# Spectral theory of nonselfadjoint Dirac operators on the circle

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# Topics and methods of analysis

- Background:
  - Integrability of the focusing nonlinear Schrödinger equation (NLS) on the circle
  - Connection to the theory of nonselfadjoint Dirac operators on the circle
- Nonselfadjoint Dirac operator with an elliptic potential
  - Discuss some general features of the spectral theory
  - Find an explicit two-parameter family of finite-gap (or finite-band) potentials
  - Semiclassical bounds on the spectrum
- Tools
  - Numerics (e.g. Hill's method)
  - Bloch-Floquet theory
  - Perturbation theory of linear operators
  - Theory of tridiagonal operators

# Outline

Seminal work in the 1970s extended the inverse scattering transform (IST) of Gardner, Greene, Kruskal and Miura to the case of $x$-periodic initial data.

- Novikov 1974
- Its and Matveev 1975, Its and Kotlyarov 1976
- Dubrovin 1975
- Lax 1975
- Kac and van Moerbeke 1975, McKean and van Moerbeke 1975
- Flaschka–McLaughlin 1976
- McKean and Trubowitz 1976
- Date and Tanaka 1976

- Evolution equation is compatibility of a "Lax pair":

$$\phi_x = X\phi\,, \qquad \phi_t = T\phi\,. \tag{1}$$

- Provides algorithmic procedure for solving the initial value problem.

# The NLS equation on the circle

- NLS with periodic initial data:

$$i\partial_t q + \partial_x^2 q - 2\kappa |q|^2 q = 0\,, \tag{2a}$$

$$q(x+l,t) = q(x,t)\,, \qquad \forall\, x \in \mathbb{R}\,, t \geq 0\,. \tag{2b}$$

  - Universal physical model for the evolution of nonlinear dispersive wavetrains.
  - Completely integrable Hamiltonian system.
  - $\kappa = -1$ (focusing) and $\kappa = 1$ (defocusing).

- Next, we briefly review the inverse scattering method for (2).
  - Direct problem and scattering data.
  - Connection to the classical theory of linear ODEs with periodic coefficients, i.e., "Bloch-Floquet theory".
  - Finite-band potentials and the finite-genus machinery.

# Zakharov-Shabat (ZS) spectral problem

- Spatial half of Lax pair:

$$L\phi = z\phi\,, \quad L := i\sigma_3(\partial_x - Q)\,, \tag{3}$$

where $L$ is a one-dimensional Dirac operator sometimes referred to as the "ZS operator", $\phi(x;z,\epsilon) = (\phi_1,\phi_2)^T$, $\sigma_3 = \mathrm{diag}(1,-1)$, and

$$Q := Q(x) = \begin{pmatrix} \frac{0}{\kappa q(x,0)} & q(x,0) \\ & 0 \end{pmatrix}\,. \tag{4}$$

- Note that $L$ is nonselfadjoint when $\kappa = -1$.
- The following sets comprise the scattering data in the IST for periodic BCs:

$$\sigma_{\mathrm{Lax}}(L) := \{z \in \mathbb{C} : L\phi = z\phi,\ \|\phi\|_\infty < \infty\} \tag{5a}$$

$$\sigma_{\mathrm{Dir}}(L) := \{z \in \mathbb{C} : L\phi = z\phi,\ \phi_1(0) = \phi_2(0),\ \phi_1(l) = \phi_2(l)\} \tag{5b}$$

# Direct scattering–Bloch-Floquet theory

- Bloch-Floquet (or normal) solution:
$$\phi(x+l;z) = \mu\phi(x;z), \tag{6}$$
where $\mu := \mu(z)$ is the Floquet multiplier.
- A *monodromy matrix* $M := M(z)$ is defined as:
$$\Phi(x+l;z) = \Phi(x;z)M(z)\,, \tag{7}$$
where $\Phi(x;z)$ is a fundamental matrix solution of ZS.
- The *Floquet discriminant* $\Delta := \Delta(z)$ is defined as:
$$\Delta(z) = \tfrac{1}{2}\,\mathrm{tr}\,M(z)\,, \tag{8}$$
- The Floquet multipliers $\mu_\pm = \Delta \pm \sqrt{\Delta^2 - 1}$ are eigenvalues of $M$.
- Importantly,
$$z \in \sigma_{\mathrm{Lax}}(L) \iff |\mu(z)| = 1 \iff \Delta(z) \in [-1,1]\,, \tag{9}$$
and $\Delta$ is isospectral.

