Large deviations for Discrete β-ensembles

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The model

Continuous β-ensembles

N ordered particles on $\mathbb{R}$: $x_N < x_{N-1} < \cdots < x_1$ with density proportional to

$$\underbrace{\prod_{1\le i<j\le N} |x_i - x_j|^\beta}_{\text{interaction}} \cdot \underbrace{\prod_{i=1}^{N} e^{-NV(x_i)}}_{\text{potential}} dx_i, \quad x_i \in \mathbb{R}, \text{ ordered}$$

Motivation:

x_i's $\longmapsto$ eigenvalues of random matrices in some cases.

$\beta = 1, 2,$ or 4, V 'nice' : orthogonal, unitary, or symplectic invariant matrices.

$V(x) = \beta x^2/4$, $\beta > 0$: Gaussian β-ensembles. [Dumitriu-Edelman '02]

Discrete log-gases [Johansson '00]:

$$\prod_{1\le i<j\le N} |\ell_i - \ell_j|^{\beta} \prod_{i=1}^{N} e^{-NV(\ell_i/N)}, \quad \ell_i \in \mathbb{Z}_{\ge 0}, \text{ ordered.}$$

Discrete β-ensembles:

$$\mathbb{P}(\vec{\ell}) \propto \prod_{1\le i<j\le N} \frac{\Gamma(\ell_i-\ell_j+1)\Gamma(\ell_i-\ell_j+\theta)}{\Gamma(\ell_i-\ell_j+1-\theta)\Gamma(\ell_i-\ell_j)} \prod_{i=1}^{N} e^{-NV(\ell_i/N)},$$

$$\ell_i = \lambda_i + (N-i)\theta, \quad 0 \le \lambda_N \le \lambda_{N-1} \le \cdots \le \lambda_1, \quad \lambda_i \in \mathbb{Z}_{\ge 0}$$

Continuous β-ensembles:

$$\prod_{1 \le i < j \le N} (x_i - x_j)^{\beta} \prod_{i=1}^{n} e^{-NV(x_i)} dx_i, \quad x_i \in \mathbb{R}, \text{ ordered.}$$

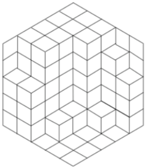
Discrete β-ensembles:

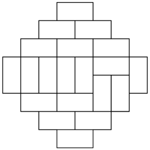
$$\mathbb{P}(\vec{\ell}) \propto \prod_{1 \le i < j \le N} \underbrace{\frac{\Gamma(\ell_i - \ell_j + 1)\Gamma(\ell_i - \ell_j + \theta)}{\Gamma(\ell_i - \ell_j + 1 - \theta)\Gamma(\ell_i - \ell_j)}}_{\approx (\ell_i - \ell_j)^{2\theta}} \prod_{i=1}^{n} e^{-NV(\ell_i/N)},$$

- Introduced in [Borodin-Gorin-Guionnet '17].
- Appeared before as special cases in several other models.

Why this discretization?

- Random Lozenge / Domino Tilings [Johansson '00].





- Measures on Signatures: zw-measures : discrete Selberg integral [Olshanski '03]
- Measures on Young diagrams: z-measures [Borodin-Olshanski '05]
- Integrability: Nekrasov's equation [Nekrasov-Pestun '12, Nekrasov-Pestun-Shatashvili '13, Borodin-Gorin-Guionnet '17]

Previous works

- $\theta = 1$. Large Deviations: [Johansson '00, Feral '08]
- Global Fluctuations: Asymptotically Gaussian [Borodin-Gorin-Guionnet '17]
- Edge Fluctuations: Tracy-Widom β [Guionnet-Huang '19]

Nekrasov's equation assumes

$$\frac{e^{-NV(x/N)}}{e^{-NV((x-1)/N)}} = \frac{\phi_N^+(x)}{\phi_N^-(x)}$$

where $\phi_N^+(x)$ and $\phi_N^-(x)$ are holomorphic on $\mathcal{M}_N \subset \mathbb{C}$.

Assumptions on potentials

- $V : [0,\infty) \to [0,\infty)$ is continuous and differentiable on $(0,\infty)$.

