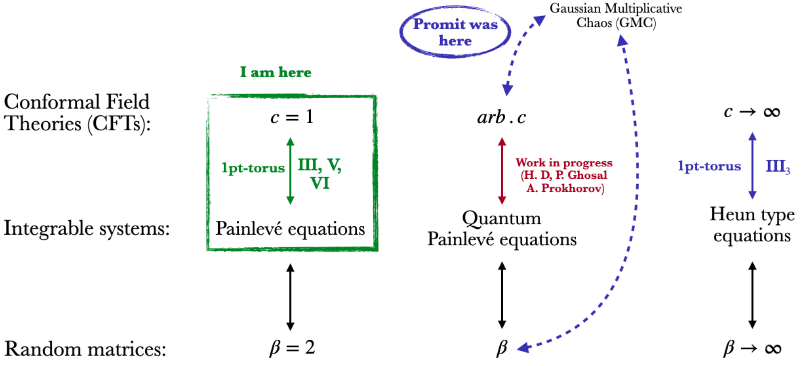
MSRI postdoc seminars

October 1, 2021

Where we left off...



The big picture...



Plan

1. Introduction: tau-functions and Conformal blocks
2. tau-function on a torus as a Fredholm determinant
3. Conformal blocks on a torus from the Fredholm determinant
4. Generalisation
5. Connection constant: Work in progress

Plan

What are tau-functions anyway?

For integrable systems

1. Generators of the Hamiltonians of integrable systems
2. Generator of the solutions of integrable hierarchies/Painlevé equations
3. Zeros of tau-functions = points where Riemann-Hilbert map is invalid = poles of Painlevé transcendents

For random matrices/statistical physics

1. Partition function of ensembles
2. Generator of transition probabilities

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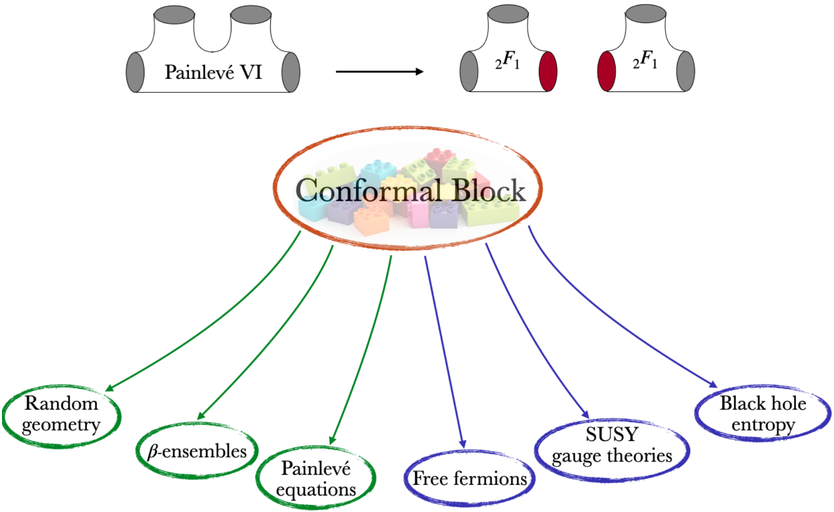
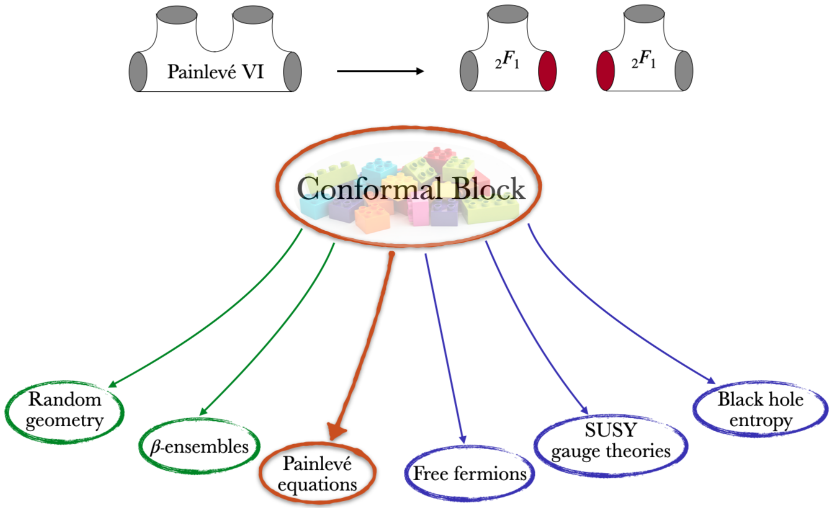
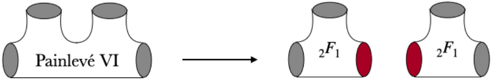
Conformal blocks



