Deformations of Toeplitz Determinants: Applications, Asymptotics, and Orthogonality Structures.

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The $N \times N$ Toeplitz matrix associated to the symbol ϕ is defined as

$$T_N[\phi] = \{\phi_{j-k}\}_{j,k=0}^{N-1} = \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-N+1} \\ \phi_1 & \phi_0 & \cdots & \phi_{-N+2} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{N-1} & \phi_{N-2} & \cdots & \phi_0 \end{pmatrix},$$

where ϕ_k's are the Fourier coefficients of ϕ

$$\phi_k = \int_{\mathbb{T}} z^{-k} \phi(z) \frac{\mathrm{d}z}{2\pi \mathrm{i} z}.$$

Let

$$D_N[\phi] := \det T_N[\phi].$$

The large-N asymptotics of the Toeplitz determinants are well known and given by the Szegö-Widom theorem by

$$D_N[\phi] \sim G[\phi]^N E[\phi],$$

$$G[\phi] = \exp\left([\log \phi]_0\right) \quad \text{and} \quad E(\phi) = \exp\left(\sum_{n \geq 1} n [\log \phi]_n [\log \phi]_{-n}\right).$$

Let Q_n and $\widehat{Q}_n$ be respectively defined by

$$Q_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix},$$

and

$$\widehat{Q}_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+1} & 1 \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+2} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n & \phi_{n-1} & \cdots & \phi_1 & z^n \end{pmatrix},$$

$$Q_n(z) = \kappa_n z^n + \sum_{\ell=0}^{n-1} \kappa_\ell^{(n)} z^\ell, \qquad \text{and} \qquad \widehat{Q}_n(z) = \kappa_n z^n + \sum_{\ell=0}^{n-1} \widehat{\kappa}_\ell^{(n)} z^\ell,$$

where

$$\kappa_n = \sqrt{\frac{D_n[\phi]}{D_{n+1}[\phi]}}.$$

Let Q_n and $\widehat{Q}_n$ be respectively defined by

$$Q_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n} \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+1} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{n-1} & \phi_{n-2} & \cdots & \phi_{-1} \\ 1 & z & \cdots & z^n \end{pmatrix},$$

and

$$\widehat{Q}_n(z) := \frac{1}{\sqrt{D_n[\phi]D_{n+1}[\phi]}} \det \begin{pmatrix} \phi_0 & \phi_{-1} & \cdots & \phi_{-n+1} & 1 \\ \phi_1 & \phi_0 & \cdots & \phi_{-n+2} & z \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_n & \phi_{n-1} & \cdots & \phi_1 & z^n \end{pmatrix},$$

One can readily observe that $\{Q_n\}_{n=0}^{\infty}$ and $\{\widehat{Q}_n\}_{n=0}^{\infty}$ form the bi-orthogonal system of polynomials on the unit circle with respect to the weight ϕ :

$$\int_{\mathbb{T}} Q_n(z)\widehat{Q}_n(z^{-1})\phi(z)\frac{\mathrm{d}z}{2\pi \mathrm{i} z} = \delta_{nk}, \qquad n,k = 0,1,2,\cdots.$$

It is due to J.Baik, P.Deift and K.Johansson that the following matrix-valued function constructed out of the polynomials Q_n and $\widehat{Q}_n$

$$X(z;n) := \begin{pmatrix} \kappa_n^{-1} Q_n(z) & \kappa_n^{-1} \int_{\mathbb{T}} \frac{Q_n(\zeta)}{(\zeta - z)} \frac{\phi(\zeta)\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta^n} \\ -\kappa_{n-1} z^{n-1} \widehat{Q}_{n-1}(z^{-1}) & -\kappa_{n-1} \int_{\mathbb{T}} \frac{\widehat{Q}_{n-1}(\zeta^{-1})}{(\zeta - z)} \frac{\phi(\zeta)\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} \end{pmatrix},$$

satisfies the following Riemann-Hilbert problem for BOPUC, which in the subsequent parts of this text will occasionally be referred to as the X-RHP:

- **RH-X1** $X : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic,
- **RH-X2** The limits of $X(\zeta)$ as ζ tends to $z \in \mathbb{T}$ from the inside and outside of the unit circle exist, and are denoted $X_{\pm}(z)$ respectively and are related by

$$X_+(z) = X_-(z) \begin{pmatrix} 1 & z^{-n}\phi(z) \\ 0 & 1 \end{pmatrix}, \qquad z \in \mathbb{T},$$

- **RH-X3** As $z \to \infty$

$$X(z) = \left(I + \mathcal{O}(z^{-1})\right) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}.$$

