Upper-tail large deviation principle for the ASEP through Lyapunov exponents

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MSRI Program Associate Short Talks

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Outline

1. **Asymmetric Simple Exclusion Process (ASEP) with step initia data**
2. **Large-time behaviours of $H_0(t)$**
3. **Main result: Lyapunov exponents and upper-tail large deviation**

Asymmetric Simple Exclusion Process (ASEP)

Definition

The ASEP is a continuous-time Markov chain on particle configurations $\mathbf{x} = (\mathbf{x}_1 > \mathbf{x}_2 > \cdots)$ in $\mathbb{Z}$.

◦**Dynamics:**

1. Each site $i \in \mathbb{Z}$ is occupied by at most one particle, which has an independent exponential clock with exponential waiting time of mean 1.
2. When the clock rings, the particle jumps to the right with probability q or to the left with probability $p = 1 - q$.
3. Jump is permitted when the target site is unoccupied.
4. We need to specify its initial state.

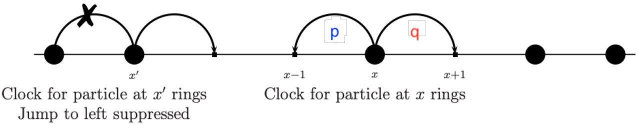
Definition

ASEP starts from the *step* initial configuration if $x_j(0) = -j$, $j = 1, 2, \ldots$.

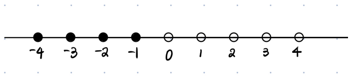
We set $\gamma = 2q - 1$ and assume $q > \frac{1}{2}$, i.e., ASEP has a drift to the right.

Example

1. Here's a demonstration of the dynamics:



2. and the step initial configuration:



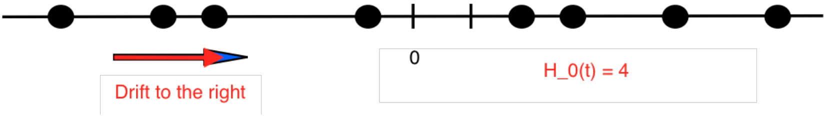
3. Special case: when q = 1, i.e. the particles only jump to the right, we obtain TASEP (totally asymmetric simple exclusion process).

Integrated ASEP current $H_0(t)$

Definition

The observable of interest in ASEP is $H_0(t)$, which is the integrated current through 0, is defined as:

$$H_0(t) := \text{ the number of particles to the right of zero at time } t. \tag{1}$$



- $\{H_0(t) \geq m\} = \{X_m(t) \geq 0\}$: the current fluctuation is related to fluctuations in the position of the m-th particle.

Large-time behaviors of $H_0(t)$

Significance of $H_0(t)$:

- $H_0(t)$ is the one-dimensional height function of the interface growth of the ASEP $\rightarrow$ ASEP is in the KPZ Universality Class$\rightarrow$ fluctuations of $H_0(t)$ exhibit universal critical behaviors

Large-time behaviors of $H_0(t)$:

- **Strong Law**

$$\frac{1}{t}H_0\big(\frac{t}{\gamma}\big) \rightarrow \frac{1}{4}, \text{ almost surely as } t \rightarrow \infty.$$

- **CLT (Tracy-Widom'09)**

$$\frac{1}{t^{1/3}}2^{4/3}\big(-H_0\big(\frac{t}{\gamma}\big)+\frac{t}{4}\big) \Longrightarrow \xi_{\mathrm{GUE}}, \tag{2}$$

  ξ_{GUE} is the **Tracy-Widom GUE** distribution.
  $t^{1/3} \rightarrow$ scaling is the signifier of the KPZ universality.
  When $q=1$, (2) recovers the same result on TASEP. ([Johannson'00])

- **What about tails of -$H_0\big(\frac{t}{\gamma}\big)+\frac{t}{4}$?**

LDP/tail behaviors

Large deviation regime:
What's the probability when $-H_0\left(\frac{t}{\gamma}\right)+\frac{t}{4}$ has a deviation of order t?

The story of two tails: we don't know if Φ_- and Φ_+ exist at this point for general ASEP model but we have the rates functions for TASEP, i.e q =1 in [Johansson '00].