Conformal blocks





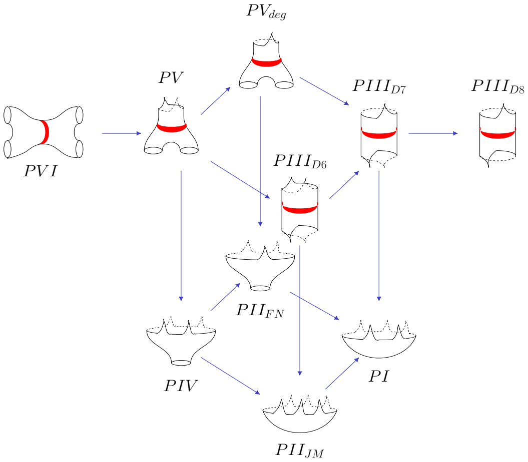


Figure: (CMR) Confluence diagram for Painlevé equations: GL'16

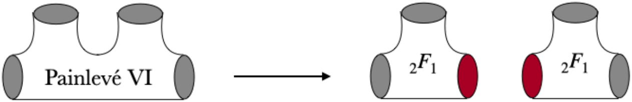
A brief history

- Jimbo-Miwa-Ueno (Kyoto school) '80s : tau-function of Painlevé VI is the correlator of 'Holonomic quantum fields'.
- Gamayun, Iorogov, Lisovyy (Kiev school) '12: re-interpreted the tau-function of Painlevé VI in terms of conformal field theory
- Cafasso, Gavrylenko, Lisovyy, '16, 17: showed that Painlevé III, V, VI tau-functions can be written as Fredholm determinants, which in turn are the discrete Fourier transforms of their respective Conformal blocks.

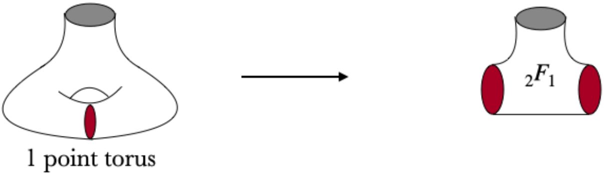
$\leadsto$ H.D '19, 20: Painlevé II tau-function can be written as a Fredholm determinant.

Pictorial summary

Painlevé VI tau-function (GL'16, CGL'17):



One point torus tau-function (F. Del Monte, **H.D**, P. Gavrylenko '20):



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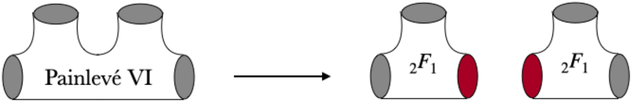

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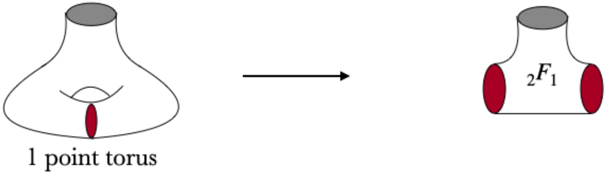