Let us first recall the two-dimensional Ising model,solved by Onsager. In this model a $2\mathcal{M}\times 2\mathcal{N}$ rectangular lattice is considered with an associated spin variable σ_{jk} taking the values 1 and -1 at each vertex (j,k), $-\mathcal{M}\leq j\leq \mathcal{M}-1$, $-\mathcal{N}\leq k\leq \mathcal{N}-1$. There are $2^{4\mathcal{M}\mathcal{N}}$ possible spin configurations $\{\sigma\}$ of the lattice (a configuration corresponds to values of all σ_{jk} fixed). By J_h and J_v we respectively denote the horizontal and vertical nearest neighbor coupling constants and with each configuration we associate its nearest-neighbor coupling energy given by

$$E(\{\sigma\}) = -\sum_{j=-\mathcal{M}}^{\mathcal{M}-1}\sum_{k=-\mathcal{N}}^{\mathcal{N}-1}\left(J_h\sigma_{j,k}\sigma_{j,k+1} + J_v\sigma_{j,k}\sigma_{j+1,k}\right), \qquad J_h, J_v > 0.$$

The probability of a spin configuration $\{\sigma\}$ is given by

$$P_{\{\sigma\}} = \frac{1}{Z(T)}\exp\left(-\frac{E(\{\sigma\})}{k_B T}\right),$$

where k_B is the Boltzmann's constant and $Z(T)$ denotes the partition function and is naturally defined as

$$Z(T) = \sum_{\{\sigma\}}\exp\left(-\frac{E(\{\sigma\})}{k_B T}\right).$$

The spin-spin correlation function between the vertices (m', n') and (m, n) is defined as the following *thermodynamic limit*

$$\langle \sigma_{m',n'}\sigma_{m,n}\rangle = \lim_{\mathcal{M},\mathcal{N}\to\infty} \frac{1}{Z(T)} \sum_{\{\sigma\}} \sigma_{m',n'}\sigma_{m,n} \exp\left(-\frac{E(\{\sigma\})}{k_B T}\right).$$

The quantity $\lim_{m^2+n^2\to\infty} \langle \sigma_{0,0}\sigma_{m,n}\rangle$ is referred to as the long-range order in the lattice at a temperature T. Indeed, the spontaneous magnetization M is defined as square of the large-n limit of *diagonal* correlations

$$M := \sqrt{\lim_{n\to\infty} \langle \sigma_{0,0}\sigma_{n,n}\rangle}.$$

Let us introduce the notations,

$$S_h = \sinh\left(\frac{2J_h}{k_B T}\right), \quad S_v = \sinh\left(\frac{2J_v}{k_B T}\right),$$

$$C_h = \cosh\left(\frac{2J_h}{k_B T}\right), \quad C_v = \cosh\left(\frac{2J_v}{k_B T}\right),$$

and

$$k = S_h S_v.$$

It is famously known that, unlike the one-dimensional case, the two-dimensional Ising model exhibits a phase transition in the spontaneous magnetization at some temperature T_c, characterized by

$$k = 1.$$

In this talk I will focus on

$$k > 1,$$

which corresponds to the low temperature regime $T < T_c$.

For the diagonal correlations $\langle \sigma_{0,0}\sigma_{N,N} \rangle$ and the horizontal correlations $\langle \sigma_{0,0}\sigma_{0,N} \rangle$, one has Toeplitz determinant representations. Indeed, for the diagonal correlations we have

$$\langle \sigma_{0,0}\sigma_{N,N} \rangle = D_N[\widehat{\phi}\,], \qquad \widehat{\phi}(z) = \sqrt{\frac{1-k^{-1}z^{-1}}{1-k^{-1}z}},$$

and for the horizontal correlations

$$\langle \sigma_{0,0}\sigma_{0,N} \rangle = D_N[\widehat{\eta}\,], \qquad \widehat{\eta}(z) = \sqrt{\frac{(1-\alpha_1 z)(1-\alpha_2 z^{-1})}{(1-\alpha_1 z^{-1})(1-\alpha_2 z)}},$$

where α_1 and α_2 are given by

$$\alpha_1 = \frac{z_h(1-z_v)}{1+z_v}, \quad \alpha_2 = \frac{1-z_v}{z_h(1+z_v)}, \qquad z_{h,v} = \tanh \frac{J_{h,v}}{k_B T}.$$

In the low temperature regime, the symbols $\widehat{\phi}$ and $\widehat{\eta}$ enjoy the regularity properties required by the strong Szegő limit theorem and the diagonal and horizontal long-range orders

$$M_D := \sqrt{\lim_{N\to\infty} \langle \sigma_{0,0}\sigma_{N,N}\rangle} \quad \text{and} \quad M_H := \sqrt{\lim_{N\to\infty} \langle \sigma_{0,0}\sigma_{0,N}\rangle},$$

both evaluate to

$$(1-k^{-2})^{1/8}.$$

In an interesting development, It was shown by Au-Yang and Perk in 1987 that the next-to-diagonal two point correlation function is given by the following *bordered Toeplitz determinant*,

$$\langle \sigma_{0,0}\sigma_{N-1,N}\rangle = D_N^B[\widehat{\phi}, \widehat{\psi}],$$

where $\widehat{\phi}$ is the symbol for diagonal correlations, and

$$\widehat{\psi}(z) = \frac{C_v z\widehat{\phi}(z) + C_h}{S_v(z-c_*)}, \qquad \text{with} \qquad c_* = -\frac{S_h}{S_v}.$$