# Direct scattering–additional properties

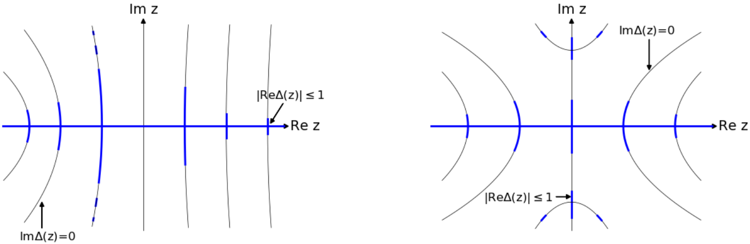
- The Floquet spectrum is defined as:

$$\Sigma_\nu = \{z \in \mathbb{C} : L\phi = z\phi,\ \phi(l) = \mathrm{e}^{i\nu l}\phi(0)\} \tag{10a}$$
$$= \{z \in \mathbb{C} : \Delta(z) = \cos(\nu l)\} \tag{10b}$$

  where $\nu \in \mathbb{R}$ and $\mu = e^{i\nu l}$.

- Importantly, $\sigma_{\text{Lax}}(L) = \cup_{\nu \in [0,2\pi/l)} \Sigma_\nu$.
  - For each fixed $\nu \in \mathbb{R}$ the Floquet spectrum is discrete.
  - Periodic spectrum $[\phi(x+l) = \phi(x)]$ when $\nu = 2n\pi/l$, or $\Delta = 1$.
  - Antiperiodic spectrum $[\phi(x+l) = -\phi(x)]$ when $\nu = 2(n+1/2)\pi/l$, or $\Delta = -1$.
- The Floquet discriminant $\Delta$ is an entire function of $z$.
- The Floquet discriminant satisifies $\Delta(\bar{z}) = \overline{\Delta(z)}$.
- If $q$ is real-valued, even, or odd then Floquet eigenvalues occur in quartets $(z, \bar{z}, -z, -\bar{z})$.

# Scattering data–spectral bands and gaps




Band and gap structure of the Lax spectrum in the nonselfadjoint case. Blue arcs are spectral bands with edges $z^{\pm}$ corresponding to periodic, and antiperiodic eigenvalues, respectively. Construct the level set $\mathcal{C} := \{z \in \mathbb{C} : \mathrm{Im}\Delta(z) = 0\}$. Then the spectral bands form an at most countable set of analytic arcs in the complex plane defined by $\{z \in \mathcal{C} : |\mathrm{Re}\Delta(z)| \leq 1\}$.

# N-band potentials and inverse scattering

**Definition 1**

If $\Delta^2 - 1$ has $2N$ simple roots, we say that the $l$-periodic potential $q$ is an $N$-band potential. The class of finite-band potentials is comprised of the set of all $N$-band potentials for all positive integer values $N$.

- Dirichlet spectra are not isospectral. Their motions are used to reconstruct the potential for $t > 0$ ("angle variables").
- The motion of the Dirichlet spectra is linearized by employing a suitable Abel transformation.
- Solutions are described in terms of $\Theta$-functions determined by hyperelliptic Riemann surfaces of genus $\mathcal{G}$ where $N = \mathcal{G} + 1$.
- In general, $N$-band potentials have $N$ noncommensurate phases.

# Outline

# Hill's method

Hill's method is a numerical technique for calculating the spectrum of a linear operator with periodic coefficients.