Plan

Elliptic form of Painlevé VI

Starting from Painlevé VI

$$u'' = \frac{1}{2}\left(\frac{1}{u} + \frac{1}{u-1} + \frac{1}{u-t}\right)(u')^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{u-t}\right)u'$$
$$+\frac{u(u-1)(u-t)}{t^2(t-1)^2}\left(\alpha_0 - \frac{\alpha_1 t}{u^2} + \frac{\alpha_2(t-1)}{(u-1)^2} - \left(\alpha_3 - \frac{1}{2}\right)\frac{t(t-1)}{(u-t)^2}\right),$$

do the following change of the dependent variable $u \to Q$, and the independent variable $t \to \tau$

$$Q(t) := \frac{1}{2(e_2 - e_1)^{1/2}} \int_\infty^u \frac{ds}{\sqrt{s(s-1)(s-t)}}, \quad t = \frac{e_3(\tau) - e_1(\tau)}{e_2(\tau) - e_1(\tau)},$$

where for $i = 0, 1, 2, 3$,

$$e_i = \wp(\omega_i), \quad \omega_0 = 0, \quad \omega_1 = \frac{1}{2}, \quad \omega_2 = \frac{1}{2} + \frac{\tau}{2}, \quad \omega_3 = \frac{\tau}{2}.$$

The elliptic form of Painlevé VI is then

$$(2\pi i)^2 \frac{d^2 Q(\tau)}{d\tau^2} = \sum_{n=0}^{3} \alpha_n \wp'(Q(\tau) + \omega_n|\tau),$$

Setting $\alpha_i = \frac{m^2}{8}$, and using that

$$\sum_{n=0}^{3} \wp'(Q + \omega_n|\tau) = 8\wp'(2Q|\tau),$$

the elliptic form of Painlevé VI reduces to

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$$\boxed{(2\pi i)^2 \frac{d^2 Q(\tau)}{d\tau^2} = m^2 \wp'(2Q|\tau).}$$

Minimal setup

Consider the equation of motion of nonautonomous Calogero-Moser system

$$(2\pi i)^2 \frac{d^2 Q(\tau)}{d\tau^2} = m^2 \wp'(2Q|\tau),$$

which arises as the consistency condition of the following system of equations

$$\partial_z Y(z,\tau) = A(z,\tau) Y(z,\tau); \quad 2\pi i \partial_\tau Y(z,\tau) = B(z,\tau) Y(z,\tau).$$

For the purposes of this talk we will only need the following information

$$A(z,\tau) = 2\pi i \frac{dQ(\tau)}{d\tau}\sigma_3 + m \frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)} \begin{pmatrix} 0 & \frac{\theta_1(z+2Q(\tau)|\tau)}{\theta_1(-2Q(\tau)|\tau)} \\ \frac{\theta_1(z-2Q(\tau)|\tau)}{\theta_1(2Q(\tau)|\tau)} & 0 \end{pmatrix}.$$

tau-function and the Hamiltonian

Isomonodromic tau-function $\mathcal{T}_{CM}(\tau)$ = generator of the Hamiltonian H:

$$2\pi i \partial_\tau \log \mathcal{T}_{CM}(\tau) := H(\tau),$$

where the Hamiltonian of the Calogero-Moser system is the a-cycle contour integral

$$H(\tau) = \oint_a dz \frac{1}{2} \operatorname{Tr} A^2(z,\tau) = \left(2\pi i \frac{dQ(\tau)}{d\tau}\right)^2 - m^2 \wp(2Q(\tau)|\tau) + 4\pi i m^2 \partial_\tau \log \eta(\tau),$$

where $\eta(\tau)$ is Dedekind's eta function

$$\eta(\tau) := \left(\frac{\theta_1'(0|\tau)}{2\pi}\right)^{1/3}.$$

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Can $\mathcal{T}_{CM}$ be written as a Fredholm determinant?

Yes! In terms of the monodromy data of the system.

Why a Fredholm determinant?

- ☑ Allows us to write the transcendent $Q(\tau)$ explicitly
- ☑ Fredholm determinant = Fourier transform of $c = 1$ conformal blocks = *charged* Nekrasov-Okounkov functions.
- ☐ Connection constant = modular transformation of $c = 1$ conformal block on a punctured torus.
- ☐ The distribution of the poles of the transcendent = zero locus of the Fredholm determinant.