The bordered Toeplitz determinant, $D_N^B[\phi;\psi]$, is defined as

$$D_N^B[\phi;\psi] := \det \begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{N-2} & \psi_{N-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{N-3} & \psi_{N-2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \phi_{2-N} & \phi_{3-N} & \cdots & \phi_0 & \psi_1 \\ \phi_{1-N} & \phi_{2-N} & \cdots & \phi_{-1} & \psi_0 \end{pmatrix}, \qquad N > 1.$$

We have the following elementary properties of bordered Toeplitz determinants:

$$D_N^B\left[\phi, \sum_{j=1}^{m} a_j \psi_j\right] = \sum_{j=1}^{m} a_j D_N^B[\phi, \psi_j],$$

$$D_N^B[\phi, \phi] = D_N[\phi],$$

and

$$D_N^B[\phi, 1] = D_{N-1}[\phi].$$

Now, let us recall the Szegő function of the symbol ϕ:

$$\alpha(z) := \exp\left[\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\ln(\phi(\tau))}{\tau - z} d\tau\right].$$

Theorem 1. Let $D_N^B[\phi;\psi]$ be the bordered Toeplitz determinant with $\psi = q_1\phi + q_2$, where

$$q_1(z) = a_0 + a_1 z + \frac{b_0}{z} + \sum_{j=1}^{m} \frac{b_j z}{z - c_j}, \quad \text{and} \quad q_2(z) = \hat{a}_0 + \hat{a}_1 z + \frac{\hat{b}_0}{z} + \sum_{j=1}^{m} \frac{\hat{b}_j}{z - c_j},$$

where all parameters are complex and the c_j are nonzero and do not lie on the unit circle, and ϕ of Szegő type. Then, as $N \to \infty$

$$D_N^B[\phi,\psi] = G[\phi]^N E[\phi] \left(F[\phi;\psi] + \mathcal{O}(e^{-\mathfrak{c}N}) \right),$$

where $\mathfrak{c} > 0$,

$$G[\phi] = \exp\left([\log\phi]_0\right) \quad \text{and} \quad E(\phi) = \exp\left(\sum_{n\geq 1} n[\log\phi]_n [\log\phi]_{-n} \right),$$

and $F[\phi;\psi]$ is given by

$$F[\phi;\psi] = a_0 + b_0[\log\phi]_1 + \sum_{\substack{j=1 \\ 0<|c_j|<1}}^{m} b_j \frac{\alpha(c_j)}{\alpha(0)} + \frac{1}{\alpha(0)} \left(\hat{a}_0 - \hat{a}_1[\log\phi]_{-1} - \sum_{\substack{j=1 \\ |c_j|>1}}^{m} \frac{\hat{b}_j}{c_j}\alpha(c_j) \right).$$

Theorem 2. Let $\langle \sigma_{0,0}\sigma_{N-1,N}\rangle$ be the next-to-diagonal two point correlation function in the Ising model. Then, in the low-temperature regime, the long-range order in the next-to-diagonal direction for the anisotropic square lattice Ising model is the same as of the diagonal and horizontal ones, i.e. is described as follows

$$\lim_{N\to\infty}\langle \sigma_{0,0}\sigma_{N-1,N}\rangle = (1-k^{-2})^{1/4}.$$

Theorem 3. The next-to-diagonal two point correlation function has, in the low-temperature regime $k>1$, the $N\to\infty$ asymptotics

$$\langle \sigma_{0,0}\sigma_{N-1,N}\rangle = (1-k^{-2})^{1/4}\left(1+\frac{1}{2\pi(1-k^{-2})}\left(\frac{1}{C_v^2}+\frac{1}{k^2-1}\right)N^{-2}k^{-2N}\left(1+O(N^{-1})\right)\right)$$

For comparison, asymptotics of the diagonal correlation function is given by

$$\langle \sigma_{0,0}\sigma_{N,N}\rangle = (1-k^{-2})^{1/4}\left(1+\frac{1}{2\pi(1-k^{-2})^2k^2}N^{-2}k^{-2N}\left(1+O(N^{-1})\right)\right),$$

as $N\to\infty$.

Theorem 4. Suppose that $\psi(z)$ admits an analytic continuation in a neighborhood of the unite circle and let ϕ be of Szegő type. Then

$$D_N^B[\phi,\psi] = G[\phi]^N E[\phi]\left(F[\phi;\psi] + \mathcal{O}(e^{-\mathfrak{c}N})\right),$$

where

$$G[\phi] = \exp\left([\log\phi]_0\right) \quad \text{and} \quad E(\phi) = \exp\left(\sum_{n\geq 1} n[\log\phi]_n[\log\phi]_{-n}\right),$$

and $F[\phi;\psi]$ is given by

$$F[\phi;\psi] = \frac{[\alpha_-\psi]_0}{\alpha(0)} \equiv \frac{1}{\alpha(0)}\int_{\mathbb{T}} \alpha_-(w)\psi(w)\frac{\mathrm{d}w}{2\pi \mathrm{i} w},$$

and $\mathfrak{c}$ is some positive constant.

