KPZ and Boltzmann-Gibbs Principle

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Stochastic Burgers Equation SBE($\bar{\mathfrak{d}}$)

$$\partial_t \mathbf{u} \;=\; \Delta \mathbf{u} + \bar{\mathfrak{d}} \partial_x \mathbf{u} + \partial_X (\mathbf{u})^2 + \partial_x \xi \quad \text{on} \quad \mathbb{R}_{\geq 0} \times \mathbb{T}^1$$

- $\mathbf{u}$ is a *universal* model for hydrodynamic fluctuations
- $\mathbf{h} = \partial_x^{-1} \mathbf{u}$ is a *universal* KPZ equation interface model
  - Kardar-Parisi-Zhang '86 in PRL – *weak KPZ universality*
  - Eden model, ballistic deposition, dynamic Ising model
- Consider $\partial_t \mathbf{u}' = \Delta \mathbf{u}' + \partial_x \mathsf{F}(\mathbf{u}') + \partial_x \xi$. It has Gaussian invariant measures so *Hemite expand*

$$\mathsf{F}(\mathbf{u}') = C_0' + C_1' \mathbf{u}' + C_2' \left((\mathbf{u}')^2 - 1 \right) + \ldots$$

  - $C_i' = \partial_\sigma^{(i)} \mathbb{E} \mathsf{F}(Z + \sigma)|_{\sigma=0}$, where $Z \sim \mathrm{N}(0,1)$
  - Under some scaling, $\mathbf{u}' \to \mathbf{u}$.

Asymmetric Exclusion Processes

- Independent RWs on $\mathbb{T}_N = \mathbb{Z}/N\mathbb{Z}$ with S_x-speed symmetry and A_x-speed asymmetry to the right. Interact through *exclusion*
- Define *density fluctuation field* $\mathbf{u}_{t,x} \in \{\pm 1\}$ depending on occupation/vacancy at x at time t.
- Large-scale behavior of $\langle \mathbf{u}_t, \Phi \rangle_{\mathbb{T}_N}$ process with any $\Phi \in \mathcal{C}^\infty(\mathbb{T}^1)$
- Bertini-Giacomin '97: if $\mathrm{S}_x = 1$ and $\mathrm{A}_x = N^{-1/2}$, then $\mathbf{u}_{N^2 t} \to \mathrm{SBE}(0)$
  - Height function $\mathbf{h}_{t,x} = \partial_x^{-1} \mathbf{u}_{t,x}$ solves exact microscopic KPZ
  - Key observation: $\mathbf{z}_{t,x} = \mathrm{Exp}(\mathbf{h}_{t,x})$ solves exact microscopic linear SHE

Universality?

- Environment-dependence (D): $\mathrm{S}_x = 1$ and $\mathrm{A}_x = N^{-1/2} + N^{-1}\mathfrak{d}_x$
  - $\mathfrak{d}_x$ a local functional of time t particle environment at x
  - "Gradient condition" for $\mathfrak{d}_x$; guarantees a one-parameter family of product invariant measures parameterized by $\mathbf{u}$-density $\sigma \in \mathbb{R}$
  - ASEP + $\mathfrak{d}$-TASEP
- Slow bond model (SB): $\mathrm{S}_x = 1$ and $\mathrm{A}_x = N^{-1/2}$, unless $x \in \{0, 1\}$
  - Random walks cross $\{0, 1\}$ with speed $N^{-\beta} + N^{-1/2}$ to the right and with speed $N^{-\beta}$ to the left. β is small and positive
- Non-simple model (NS): allow for jumps of arbitrary length
  - Require RW step distributions have enough moments

Theorem: [Y '20, Y'21, Y'21+]

$\mathbf{u}_{N^2 t} \to \mathrm{SBE}(\bar{\mathfrak{d}})$, where $\bar{\mathfrak{d}} = 0$ for SB and NS and $\bar{\mathfrak{d}} = \partial_\sigma \mathbb{E}^\sigma \mathrm{F}^{\mathfrak{d}}|_{\sigma=0}$, where $\mathrm{F}^{\mathfrak{d}}$ is the instantaneous flux of $\mathfrak{d}$-TASEP

- Big Picture Questions from AIM workshop on KPZ
- Stationary case – Goncalves-Jara '14, '17; Franco-Goncalves-Simon '17
- Done via Cole-Hopf transform

Boltzmann-Gibbs Principle and the D Model

Non-integrability of ASEP + $\mathfrak{d}$-TASEP reflected in order $N^{\frac{1}{2}}\nabla \mathsf{F}^{\mathfrak{d}}$

Problem – homogenization of $N^{1/2}\bar{\mathsf{F}}^{\mathfrak{d}}_{t,x} = N^{1/2}\mathsf{F}^{\mathfrak{d}}_{T,x} - N^{1/2}C_0 - N^{1/2}\bar{\mathfrak{d}}\mathbf{u}_{T,x}$

Theorem: "Boltzmann-Gibbs Principle" [Y'21+]

$N^{1/2}\bar{\mathsf{F}}^{\mathfrak{d}}_{t,x} \to 0$ in the large-N limit *in a weak sense*, in particular after integrating against smooth space-time test functions

- "Fluctuations of local statistics are asymptotically linear projections onto conserved quantities of the particle system"
- Nothing special about $\mathsf{F}^{\mathfrak{d}}$; can compute linear asymptotics for *any* local environment-dependent statistics
- Succeeds even when multiplying $\bar{\mathsf{F}}^{\mathfrak{d}}$ by Gartner transform

Boltzmann-Gibbs Principle – Applications

Applications/Open Problems:

- General mechanism to derive fluctuations around hydrodynamic limits for many interacting particle systems
  - Brox-Rost '84, Chang-Yau '92, Jara-Menezes '18
  - Also for geometric flows instead of hydrodynamic equations
- *Local/Analytic Method* – fluctuation scaling limits for models without explicit invariant measures or integrable structure
  - Models with boundary, smoothly inhomogeneous/random environment models, e.g. geometric flows with nontrivial topology/metric

Thank you for your attention!

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