- Hill's method is spectrally accurate as a result of using Fourier series approximations.
- The method is limited to the number of Fourier modes chosen and an eigenvalue solver such as the QR algorithm.

Note $Q(x+l) = Q(x)$ and so by Floquet's theorem one gets:

$$z \in \sigma_{\text{Lax}}(L) \iff \phi(x;z) = \mathrm{e}^{i\nu x}\, w(x;z)\,, \tag{11}$$

where $w(x+l;z) = w(x;z)$ and $\nu \in \mathbb{R}$.

Rewrite ZS in the form of a modified eigenvalue problem:

$$L^{\nu} w = zw\,, \qquad L^{\nu} := \sigma_3\big((i\partial_x - \nu) - iQ\big)\,. \tag{12}$$

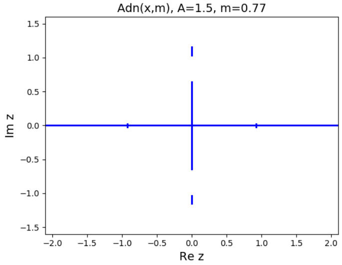
# Elliptic potentials

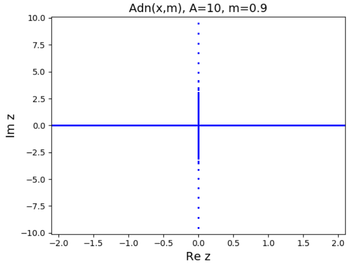
Consider

$$Q(x;m,A) = \begin{pmatrix} 0 & A\,\mathrm{dn}(x;m) \\ -A\,\mathrm{dn}(x;m) & 0 \end{pmatrix}, \qquad (13)$$

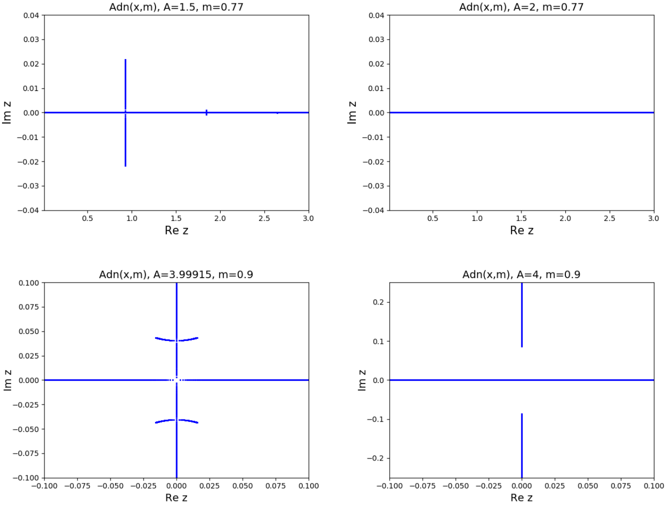
where dn is a Jacobi elliptic function.

- $m \in (0,1)$ is the elliptic parameter and $A \in \mathbb{R}$
- $l = 2K$ with $K := K(m)$ the complete elliptic integral of the first kind



Through various numerical simulations an interesting property emerged for this family of potentials, namely, for $A \in \mathbb{Z}$ there appears to be no bands intersecting the real or imaginary $z$-axis.



# Main result

Consider the focusing ZS eigenvalue problem:

$$L\phi = z\phi\,, \qquad L := i\sigma_3(\partial_x - Q)\,, \tag{14}$$

and

$$q := q(x; m, A) = A\,\mathrm{dn}(x; m) \tag{15}$$

**Theorem 2**

*If $A \in \mathbb{Z}$ and $m \in (0,1)$, then $\sigma_{\mathrm{Lax}}(L) \subset \mathbb{R} \cup (-iA, iA)$.*

**Theorem 3**

*If $A \in \mathbb{Z}$ and $m \in (0,1)$, then $q$ is a two-parameter family of finite-gap (or finite-band) potentials of the focusing ZS eigenvalue problem* (14).