The bordered Toeplitz determinants $D^B_{n+1}[\phi, \dfrac{1}{z-c}]$, and $D^B_{n+1}[\phi, \dfrac{\phi}{z-c}]$ are encoded into X-RHP data described by

$$D^B_{n+1}[\phi, \frac{1}{z-c}] = \begin{cases} 0, & |c|<1, \\ -c^{-n-1}D_n[\phi]X_{11}(c;n), & |c|>1, \end{cases}$$

and

$$D^B_{n+1}[\phi, \frac{\phi}{z-c}] = -\frac{1}{c}D_{n+1}[\phi] + \frac{1}{c}D_n[\phi]X_{12}(c,n), \qquad c \neq 0,$$

$$D^B_n[\phi, \frac{\phi}{z}] = -D^B_{n+1}[\tilde{\phi}, z],$$

$$D^B_{n+1}[\phi, z] = D_n[\phi]\lim_{z\to\infty}\left(\frac{X_{11}(z;n)-z^n}{z^{n-1}}\right) \equiv D_n[\phi]\frac{\kappa^{(n)}_{n-1}}{\kappa_n},$$

$$D^B_{n+1}[\phi, z^2] = D_n[\phi]\lim_{z\to\infty}\left(\frac{X_{11}(z;n)-z^n-\frac{\kappa^{(n)}_{n-1}}{\kappa_n}z^{n-1}}{z^{n-2}}\right) \equiv D_n[\phi]\frac{\kappa^{(n)}_{n-2}}{\kappa_n},$$

and so on. Note that

$$D^B_{n+1}[\phi, z^k] = 0, \qquad k \in \mathbb{Z}\setminus\{0,1,\cdots n\}.$$

Let $\vec{x} = (x_0, x_1, \cdots, x_{N-1})^T$ and $\vec{\psi} = (\psi_{N-1}, \psi_{N-2}, \cdots, \psi_0)^T$. Applying the Cramer's rule to the linear system $T_n[\tilde{\phi}]\vec{x} = \vec{\psi}$ gives

$$x_{N-1} = \frac{1}{D_N[\tilde{\phi}]} \det \begin{pmatrix} \tilde{\phi}_0 & \tilde{\phi}_{-1} & \cdots & \tilde{\phi}_{-N+2} & \psi_{N-1} \\ \tilde{\phi}_1 & \tilde{\phi}_0 & \cdots & \tilde{\phi}_{-N+3} & \psi_{N-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \tilde{\phi}_{N-1} & \tilde{\phi}_{N-2} & \cdots & \tilde{\phi}_1 & \psi_0 \end{pmatrix}$$

$$= \frac{1}{D_N[\phi]} \det \begin{pmatrix} \phi_0 & \phi_1 & \cdots & \phi_{N-2} & \psi_{N-1} \\ \phi_{-1} & \phi_0 & \cdots & \phi_{N-3} & \psi_{N-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ \phi_{1-N} & \phi_{2-N} & \cdots & \phi_{-1} & \psi_0 \end{pmatrix}.$$

Therefore

$$D_N^B[\phi;\psi] = D_N[\phi]x_{N-1} = D_N[\phi] \sum_{\ell=0}^{N-1} \left(T_N^{-1}[\tilde{\phi}]\right)_{N-1,\ell} \psi_{N-1-\ell}$$

$$\left(T_n^{-1}[\phi]\right)_{j,k} = \delta_{jk} + \langle \boldsymbol{\mathcal{R}_n^{(\phi)}}[z^k], z^j\rangle, \qquad 0 \le j,k \le n-1,$$

where δ_{jk} is the Kronecker delta function,

$$\langle f(z), g(z)\rangle = \int_{\mathbb{T}} f(z)\overline{g(z)}\frac{\mathrm{d}z}{2\pi \mathrm{i} z},$$

and

$$\boldsymbol{\mathcal{R}_n^{(\phi)}} : f(z) \mapsto \int_{\mathbb{T}} \mathcal{R}_n^{(\phi)}(z,w) f(w)\mathrm{d}w$$

is the *Resolvent* operator with the kernel

$$\mathcal{R}_n^{(\phi)}(z,w) = \frac{X_{11}^{(\phi)}(z)X_{21}^{(\phi)}(w) - X_{11}^{(\phi)}(w)X_{21}^{(\phi)}(z)}{z-w}\,\frac{\phi(w)-1}{2\pi \mathrm{i} w^n},$$

Using this approach we can show

$$D_N^B[\phi,\psi] = G[\phi]^N E[\phi] F[\phi;\psi]\left(1 + \mathcal{O}(e^{-\mathfrak{c}N})\right), \qquad \mathfrak{c} > 0.$$

where $F[\phi;\psi]$ is given by

$$F[\phi;\psi] = \frac{[\alpha_-\psi]_0}{\alpha(0)} \equiv \frac{1}{\alpha(0)}\int_{\mathbb{T}} \alpha_-(w)\psi(w)\frac{\mathrm{d}w}{2\pi \mathrm{i} w}.$$