Monodromy data

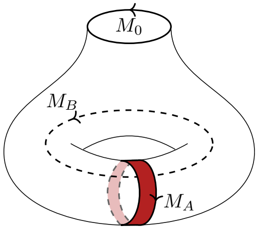
The Lax matrix A behaves as

$$A(z+1,\tau) = A(z,\tau), \qquad A(z+\tau,\tau) = e^{-2\pi i Q(\tau)\sigma_3} A(z,\tau) e^{2\pi i Q(\tau)\sigma_3}.$$

In turn, the monodromies around the a,b-cycles, and the puncture at $z=0$ read

$$Y(z+1,\tau) = Y(z,\tau) M_A, \qquad Y(z+\tau,\tau) = e^{2\pi i Q(\tau)\sigma_3} Y(z,\tau) M_B,$$

$$Y(e^{2\pi i} z,\tau) = Y(z,\tau) M_0,$$



1. The monodromy matrices satisfy the constraint

$$M_0 = M_A^{-1} M_B^{-1} M_A M_B.$$

2. Explicitly, the monodromies are

$$M_A = e^{2\pi i a \sigma_3}, \qquad M_0 \sim e^{2\pi i m \sigma_3},$$

$$M_B = e^{2\pi i \rho} \begin{pmatrix} \frac{\sin \pi (2a - m)}{\sin 2\pi a} e^{-i\nu/2} & \frac{\sin \pi m}{\sin 2\pi a} \\ -\frac{\sin \pi m}{\sin 2\pi a} & \frac{\sin \pi (2a+m)}{\sin 2\pi a} e^{i\nu/2} \end{pmatrix}.$$

Note: The parameter

1. m is the parameter in the equation of motion of the Calogero-Moser system,
2. a, ν give the monodromy data,
3. ρ is a symmetry parameter.

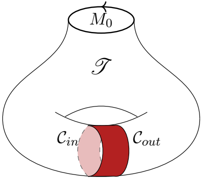
Pants decomposition

Cutting the torus along its a-cycle gives us a pair of pants $\mathscr{T}$ and its monodromies follow from the relation

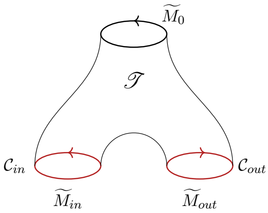
$$M_A M_0 M_B^{-1} M_A^{-1} M_B = 1 = (M_A) M_0 (M_B^{-1} M_A M_B)^{-1} := \widetilde{M}_{in} \widetilde{M}_0 \widetilde{M}_{out},$$

and the pair of pants linear system reads

$$\partial_z \widetilde{Y}(z) = -2\pi i \left(\widetilde{A}_{in} + \frac{\widetilde{A}_0}{1 - e^{2\pi i z}} \right) \widetilde{Y}(z), \qquad \widetilde{A}_{in} \sim a\sigma_3, \qquad \widetilde{A}_0 \sim m\sigma_3.$$



(a) 1 point Torus



(b) Pair of pants

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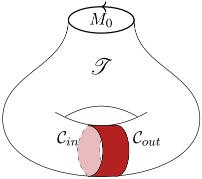

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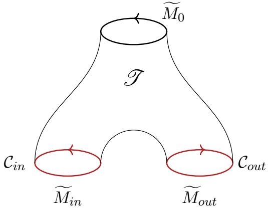
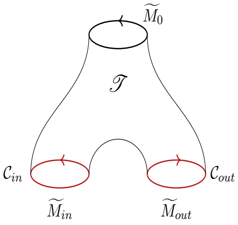


(b) Pair of pants

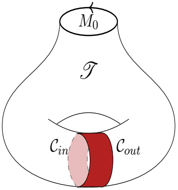
The local solutions of $\widetilde{Y}(z)$ are described by hypergeometric functions!