# Outline

Main ideas:

- Discuss some results in the spectral theory of the nonselfadjoint ZS operator on the circle.
- Show the periodic and antiperiodic eigenvalues are real and purely imaginary only and that this is sufficient to claim that the entire spectrum is real and purely imaginary only.
- Relate the ZS eigenvalue problem to an eigenvalue problem for a tridiagonal operator.

The proof of the the above theorems involves several steps:

- Map the focusing ZS eigenvalue problem into a second-order ODE with trigonometric coefficients (and $\lambda = z^2$).
- Map the trigonometric ODE into a three-term recurrence relation for the Fourier coefficients.
- Demonstrate the existence of ascending and descending half-infinite Fourier series solutions.
- Map the trigonometric ODE into Heun's equation and relate the eigenvalues of the ZS problem to the connection problem for Heun's equation.
- Establish reality of eigenvalues of finite truncations of the associated Heun matrices.
- Establish continuity of eigenvalues as the truncation becomes infinite.

# Reduction to $m = 0$, $m = 1$, and the Lamé equation

- When $m = 0$ it follows $A\,\mathrm{dn}(x, 0) \equiv A$. Thus, the ZS problem reduces to that of a constant background and is exactly solvable.
- Whem $m = 1$ it follows $A\,\mathrm{dn}(x, 1) \equiv A\mathrm{sech}\,\mathrm{x}$. Thus, the ZS problem reduces to the case studied by Satsuma and Yajima (1974). For $A \in \mathbb{Z}$ one gets $N$-soliton solution of focusing NLS. Moreover, discrete eigenvalues occur at the half-integers along the imaginary $z$-axis.
- The invertible change of dependent variable:

$$y^{\pm} = \phi_1 \pm i\phi_2 \tag{16}$$

maps the ZS eigenvalue problem into

$$y_{xx} + (iAm\,\mathrm{sn}(x, m)\,\mathrm{cn}(x, m) + \lambda + A^2\,\mathrm{dn}^2(x, m))y = 0\,, \tag{17}$$

where $y := y^-$ and $\lambda := z^2$. Since $\mathrm{dn}^2(x; m) = 1 - m\ \mathrm{sn}^2(x; m)$, (17) is a complex perturbation of the celebrated Lamé equation.

# Outline

# Large $z$ asymptotics of $\Delta$

We find the large $z$ asymptotics for $\Delta(z)$ and $\Delta'(z)$.

**Lemma 4**

*If $q \in L^\infty(\mathbb{R})$, then*

$$\Delta(z) = \cos(zl) + e^{\operatorname{Im} zl}\, o(1)\,, \qquad \text{as } z \to \infty\,,\ \operatorname{Im} z \geq 0\,, \tag{18a}$$

$$\Delta'(z) = -l\sin(zl) + e^{\operatorname{Im} zl}\, o(1)\,, \qquad \text{as } z \to \infty\,,\ \operatorname{Im} z \geq 0\,, \tag{18b}$$

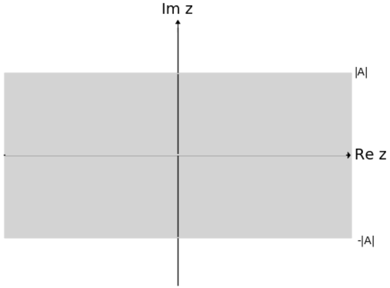
*where $l$ is the period of $q$.*

- The behavior for $\operatorname{Im} z < 0$ follows from $\Delta(\bar{z}) = \overline{\Delta(z)}$.
- The real $z$-axis is one infinitely long band, i.e., $\mathbb{R} \subset \sigma_{\text{Lax}}$. Moreover this is the only band extending to infinity.

# Bound on the spectrum

## Lemma 5

*Take* $q(x;m,A) = A\,\mathrm{dn}(x;m)$ *with* $A \in \mathbb{C}$ *and* $m \in (0,1)$. *If* $z \in \sigma_{\mathrm{Lax}}(L)$, *then* $|\operatorname{Im} z| < |A|$.