Let $G(2N)$ represent one of $USp(2N), SO(2N)$, or $O^{-}(2N)$. For $A \in G(2N)$, let $\Lambda_A(z) = \det(I - Az)$ denote its characteristic polynomial and consider the k-th moment of $\Lambda_A^{(m)}(1)$:

$$M_k(G(2N), m) := \int_{G(2N)} \left(\Lambda_A^{(m)}(1)\right)^k \mathrm{d}A,$$

where $\mathrm{d}A$ is the Haar measure on $G(2N)$. In

S. A. Altug, S. Bettin, I. Petrow, Rishikesh, and I. Whitehead. A recursion formula for moments of derivatives of random matrix polynomials.

the authors study large-N asymptotics of $M_n(G(2N), m)$. Here is where the $2j-k$ determinants come into the picture. Let

$$\mathcal{T}_{k,\ell}(u) = \det_{k\times k}\left(f_{2j-k+\ell}(u)\right)$$

$$f(z;u) = \exp\left(z + \frac{u}{z^2}\right)$$

Then

$$M_k(USp(2N),2) = b_k(USp(2N),2)\cdot(2N)^{\frac{k^2+5k}{2}} + O(N^{\frac{k^2+3k}{2}}),$$

where

$$b_k(USp(2N),2) = 2^{-\frac{k^2+5k}{2}}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\left(e^u\mathcal{T}_{k,0}(2u)\right)\Big|_{u=0}.$$

$$M_k(SO(2N),2) = b_k(SO(2N),2)\cdot(2N)^{\frac{k^2+3k}{2}} + O(N^{\frac{k^2+k}{2}}),$$

where

$$b_k(SO(2N),2) = 2^{-\frac{k^2+k}{2}}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\left(e^u\mathcal{T}_{k,-1}(2u)\right)\Big|_{u=0}.$$

$$M_k(O^-(2N),3) = b_k(O^-(2N),3)\cdot(2N)^{\frac{k^2+5k}{2}} + O(N^{\frac{k^2+3k}{2}}),$$

where

$$b_k(O^-(2N),3) = 3\cdot 2^{-\frac{k^2+3k}{2}}\frac{\mathrm{d}^k}{\mathrm{d}u^k}\left(e^u\mathcal{T}_{k,0}(2u)\right)\Big|_{u=0}.$$

For a general symbol w we consider the $2j-k$ and $j-2k$ determinants with offsets $r, s \in \mathbb{Z}$:

$$
\begin{aligned}
D_n^{(r)} &:= \det_{0\le j,k\le n-1}\left(w_{r+2j-k}\right) \\
&= \det\begin{pmatrix} w_r & w_{r-1} & w_{r-2} & \cdots & w_{r-n+1} \\ w_{r+2} & w_{r+1} & w_r & \cdots & w_{r-n+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{r+2n-2} & w_{r+2n-3} & w_{r+2n-4} & \cdots & w_{r+n-1} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
E_n^{(s)} &:= \det_{0\le j,k\le n-1}\left(w_{s+j-2k}\right) \\
&= \det\begin{pmatrix} w_s & w_{s-2} & w_{s-4} & \cdots & w_{s-2n+2} \\ w_{s+1} & w_{s-1} & w_{s-3} & \cdots & w_{s-2n+3} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{s+n-1} & w_{s+n-3} & w_{s+n-5} & \cdots & w_{s-n+1} \end{pmatrix}.
\end{aligned}
$$

$$\mathcal{D}_n[f(\zeta)] := \frac{1}{n!}\int_{\mathbb{T}}\frac{\mathrm{d}\zeta_1}{2\pi\mathrm{i}\zeta_1}\int_{\mathbb{T}}\frac{\mathrm{d}\zeta_2}{2\pi\mathrm{i}\zeta_2}\cdots\int_{\mathbb{T}}\frac{\mathrm{d}\zeta_n}{2\pi\mathrm{i}\zeta_n}\prod_{j=1}^{n}f(\zeta_j)\prod_{1\le j<k\le n}(\zeta_k-\zeta_j)(\zeta_k^{-2}-\zeta_j^{-2}),$$

and

$$\mathcal{E}_n[f(\zeta)] := \frac{1}{n!}\int_{\mathbb{T}}\frac{\mathrm{d}\zeta_1}{2\pi\mathrm{i}\zeta_1}\int_{\mathbb{T}}\frac{\mathrm{d}\zeta_2}{2\pi\mathrm{i}\zeta_2}\cdots\int_{\mathbb{T}}\frac{\mathrm{d}\zeta_n}{2\pi\mathrm{i}\zeta_n}\prod_{j=1}^{n}f(\zeta_j)\prod_{1\le j<k\le n}(\zeta_k^2-\zeta_j^2)(\zeta_k^{-1}-\zeta_j^{-1}).$$