Cauchy operators

(in cylindrical coordinates)



$$(\mathcal{P}_{\oplus} f)(z) := \oint_{\mathcal{C}_{in} \cup \mathcal{C}_{out}} \frac{\widetilde{Y}(z)\widetilde{Y}(w)^{-1}}{1 - e^{-2\pi i (z-w)}} f(w) dw$$



$$(\mathcal{P}_{\Sigma} f)(z) := \oint_{\mathcal{C}_{in} \cup \mathcal{C}_{out}} Y(z) \Xi(z,w) Y(w)^{-1} f(w) dw,$$

the Cauchy kernel on the torus $\Xi(z,w)$ is

$$\Xi(z,w) = \frac{\theta_1'(0)}{\theta_1(z-w)} \mathrm{diag}\left(\frac{\theta_1(z - w + Q(\tau) - \rho)}{\theta_1(Q(\tau) - \rho)}, \frac{\theta_1(z - w - Q(\tau) - \rho)}{\theta_1(Q(\tau) + \rho)} \right)$$

Explicitly,

$$\mathcal{P}_{\oplus} : f(z) = \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix} \oplus \begin{pmatrix} f_{in,+} \\ f_{out,-} \end{pmatrix} \to \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix} \oplus \begin{pmatrix} \mathsf{a} & \mathsf{b} \\ \mathsf{c} & \mathsf{d} \end{pmatrix} \begin{pmatrix} f_{in,-} \\ f_{out,+} \end{pmatrix},$$

In terms of the local solutions

$$\widetilde{Y}_{in}(z) := \widetilde{Y}(z)|_{\mathcal{C}_{in}}, \qquad \widetilde{Y}_{out}(z) := e^{2\pi i \nu} \sigma_1 \widetilde{Y}_{in}(-z) \sigma_1,$$

$$(\mathsf{a}g)(z) = \oint_{\mathcal{C}_{in}} dw \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{in}(w)^{-1} - \mathbb{1}}{1 - e^{-2\pi i(z-w)}} g(w), \ \ z \in \mathcal{C}_{in},$$

$$(\mathsf{b}g)(z) = \oint_{\mathcal{C}_{out}} dw \frac{\widetilde{Y}_{in}(z)\widetilde{Y}_{out}(w)^{-1}}{1 - e^{-2\pi i(z-w)}} g(w), \ \ z \in \mathcal{C}_{in},$$

$$(\mathsf{c}g)(z) = \oint_{\mathcal{C}_{in}} dw \frac{\widetilde{Y}_{out}(z)\widetilde{Y}_{in}(w)^{-1}}{1 - e^{-2\pi i(z-w)}} g(w), \ \ z \in \mathcal{C}_{out},$$

$$(\mathsf{d}g)(z) = \oint_{\mathcal{C}_{out}} dw \frac{\widetilde{Y}_{out}(z)\widetilde{Y}_{out}(w)^{-1} - \mathbb{1}}{1 - e^{-2\pi i(z-w)}} g(w), \ \ z \in \mathcal{C}_{out}.$$

Main statement 1/2: Fredholm determinant

The following statement can be verified

$$\det_{\mathcal{H}_+}\left[\mathcal{P}_\Sigma^{-1}\mathcal{P}_\oplus\right] = \det_{\mathcal{H}}\left[\mathbb{1} - K_{1,1}(\tau)\right] := \det\left[\mathbb{1} - \begin{pmatrix} \nabla^{-1}\mathsf{c} & \nabla^{-1}\mathsf{d}\nabla \\ \mathsf{a} & \mathsf{b}\nabla \end{pmatrix}\right],$$

where ∇ is a shift operator

$$\nabla g(z) = e^{2\pi i \rho} g(z-\tau).$$

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- The main trick to obtain Fredholm determinants (that applies also to Riemann-Hilbert problems) is to *take the ratio of a global object and a local object.*
- Here, $\mathcal{P}_\Sigma$ is the global object and $\mathcal{P}_\oplus$ is the local object.
- The structure of this determinant generalizes the construction of GL'16 to the torus case.