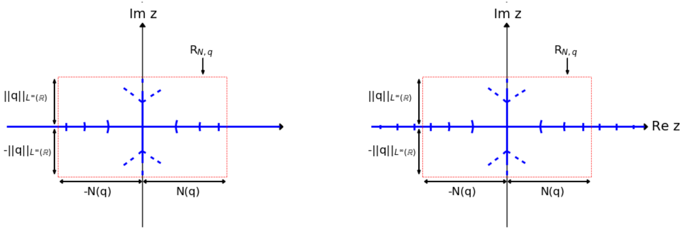
# Finite–gap potentials of the focusing ZS operator

## Theorem 6

*Let $q \in L^\infty(\mathbb{R})$. Then $q$ is a finite-gap potential if and only if $\exists N = N(q) \in \mathbb{N}$ such that $(\sigma_{\text{Lax}}(L) \setminus \mathbb{R}) \subset R_{N,q}$.*

## Theorem 7

*Let $q \in L^\infty(\mathbb{R})$ be real, even, or odd. If the periodic and antiperiodic eigenvalues are real and purely imaginary only then $\sigma_{\text{Lax}}(L) \subset \mathbb{R} \cup [-i\|q\|_{L^\infty(\mathbb{R})}, i\|q\|_{L^\infty(\mathbb{R})}]$ and $q$ is a finite-gap potential.*



# Outline

$$\text{ZS} \xrightarrow{\varphi_1} \text{Hill with complex potential}$$

$$y_{xx} + (iAm\,\mathrm{sn}(x,m)\,\mathrm{cn}(x,m) + \lambda + A^2\,\mathrm{dn}^2(x,m))y = 0 \tag{19}$$

$$\text{Hill with complex potential} \xrightarrow{\varphi_2} \text{ODE with trig. coefficients}$$

$$\begin{aligned} 4(1 - m\sin^2(t/2))y_{tt} - (m\sin t)y_t \\ +(\lambda + A^2(1 - m\sin^2(t/2)) + \tfrac{i}{2}Am\sin t)y = 0 \end{aligned} \tag{20}$$

- By Bloch-Floquet theory, all bounded solutions have the form $y = \mathrm{e}^{i\nu t}\,w$ with $w(t + 2\pi; \lambda, m, A) = w(t; \lambda, m, A)$.
- Thus consider the Fourier series expansion:

$$y(t;\lambda,m,A) = \mathrm{e}^{i\nu t}\sum_{n\in\mathbb{Z}} c_n\,\mathrm{e}^{int} \tag{21}$$

# Three-term recurrence relation

Plugging (21) into (20) gives the following recurrence relation:

$$\alpha_n c_{n-1} + (\beta_n - \lambda) c_n + \gamma_n c_{n+1} = 0\,, \tag{22}$$

where

$$\alpha_n = -\frac{m}{4}[A - (2n + 2\nu - 2)][A + (2n + 2\nu - 1)]\,, \tag{23a}$$

$$\beta_n = (1 - \frac{m}{2})[(2n + 2\nu)^2 - A^2]\,, \tag{23b}$$

$$\gamma_n = -\frac{m}{4}[A - (2n + 2\nu + 2)][A + (2n + 2\nu + 1)]\,. \tag{23c}$$

Equivalently, one can express (22) as the eigenvalue problem:

$$B_\nu^A c = \lambda c\,, \tag{24}$$

where $c = \{c_n\}_{n\in\mathbb{Z}}$,

$$B_\nu^A := \begin{pmatrix} \ddots & \ddots & \ddots & & \\ & \alpha_n & \beta_n & \gamma_n & \\ & & \ddots & \ddots & \ddots \end{pmatrix}, \tag{25}$$

$$\mathrm{dom}(B_\nu^A) = \{c \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |n|^4 |c_n|^2 < \infty\}\,. \tag{26}$$

# Ascending and descending Fourier series solutions

- A tridiagonal matrix is "reducible" when there exists a zero along the subdiagonal, (or superdiagonal).
- Recall $\nu \in \mathbb{Z}$ corresponds to periodic eigenvalues, and $\nu \in \mathbb{Z} + \frac{1}{2}$ corresponds to antiperiodic eigenvalues.
- Let $A \in \mathbb{N}$. Then for $\nu \in \mathbb{Z}$ or $\nu \in \mathbb{Z} + \frac{1}{2}$ there exists a zero along both the subdiagonal and superdiagonal.