$$\mathcal{D}_n[w(\zeta)\zeta^{-r}] = D_n^{(r)}, \quad \text{and} \quad \mathcal{E}_n[w(\zeta)\zeta^{-s}] = E_n^{(s)}.$$

or each offset value $r \in \mathbb{Z}$, let us consider two sequences of *monic* polynomials $\{P_n(z;r)\}_{n=0}^{\infty}$ and $\{Q_n(z;r)\}_{n=0}^{\infty}$, $\deg P_n(z;r) = \deg Q_n(z;r) = n$, with the *biorthogonality* condition:

$$\int_{\mathbb{T}} P_m(\zeta;r) Q_n(\zeta^{-2};r) \zeta^{-r} \frac{\mathrm{d}\mu(\zeta)}{2\pi \mathrm{i} \zeta} = h_n^{(r)} \delta_{mn},$$

Similarly, for each offset value $s \in \mathbb{Z}$, we consider two sequences of monic polynomials $\{R_n(z;s)\}_{n=0}^{\infty}$ and $\{S_n(z;s)\}_{n=0}^{\infty}$, $\deg R_n(z;s) = \deg S_n(z;s) = n$, with the following biorthogonality condition:

$$\int_{\mathbb{T}} R_m(\zeta^2;s) S_n(\zeta^{-1};s) \zeta^{-s} \frac{\mathrm{d}\mu(\zeta)}{2\pi \mathrm{i} \zeta} = g_n^{(s)} \delta_{mn},$$

Theorem

If $D_n^{(r)} \neq 0$, the polynomials $P_n(z;r)$ and $Q_n(z;r)$ exist and are uniquely given by

$$P_n(z;r) = \frac{1}{D_n^{(r)}} \det \begin{pmatrix} w_r & w_{r-1} & w_{r-2} & \cdots & w_{r-n} \\ w_{r+2} & w_{r+1} & w_r & \cdots & w_{r-n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{r+2n-2} & w_{r+2n-3} & w_{r+2n-4} & \cdots & w_{r+n-2} \\ 1 & z & z^2 & \cdots & z^n \end{pmatrix},$$

and

$$Q_n(z;r) = \frac{1}{D_n^{(r)}} \det \begin{pmatrix} w_r & w_{r-1} & w_{r-2} & \cdots & w_{r-n+1} & 1 \\ w_{r+2} & w_{r+1} & w_r & \cdots & w_{r-n+3} & z \\ w_{r+4} & w_{r+3} & w_{r+2} & \cdots & w_{r-n+5} & z^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{r+2n} & w_{r+2n-1} & w_{r+2n-2} & \cdots & w_{r+n+1} & z^n \end{pmatrix},$$

from which one can observe that $h_n^{(s)}$ exists and can be written as

$$h_n^{(r)} = \frac{D_{n+1}^{(r)}}{D_n^{(r)}}, \qquad n \in \mathbb{N} \cup \{0\}, \qquad D_0^{(r)} \equiv 1.$$

Theorem

If $E_n^{(s)} \neq 0$, the polynomials $R_n(z;s)$ and $S_n(z;s)$ exist and are uniquely given by

$$R_n(z;s) = \frac{1}{E_n^{(s)}} \det \begin{pmatrix} w_s & w_{s-2} & w_{s-4} & \cdots & w_{s-2n} \\ w_{s+1} & w_{s-1} & w_{s-3} & \cdots & w_{s-2n+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ w_{s+n-1} & w_{s+n-3} & w_{s+n-5} & \cdots & w_{s-n-1} \\ 1 & z & z^2 & \cdots & z^n \end{pmatrix},$$

and

$$S_n(z;s) = \frac{1}{E_n^{(s)}} \det \begin{pmatrix} w_s & w_{s-2} & w_{s-4} & \cdots & w_{s-2n+2} & 1 \\ w_{s+1} & w_{s-1} & w_{s-3} & \cdots & w_{s-2n+3} & z \\ w_{s+2} & w_s & w_{s-2} & \cdots & w_{s-2n+4} & z^2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ w_{s+n} & w_{s+n-2} & w_{s+n-4} & \cdots & w_{s-n+2} & z^n \end{pmatrix},$$

from which one can observe that $g_n^{(s)}$ exists and can be written as

$$g_n^{(s)} = \frac{E_{n+1}^{(s)}}{E_n^{(s)}}, \qquad n \in \mathbb{N} \cup \{0\}, \qquad E_0^{(s)} \equiv 1.$$