Main statement 2/2: Relating to the Hamiltonian

Theorem (F. Del Monte, H.D, P. Gavrylenko; 2020)

The logarithmic derivative of the Fredholm determinant gives back the Hamiltonian

$$2\pi i \partial_\tau \log \det \left[\mathbb{1} - K_{1,1} \right] = 2\pi i \partial_\tau \log \mathcal{T}_{CM} - (2\pi i)^2 a^2 - \frac{(2\pi i)^2}{6} + 2\pi i \frac{d}{d\tau} \log \left(\frac{\theta_1(Q-\rho)\theta_1(Q+\rho)}{\eta(\tau)^2} \right).$$

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The transcendent $Q(\tau)$ is then expressed in terms of the Fredholm determinant as

$$\frac{\theta_3(2Q(\tau)|2\tau)}{\theta_2(2Q(\tau)|2\tau)} = ie^{3i\pi\tau/2} \frac{\det\left(\mathbb{1} - K_{1,1}|_{\rho=\frac{1}{4}+\frac{\tau}{2}}\right)}{\det\left(\mathbb{1} - K_{1,1}|_{\rho=\frac{1}{4}}\right)}.$$

In this sense, the determinant is the true tau-function of the system.

Plan

Minor expansion

The minor expansion of the Fredholm determinant is a series labelled by two tuples of charged partitions $(\vec{\mathsf{Q}}, \vec{\mathsf{Y}})$

$$\det\left[\mathbb{1} - K_{1,1}\right] = \sum_{\vec{\mathsf{Q}}} \sum_{\vec{\mathsf{Y}} \in \mathbb{Y}^2} e^{2\pi i \tau \left[\frac{1}{2}\left(\vec{\mathsf{Q}} + \sigma_1\right)^2 - \frac{1}{2}\sigma_1^2 + |\vec{\mathsf{Y}}|\right] - 2\pi i \left(\rho - \frac{\tau}{2} - m\tau\right)\mathsf{Q}} \underbrace{\times (-1)^{|I_1|} \det \begin{pmatrix} (\mathsf{a})^{I_1}_{J_1} & (\mathsf{b})^{I_1}_{I_1} \\ (\mathsf{c})^{J_1}_{J_1} & (\mathsf{d})^{J_1}_{I_1} \end{pmatrix}}_{Z^{\vec{\mathsf{Y}},\vec{\mathsf{Q}}}_{\vec{\mathsf{Y}},\vec{\mathsf{Q}}}(\mathscr{T})}.$$

1. The 'charge' of the partition Q is topological in nature and comes into the picture due to the b-cycle monodromy.
2. $Z^{\vec{\mathsf{Y}},\vec{\mathsf{Q}}}_{\vec{\mathsf{Y}},\vec{\mathsf{Q}}}(\mathscr{T})$ is $c = 1$ conformal block $=$ *charged* Nekrasov-Okounkov function.

Combinatorial expression of the tau-function

Theorem (F.Del Monte, H.D, P. Gavrylenko; 2020)

$$\mathcal{T}_{CM}(\tau) = \frac{\left(\eta(\tau)e^{-i\pi\tau/12}\right)^{2(1-m^2)} e^{-2\pi i\left[\rho - \frac{\tau}{2}\left(m+\frac{1}{2}\right) - \frac{m}{2}\right]}}{\theta_1\left(Q(\tau)+\rho-\frac{m(\tau+1)}{2}\right)\theta_1\left(Q(\tau)-\rho+\frac{m(\tau+1)}{2}\right)}$$

$$\times \sum_{\vec{\mathsf{Q}}} \sum_{\vec{\mathsf{Y}}\in\mathbb{Y}^2} e^{2\pi i\tau\left[\frac{1}{2}(\vec{\mathsf{Q}}+\vec{a})^2+|\vec{\mathsf{Y}}|\right]} e^{2\pi i\left[\vec{\mathsf{Q}}\cdot\vec{\nu}-\mathsf{Q}\left(\rho-\frac{m(\tau+1)}{2}-\frac{\tau}{2}\right)\right]}$$

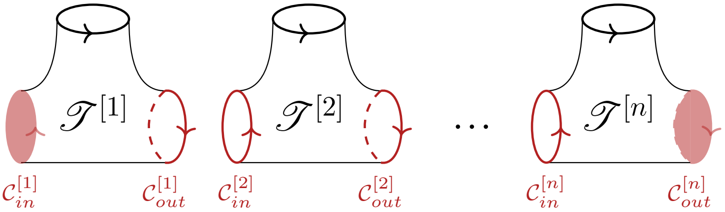
$$\times \frac{Z_{pert}\left(\vec{a}+\vec{\mathsf{Q}}, \vec{a}+\vec{\mathsf{Q}}+m\right)}{Z_{pert}\left(\vec{a}, \vec{a}+m\right)} Z_{inst}\left(\vec{a}+\vec{\mathsf{Q}}, \vec{a}+\vec{\mathsf{Q}}+m|\vec{\mathsf{Y}}, \vec{\mathsf{Y}}\right),$$

where Z_{pert}, Z_{inst} are combinatorial objects.