- Growth Condition:

$$V(x) \geq 2\theta \log(1+x^2), \quad x \geq 0$$

- Derivative Condition:

$$|V'(x)| \leq \underbrace{F(a)}_{\text{inc.}} + C|\log x|, \quad x \in (0,a],\ a > 0$$

- Examples:

$$V(x) = x^2, \quad V(x) = x, \quad V(x) = x\ln(x) + 1.$$

Assumptions on potentials

- $V_N, V : [0,\infty) \to [0,\infty)$ are continuous and V is differentiable on $(0,\infty)$.
- Convergence Condition:

$$\sup_{x\in[0,a]} N^{3/4}|V_N(x) - V(x)| \to 0, \quad a > 0.$$

- Growth Condition:

$$V_N(x) \geq 2\theta \log(1+x^2), \quad x \geq 0$$

- Derivative Condition:

$$|V'(x)| \leq \underbrace{F(a)}_{\text{inc.}} + C|\log x|, \quad x \in (0,a],\ a > 0.$$

Results and Intuitions

Connections to Potential Theory

$$\mathbb{P}(\vec{\ell}) \propto \prod_{1\le i<j\le N} \underbrace{\frac{\Gamma(\ell_i-\ell_j+1)\Gamma(\ell_i-\ell_j+\theta)}{\Gamma(\ell_i-\ell_j+1-\theta)\Gamma(\ell_i-\ell_j)}}_{|\ell_i-\ell_j|^{2\theta}} \prod_{i=1}^{N} e^{-NV(\ell_i/N)}$$

$$\approx \exp(-N^2 I_V(\mu_N)), \qquad \mu_N := \frac{1}{N}\sum_{i=1}^{N}\delta_{\ell_i/N}.$$

$$I_V(\mu) := \underbrace{-\theta \iint_{x\neq y} \log|x-y| d\mu(x)d\mu(y)}_{\text{Logarithmic Interaction}} + \underbrace{\int V(x)d\mu(x)}_{\text{External Potential}}$$

$= \text{Energy of the system}$

Law of Large Numbers

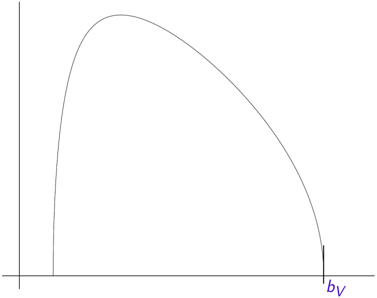
$$\boxed{\frac{1}{N}\sum_{i=1}^{N}\delta_{\ell_i/N} \xrightarrow{\text{weakly in prob.}} \mu_{\text{eq}}}$$

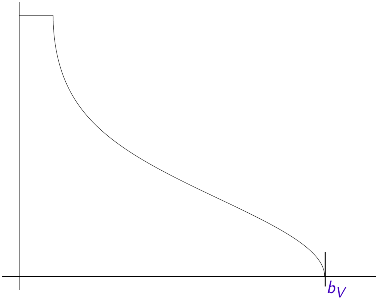
μ_{eq} is the unique minimizer of $I_V(\cdot)$ over all measures with density $0 \leq \phi(x) \leq \theta^{-1}$, supported on $[0,\infty)$.

- Continuous case: [Ben-Arous-Guionnet '97], [Ben-Arous-Zeitouni '98].
- Discrete case: [Borodin-Gorin-Guionnet '17], [D.-Dimitrov '21].

Equilibrium Measure

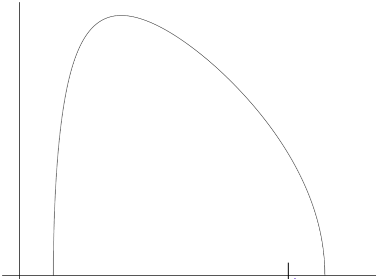
μ_{eq} exists uniquely and its density is compactly supported [Dragnev-Saff '97].





$$\frac{\ell_1}{N} \to b_V \implies \underbrace{\mathbb{P}(\ell_1 \leq (b_V - y)N)}_{\text{Lower Tail}} \qquad \underbrace{\mathbb{P}(\ell_1 \geq (b_V + y)N)}_{\text{Upper Tail}}$$

Lower Tail: $\mathbb{P}(\ell_1 \leq (b_V - y)N)$



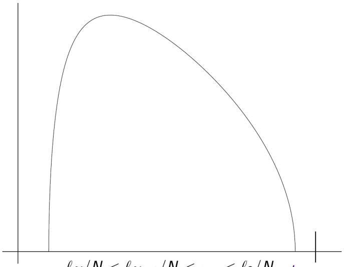
$$I_V(\mu) := -\theta \iint_{x \neq y} \log|x-y| d\mu(x) d\mu(y) + \int V(x) d\mu(x)$$

A_y : measures with density $0 \leq \phi(x) \leq \theta^{-1}$ supported on $[0, b_V - y]$.