Theorem

The pure-degree recurrence relations for the $2j-k$ polynomials are given by

$$P_{n+3}(z;r)-(\delta_{n+2}^{(r)}+\delta_{n+1}^{(r-1)})P_{n+2}(z;r)+$$
$$(\delta_{n+1}^{(r-1)}\delta_{n+1}^{(r)}-z^2)P_{n+1}(z;r)+(\delta_n^{(r)}+\eta_n^{(r-2)})z^2P_n(z;r)=0$$

and

$$Q^*_{n+3}(z;r)-(1+\beta_{n+2}^{(r)}z)Q^*_{n+2}(z;r)+(\beta_{n+1}^{(r)}+\alpha_{n+1}^{(r+1)}+\beta_{n+1}^{(r+1)}+\alpha_{n+1}^{(r+2)})zQ^*_{n+1}(z;r)$$
$$-(\beta_{n+1}^{(r+1)}+\alpha_{n+1}^{(r+2)})(\beta_n^{(r)}+\alpha_n^{(r+1)})z^2Q^*_n(z;r)=0$$

where

$$\delta_n^{(r)}=-\frac{h_n^{(r-1)}}{h_n^{(r)}},\qquad \eta_n^{(r)}=\frac{D_n^{(r+2)}D_{n+1}^{(r-1)}}{D_{n+1}^{(r)}D_n^{(r+1)}}$$

$$\beta_n^{(r)}=-\frac{h_n^{(r+2)}}{h_n^{(r)}},\qquad \alpha_n^{(r)}=\frac{D_n^{(r-1)}D_{n+1}^{(r+2)}}{D_{n+1}^{(r)}D_n^{(r+1)}}.$$

Theorem

The pure-offset recurrence relations for the $2j-k$ and $j-2k$ polynomials are given by

$$\eta_n^{(r+1)} P_n(z;r+3) - zP_n(z;r+2) + \delta_n^{(r+1)} P_n(z;r+1) + zP_n(z;r) = 0,$$

$$Q_n^*(z;r+3) - Q_n^*(z;r+2) + \beta_n^{(r+1)} zQ_n^*(z;r+1) + \alpha_n^{(r+1)} zQ_n^*(z;r) = 0,$$

$$\rho_n^{(s+2)} R_n(z;s+3) + \varkappa_n^{(s+2)} R_n(z;s+2) - zR_n(z;s+1) + zR_n(z;s) = 0,$$

and

$$S_n^*(z;s+3) + \gamma_n^{(s+2)} zS_n^*(z;s+2) - S_n^*(z;s+1) + \theta_n^{(s+2)} zS_n^*(z;s) = 0.$$

It holds that

$$Q_n(z^2;r) = \frac{D_n^{(r+2)}}{D_n^{(r)}} \frac{z^{2n+1}}{2} \left[P_{n+1}(z^{-1};r+2)P_n(-z^{-1};r+2) - P_{n+1}(-z^{-1};r+2)P_n(z^{-1};r+2) \right],$$

$$R_n(z^2;s) = \frac{E_n^{(s-2)}}{E_n^{(s)}} \frac{z^{2n+1}}{2} \left[S_{n+1}(z^{-1};s-2)S_n(-z^{-1};s-2) - S_{n+1}(-z^{-1};s-2)S_n(z^{-1};s-2) \right].$$

Compare with the Toeplitz $(j-k)$ Case:

$$\widehat{\phi}_{n+1}(z) = \frac{z^{n+1}}{\phi_{n+1}(0)} \left[\kappa_{n+1}\phi_{n+1}(z^{-1}) - \kappa_n z^{-1}\phi_n(z^{-1}) \right]$$

$$\phi_{n+1}(z) = \frac{z^{n+1}}{\widehat{\phi}_{n+1}(0)} \left[\kappa_{n+1}\widehat{\phi}_{n+1}(z^{-1}) - \kappa_n z^{-1}\widehat{\phi}_n(z^{-1}) \right]$$

Theorem

We have the following multiple integral representations for $2j-k$ and $j-2k$ biorthogonal polynomials:

$$P_n(z;r) = \frac{1}{D_n^{(r)}}\mathcal{D}_n[w(\zeta)\zeta^{-r}(z-\zeta)],$$

$$Q_n(z;r) = \frac{1}{D_n^{(r)}}\mathcal{D}_n[w(\zeta)\zeta^{-r}(z-\zeta^{-2})],$$

$$R_n(z;s) = \frac{1}{E_n^{(s)}}\mathcal{E}_n[w(\zeta)\zeta^{-s}(z-\zeta^{2})],$$

$$S_n(z;s) = \frac{1}{E_n^{(s)}}\mathcal{E}_n[w(\zeta)\zeta^{-s}(z-\zeta^{-1})].$$

Define the *reproducing kernel* for the $2j-k$ and $j-2k$ systems respectively as

$$K_n(z,\tau;r) := \sum_{j=0}^{n} \frac{1}{h_j^{(r)}} Q_j(z;r) P_j(\tau;r),$$

and

$$L_n(z,\tau;s) := \sum_{j=0}^{n} \frac{1}{g_j^{(s)}} S_j(z;s) R_j(\tau;s).$$

It is easy to see that the following *reproducing properties* hold:

$$\int_{\mathbb{T}} K_n(z,\zeta;r) Q_\ell(\zeta^{-2};r)\zeta^{-r} w(\zeta) \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} = \begin{cases} Q_\ell(z;r), & 0 \le \ell \le n, \\ 0, & \ell > n, \end{cases}$$