Plan

Torus with n punctures

The construction for the 1- point torus can be generalised to a torus with any number of simple poles



and the tau-function will be defined by an operator which is also explicitly described by hypergeometric functions.

- Isomonodromic tau functions on a torus as Fredholm determinants, and charged partitions. (*F. Del Monte, H.D, P. Gavrylenko* ***(2011.06292v2)***)

Kernel $K_{1,n}$

$$\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \nabla^{-1}\mathsf{c}^{[n]} & \nabla^{-1}\mathsf{d}^{[n]}\nabla \\
0 & U_1 & V_1 & & & & 0 \\
\mathsf{c}^{[1]} & & & & \mathbf{0} & & 0 \\
0 & W_1 & U_2 & V_2 & & & \\
0 & & W_2 & & & & \vdots \\
& & & & \ddots & & \\
\vdots & & & \ddots & & & \\
& & & \ddots & & & \\
& \mathbf{0} & & & V_{n-2} & & 0 \\
& & & W_{n-2} & U_{n-1} & & \mathsf{b}^{[n]}\nabla \\
0 & & & & & & 0 \\
\mathsf{a}^{[1]} & \mathsf{b}^{[1]} & 0 & 0 & \cdots & & 0
\end{pmatrix}$$

$$U_k = \begin{pmatrix} 0 & \mathsf{a}^{[k+1]} \\ \mathsf{d}^{[k]} & 0 \end{pmatrix}, \quad V_k = \begin{pmatrix} \mathsf{b}^{[k+1]} & 0 \\ 0 & 0 \end{pmatrix}, \quad W_k = \begin{pmatrix} 0 & 0 \\ 0 & \mathsf{c}^{[k+1]} \end{pmatrix}. \quad (2)$$

The operators $\mathsf{a}^{[k]}, \mathsf{b}^{[k]}, \mathsf{c}^{[k]}, \mathsf{d}^{[k]}$ are given by some hypergeometric functions and ∇ is the shift operator.

Structure of the kernel

$$
\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 & \nabla^{-1}\mathsf{c}^{[n]} & \nabla^{-1}\mathsf{d}^{[n]}\nabla \\
0 & U_1 & V_1 & & 0 & & 0 \\
\mathsf{c}^{[1]} & & & & & & 0 \\
0 & W_1 & U_2 & V_2 & & & \\
0 & & W_2 & & \ddots & & \vdots \\
\vdots & & & \ddots & & & \\
 & & 0 & & \ddots & V_{n-1} & 0 \\
 & & & & & & \mathsf{b}^{[n]}\nabla \\
0 & & & & W_{n-1} & U_n & 0 \\
\mathsf{a}^{[1]} & \mathsf{b}^{[1]} & 0 & 0 & \cdots & & 0
\end{pmatrix}
$$

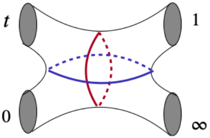
$\det[\mathbb{1} - \square]$ is the tau-function of Garnier system (sphere with $n+2$ punctures)!

Main message:

1. We are able to explicitly formulate the isomonodromic tau-function for genus 1 surfaces.
2. We rigorously establish the connection between isomonodromic deformations on a torus, $c = 1$ conformal blocks, and Nekrasov partition functions.