- $$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right)+\frac{t}{4}<-\tfrac{t}{4}y\right)\approx e^{-t^2\Phi_-(y)}; \qquad \text{(Lower Tail)}$$
- $$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right)+\frac{t}{4}>\tfrac{t}{4}y\right)\approx e^{-\mathbf{t}\Phi_+(y)}. \qquad \text{(Upper Tail)}$$
- The upper tail corresponds to the ASEP being "too slow"
- The lower tail corresponds to the ASEP being "too fast"
- We recall a similar phenomenon with the KPZ upper/lower tails ([Tsai'18], [Das-Tsai'19])

Tail behaviors: speed differentials

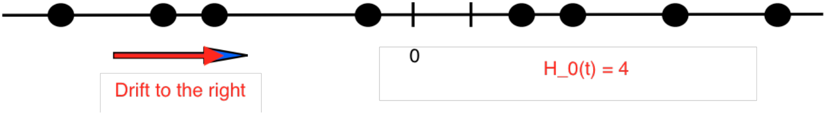
Heuristics for the speed differentials:

- Lower tail corresponds to the ASEP being "too fast":

$$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \frac{t}{4} < -\tfrac{t}{4}y\right) \approx e^{-t^2\Phi_-(y)}; \qquad \text{(Lower Tail)}$$

- Upper tail corresponds to the ASEP being "too slow":

$$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \frac{t}{4} > \tfrac{t}{4}y\right) \approx e^{-\mathbf{t}\Phi_+(y)}. \qquad \text{(Upper Tail)}$$



- $\{-H_0(t/\gamma) + \frac{t}{4} > \frac{t}{4}y\} = \{H_0(t/\gamma) - \frac{t}{4} < -\frac{t}{4}y\}$

Previous result

- Johansson proved both tail large deviation problems for the TASEP in a variational formula. ([Johansson '00])
- For ASEP with step initial data, [Damron-Petrov-Sivakoff '18] produced the following exponential bound:

Theorem (Damron-Petrov-Sivakoff '18)

For $\widetilde{\Phi}_+(y) = \sqrt{y} - (1-y)\tanh^{-1}(\sqrt{y})$ *for* $y \leq y_0 = \frac{1-2\sqrt{q(1-q)}}{1+2\sqrt{q(1-q)}}$ *, we have*

$$\mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > \tfrac{t}{4}y\right) \leq e^{-t\widetilde{\Phi}_+(y)}.$$

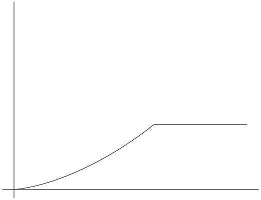


Figure: $\widetilde{\Phi}_+(y)$

Main results: upper-tail LDP through Lyapunov exponents

We present the first proof of the precise upper-tail LDP for the ASEP with step initial data

Theorem (Das - Z. '21)

Fix $q \in (\frac{1}{2}, 1)$. *For any* $y \in (0,1)$ *we have*

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > \tfrac{t}{4}y\right) = -[\sqrt{y} - (1-y)\tanh^{-1}(\sqrt{y})] =: -\Phi_+(y), \quad (3)$$

where $\gamma = 2q - 1$.

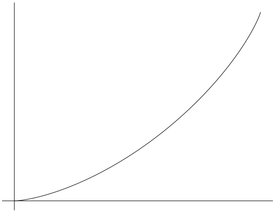


Figure: $\Phi_+(y)$

Proof idea: Lyapunov exponents

- *Lyapunov exponents*: $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[\tau^{sH_0(t)}]$- access to the exponential moments

- *Connection to large deviation*
  1. Using Markov inequality and tilting, we can show that the upper-tail large deviation principle of $\log \tau^{H_0(t)}$ is the Legendre-Fenchel dual of the Lyapunov exponent

We have the following theorem that computes the sth-*Lyapunov exponent* of $\tau^{H_0(t)}$

Theorem (Das - Z. '21)

Let $\tau = \frac{p}{q} < 1$. For $s \in (0, \infty)$ we have

$$\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[\tau^{sH_0(t)}] =: -(q-p)\frac{1-\tau^{\frac{s}{2}}}{1+\tau^{\frac{s}{2}}}. \tag{4}$$

Lyapunov exponents

- It has been a recent popular approach in studying the large deviations of integrable models in the KPZ universality class
- *Recent works*
  1. Das-Tsai'19 - Stochastic Heat Equation with narrow-wedge initial data $\to$ KPZ upper-tail
  2. Ghosal-Lin'20 - SHE with a large class of initial data, including any bounded deterministic positive initial data and the stationary initial data
  3. Lin'20 - half-line SHE
- *Why Lyapunov exponents?*

  [Damron-Petrov-Sivakoff '18] obtained their exponential bound from steepest descent analysis on the exact formula of the distribution of $H_0(t)$ in the form of Fredholm determinants. This formula comes from [Tracy-Widom '09].