$$\int_{\mathbb{T}} K_n(\zeta^{-2},\tau;r) P_\ell(\zeta;r)\zeta^{-r} w(\zeta) \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} = \begin{cases} P_\ell(\tau;r), & 0 \le \ell \le n, \\ 0, & \ell > n, \end{cases}$$

$$\int_{\mathbb{T}} L_n(z,\zeta^2;s) S_\ell(\zeta^{-1};s)\zeta^{-s} w(\zeta) \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} = \begin{cases} S_\ell(z;s), & 0 \le \ell \le n, \\ 0, & \ell > n, \end{cases}$$

$$\int_{\mathbb{T}} L_n(\zeta^{-1},\tau;r) R_\ell(\zeta^2;s)\zeta^{-s} w(\zeta) \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}\zeta} = \begin{cases} R_\ell(\tau;s), & 0 \le \ell \le n, \\ 0, & \ell > n. \end{cases}$$

Define the *reproducing kernel* for the $2j-k$ and $j-2k$ systems respectively as

$$K_n(z,\tau;r) := \sum_{j=0}^{n} \frac{1}{h_j^{(r)}} Q_j(z;r) P_j(\tau;r),$$

and

$$L_n(z,\tau;s) := \sum_{j=0}^{n} \frac{1}{g_j^{(s)}} S_j(z;s) R_j(\tau;s).$$

$$K_n(z_2,z_1;r) = \frac{1}{D_{n+1}^{(r)}} \mathcal{D}_n[w(\zeta)\zeta^{-r}(z_1-\zeta)(z_2-\zeta^{-2})],$$

$$L_n(z_2,z_1;s) = \frac{1}{E_{n+1}^{(s)}} \mathcal{E}_n[w(\zeta)\zeta^{-s}(z_1-\zeta^2)(z_2-\zeta^{-1})].$$

$$K_n(z_2^2, z_1; r) = \frac{1}{2}\frac{D_n^{(r+2)}}{D_{n+1}^{(r)}}\frac{z_2^{2n+1}}{z_1^2 - z_2^{-2}} \times$$
$$\det\begin{pmatrix} P_n(-z_2^{-1}; r+2) & P_{n+1}(-z_2^{-1}; r+2) & P_{n+2}(-z_2^{-1}; r+2) \\ P_n(z_2^{-1}; r+2) & P_{n+1}(z_2^{-1}; r+2) & P_{n+2}(z_2^{-1}; r+2) \\ P_n(z_1; r+2) & P_{n+1}(z_1; r+2) & P_{n+2}(z_1; r+2) \end{pmatrix},$$

and

$$L_n(z_2, z_1^2; s) = \frac{1}{2}\frac{E_n^{(s-2)}}{E_{n+1}^{(s)}}\frac{z_1^{2n+1}}{z_1^{-2} - z_2^{2}} \times$$
$$\det\begin{pmatrix} S_n(z_1^{-1}; s-2) & S_{n+1}(z_1^{-1}; s-2) & S_{n+2}(z_1^{-1}; s-2) \\ S_n(-z_1^{-1}; s-2) & S_{n+1}(-z_1^{-1}; s-2) & S_{n+2}(-z_1^{-1}; s-2) \\ S_n(z_2; s-2) & S_{n+1}(z_2; s-2) & S_{n+2}(z_2; s-2) \end{pmatrix}.$$

Compare with the Christoffel-Darboux identity for Toeplitz $(j-k)$ case

$$\widehat{K}_n(z_1, z_2) = -\frac{z_1^{n+1}\phi_{n+1}(z_1^{-1})z_2^{n+1}\widehat{\phi}_{n+1}(z_2^{-1}) - \widehat{\phi}_{n+1}(z_1)\phi_{n+1}(z_2)}{1 - z_1 z_2}.$$

$$D_n^{(r)} = \frac{(2\pi)^{\frac{n}{2}} G(\frac{r+1}{2}) G(\frac{r+2}{2}) G(n+1)}{2^{\frac{n}{2}(2r+n)} G(n+\frac{r+1}{2}) G(n+\frac{r+2}{2})}$$

$$h_n^{(r)} = \frac{2^n n!}{(2n+r)!}.$$

$$\delta_n^{(r)} = -2n - r, \qquad \eta_n^{(r)} = r$$

$$\beta_n^{(r)} = -\frac{1}{(2n+r+1)(2n+r+2)} \qquad \alpha_n^{(r)} = \frac{r}{(2n+r)(2n+r+1)(2n+r+2)}.$$

$$P_n(z;r) = 2^n \sum_{\ell=0}^{n} \frac{1}{\ell!} \sum_{m=0}^{\ell} (-1)^{m+n+\ell} \binom{\ell}{m} \left(\frac{r-m}{2}\right)_n z^\ell,$$

and

$$Q_n(z;r) = \frac{(-1)^n}{4^n \left(\frac{r+1}{2}\right)_n \left(\frac{r+2}{2}\right)_n} {}_3F_0(-n, \frac{r+1}{2}, \frac{r+2}{2};;4z),$$

Thank you!