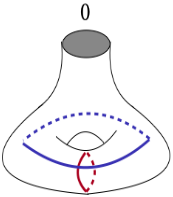
Plan

Pictorial representation of the connection constant



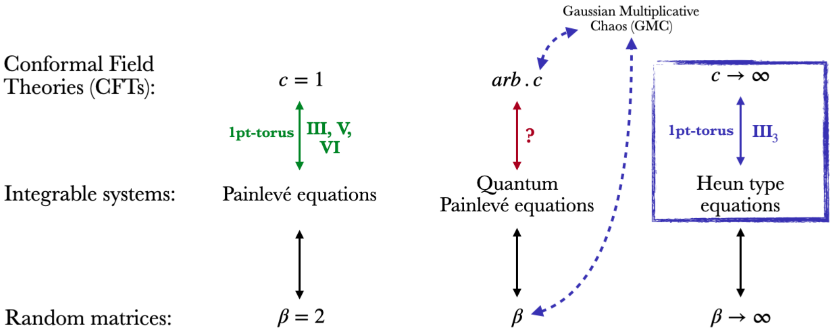
$$= \frac{\mathcal{T}(t \to 0)}{\mathcal{T}(t \to \infty)} = \text{connection constant for Painlevé VI}$$

Work in progress (with F. Del Monte, P. Gavrylenko)



$$= \frac{\mathcal{T}(\tau \to i\infty)}{\mathcal{T}(\tau \to 0)} = \text{connection constant for 1 pt torus}$$

More connection constant surprises



M. Bershtein, P. Gavrylenko, A. Grassi '20:

1. **Heun type equation** The scalar form of the ODE

$$\partial_z Y(z) Y(z)^{-1} = 2\pi i \frac{dQ(\tau)}{d\tau}\sigma_3 + m \frac{\theta_1'(0|\tau)}{\theta_1(z|\tau)} \begin{pmatrix} 0 & \frac{\theta_1(z+2Q(\tau)|\tau)}{\theta_1(-2Q(\tau)|\tau)} \\ \frac{\theta_1(z-2Q(\tau)|\tau)}{\theta_1(2Q(\tau)|\tau)} & 0 \end{pmatrix}$$

near the zeros of $Q(\tau_\star) = 0$ gives a Hamiltonian with a Weierstass potential, which is BPZ equation for a one point torus in the semiclassical limit!

M. Bershtein, P. Gavrylenko, A. Grassi '20:

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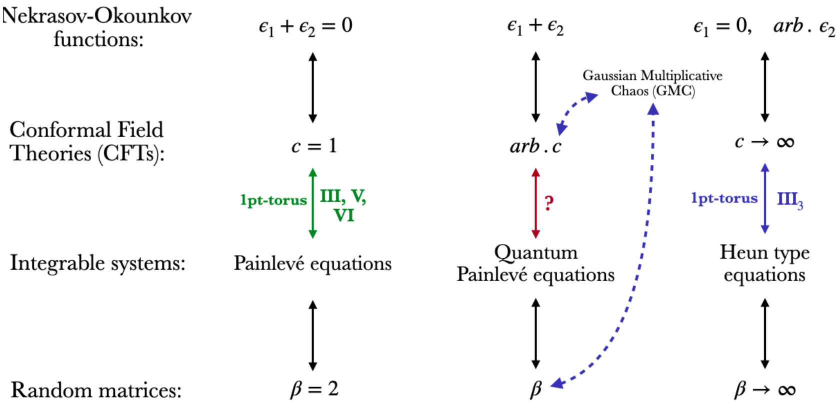
2. **Relation to CFT**

$$\boxed{\frac{\mathcal{T}(\tau\to\tau_\star)}{\mathcal{T}(\tau\to i\infty)} = c \to \infty \text{ conformal block for 1 pt torus}}$$

Work in progress (with P. Ghosal, A. Prokhorov)


Nekrasov-Okounkov functions:
$\epsilon_1 + \epsilon_2 = 0$
$\epsilon_1 + \epsilon_2$
$\epsilon_1 = 0, \quad arb.\ \epsilon_2$
Gaussian Multiplicative Chaos (GMC)
Conformal Field Theories (CFTs):
$c = 1$
$arb.\ c$
$c \to \infty$
1pt-torus
III, V, VI
?
1pt-torus
III_3
Integrable systems:
Painlevé equations
Quantum Painlevé equations
Heun type equations
Random matrices:
$\beta = 2$
β
$\beta \to \infty$