Theorem (D.-Dimitrov '21)

For any $y \in (0, b_V - \theta]$, we have

$$\lim_{N\to\infty} \frac{1}{N^2} \log \mathbb{P}(\ell_1 \leq (b_V - y)N) = -\Big(\inf_{\mu \in A_y} I_V(\mu) - I_V(\mu_{eq}) \Big).$$



Upper Tail: $\mathbb{P}(\ell_1 \geq (b_V + y)N)$

$$G_V(x) := \underbrace{-2\theta \int \log|x-t|\,d\mu_{\text{eq}}(t) + V(x)}_{\text{potential at x}}.$$

$$J_V(y) := \inf_{x \geq b_V + y} G_V(x) - G_V(b_V).$$

Theorem (D.-Dimitrov '21)

Suppose $J_V(y) > 0$ for all $y > 0$. For any $y > 0$, we have

$$\lim_{N\to\infty} \frac{1}{N} \log \mathbb{P}(\ell_1 \geq (b_V + y)N) = -J_V(y).$$

Proof Idea

Key ingredients

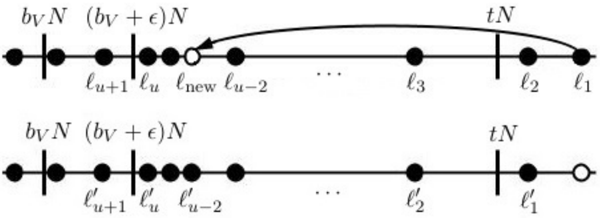
- Exp Tightness: For any $A > 0$, there exists $C_A > 0$ such that

$$\mathbb{P}(\ell_1 > C_A N) \leq e^{-AN}.$$

- Conditional Global LDP estimates:

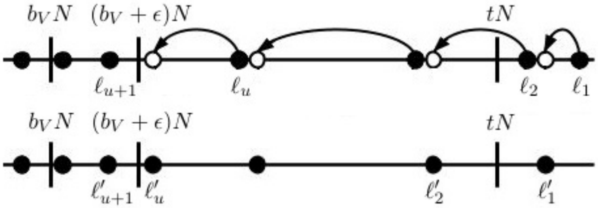
$$\mathbb{P}\left(\left| \int g(x)\mu_N(dx) - \int g(x)\mu_{\mathrm{eq}}(dx) \right| \geq \gamma ||g|| \,\middle|\, \ell_1 \leq CN \right) \leq \exp\left(-c\gamma^2\theta N^2 + O(N \log N)\right)$$

$\theta = 1$ case



$$\frac{\mathbb{P}(\vec{\ell})}{\mathbb{P}(\vec{\ell'})} = \exp\left(2\sum_{i=2}^{N}\log\left|\frac{\ell_i}{N}-\frac{\ell_1}{N}\right| - 2\sum_{i=2}^{N}\log\left|\frac{\ell_i}{N}-\frac{\ell_{\text{new}}}{N}\right|\right) \cdot \exp\left(-NV\left(\frac{\ell_1}{N}\right)+NV\left(\frac{\ell_{\text{new}}}{N}\right)\right)$$

$$= \exp\left(-N\left(G_V\left(\frac{\ell_1}{N}\right)-G_V\left(\frac{\ell_{\text{new}}}{N}\right)\right)\right)$$



- Exponential tightness to discard too high values.
- Global large deviation estimates: not many particles above $b_V N$.
- Fine control on Gamma functions.

Summary

- Proofs are probabilistic.
- Technical Challenges: $\ell_i \in \mathbb{Z}_{\geq 0} + (N - i)\theta$.
- Applications: Krawtchouk ensemble and measures related to Jack symmetric functions.
- Explicit upper tail rate function in the two cases.
- Recovered the $\frac{3}{2}$ power law shallow upper tail asymptotics akin to Tracy-Widom GUE distribution.

$$J_V(y) \sim c \cdot y^{3/2}, \quad \text{as } y \downarrow 0.$$

Thank you!