  1. Choose an appropriate contour that passes through its critical points and this choice of contour imposes restrictions on the range of y.
  2. Improvement is possible theoretically but it will require much finer analysis.

Proof Idea

- How do we obtain the Lyapunov exponent? $\lim_{t\to\infty} \frac{1}{t} \log \mathbb{E}[\tau^{sH_0(t)}]$

1. Exact formula for integer moments of $\tau^{H_0(t)}$ ([Borodin-Corwin-Sasamoto '14]) exists but doesn't extend to fractional moments:

2. Integrability lends us ([Borodin-Corwin-Sasamoto '14])

Theorem

Fix any $\delta \in (0,1)$. *For* $\zeta > 0$ *we have*

$$\mathbb{E}\left[F_q(\zeta\tau^{H_0(t)})\right] = \det(I + K_{\zeta,t}), \quad F_q(\zeta) := \prod_{n=0}^{\infty} \frac{1}{1+\zeta\tau^n}. \tag{5}$$

Here $\det(I + K_{\zeta,t})$ is the Fredholm determinant of some integral operator $K_{\zeta,t}$.

Proof Idea

- (Borodin-Corwin-Sasamoto '14)

$$\mathbb{E}\left[F_q(\zeta\tau^{H_0(t)})\right] = \det(I + K_{\zeta,t}), \quad F_q(\zeta) := \prod_{n=0}^{\infty} \frac{1}{1+\zeta\tau^n}. \tag{6}$$

- Elementary identity

$$\mathbb{E}[U^{n-1+\alpha}] = \frac{\int_0^\infty \zeta^{-\alpha}\mathbb{E}[U^n F^{(n)}(\zeta U)]d\zeta}{\int_0^\infty \zeta^{-\alpha} F^{(n)}(\zeta)d\zeta} = \frac{\int_0^\infty \zeta^{-\alpha} \frac{d^n}{d\zeta^n}\mathbb{E}[F(\zeta U)]d\zeta}{\int_0^\infty \zeta^{-\alpha} F^{(n)}(\zeta)d\zeta}. \tag{7}$$

- Let $U = \tau^{H_0(t)}$. Combining both identities allows us to obtain good control on the fractional moment $\mathbb{E}[\tau^{sH_0(t)}]$. A continuity argument extends the result to the integer moments.
- Compared to the work on KPZ upper tails by [Das-Tsai '19] and [Lin '20], the analysis of our moments is much more intricate given the complexity of our kernel.

$$K_{\zeta,t}(w,w') := \frac{1}{2\pi\imath}\int_{\delta-\imath\infty}^{\delta+\imath\infty} \Gamma(-u)\Gamma(1+u)\zeta^u \frac{g_t(w)}{g_t(\tau^u w)} \frac{du}{w'-\tau^u w}, \tag{8}$$

$$\text{for } g_t(z) = \exp\left(\frac{(q-p)t}{1+\frac{z}{\tau}}\right). \tag{9}$$

Main Theorem

Theorem (Das - Z. '21)

Let $\tau = \frac{p}{q} < 1$*. For* $s \in (0, \infty)$ *we have*

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{E}[\tau^{sH_0(t)}] =: -(q-p) \frac{1 - \tau^{\frac{s}{2}}}{1 + \tau^{\frac{s}{2}}}. \quad (10)$$

Consequently, for any $y \in (0, 1)$*, we have*

$$\lim_{t \to \infty} \frac{1}{t} \log \mathbb{P}\left(-H_0\left(\tfrac{t}{\gamma}\right) + \tfrac{t}{4} > \tfrac{t}{4} y\right) = -[\sqrt{y} - (1-y) \tanh^{-1}(\sqrt{y})] =: -\Phi_+(y), \quad (11)$$

where $\gamma = 2q - 1$*. Furthermore, we have the following asymptotics near zero:*

$$\lim_{y \to 0^+} y^{-3/2} \Phi_+(y) = \tfrac{2}{3}. \quad (12)$$

Thank you!