This leads to the following result:

**Theorem 8**

*Let $A \in \mathbb{N}$. If $\lambda \in \mathbb{C}$ is a periodic or antiperiodic eigenvalue of the trigonometric operator* (20)*, then there is an associated eigenfunction generated by either an ascending or descending Fourier series.*

# Heun's equation

- The change of independent variable
$$\zeta := \mathrm{e}^{it} \tag{27}$$
maps (20) into a second-order Heun ODE:
$$\zeta^2 F(\zeta;m)y_{\zeta\zeta} + \zeta G(\zeta;m)y_{\zeta} + H(\zeta;m,A)y = 0 \tag{28}$$
$$\begin{aligned} F(\zeta;m) &= -m\zeta^2 + (2m-4)\zeta - m\,, \\ G(\zeta;m) &= -3m\zeta/2 + (2m-4)\zeta - m/2\,, \\ H(\zeta;m,A) &= (A^2m/4 + Am/4)\zeta^2 + (\lambda + A^2 - A^2m/2)\zeta + A^2m/4 - Am/4\,. \end{aligned}$$
- The Heun ODE has four regular singular points:
$$z_o = 0, \qquad z_{1,2} = \left(m - 2 \pm 2\sqrt{1-m}\right)/m, \qquad z_\infty = \infty\,. \tag{29}$$
- Frobenius series solution:
$$y(\zeta;\lambda,m,A) = \zeta^{\rho}\sum_{n=0}^{\infty} C_n \zeta^n\,. \tag{30}$$

# Frobenius analysis and recurrence relations

- Frobenius exponents at $\zeta = 0$: $\rho_1^o = \frac{1}{2}A$ & $\rho_2^o = \frac{1}{2}(1-A)$ with three-term recurrence relations:

$$R_n C_{n-1} + (S_n - \lambda) C_n + P_n C_{n+1} = 0\,, \tag{31a}$$

$$\widetilde{R}_n C_{n-1} + (\widetilde{S}_n - \lambda) C_n + \widetilde{P}_n C_{n+1} = 0\,. \tag{31b}$$

  Let $T_o^{\pm}$ be the associated tridiagonal operators.

- Frobenius exponents at $\zeta = \infty$: $\rho_1^\infty = -\frac{1}{2}A$ & $\rho_2^\infty = \frac{1}{2}(1+A)$ with three-term recurrence relations:

$$X_n C_{n-1} + (Y_n - \lambda) C_n + Z_n C_{n+1} = 0\,, \tag{32a}$$

$$\widetilde{X}_n C_{n-1} + (\widetilde{Y}_n - \lambda) C_n + \widetilde{Z}_n C_{n+1} = 0\,. \tag{32b}$$

  Let $T_\infty^{\pm}$ be the associated tridiagonal operators.

- Letting $\nu = \rho_{1,2}^o$ or $\nu = \rho_{1,2}^\infty$ with $A \in \mathbb{N}$ maps the Frobenius recurrence relations to the ascending/descending Fourier series recurrence relations.

# Real eigenvalues of the truncated Heun matrices

- For $\lambda$ a periodic or antiperiodic eigenvalue the corresponding Frobenius series solution converges on the unit circle.
- Periodic and antiperiodic eigenvalues of ZS correspond to the union of eigenvalues of the tridiagonal operators $T_o^{\pm}$, $T_\infty^{\pm}$ generated by three-term recurrence relations of the Frobenius series.
- The $N \times N$ finite truncations $T_{o,N}^{+}$ and $T_{\infty,N}^{\pm}$ are similar to real symmetric matrices. Thus, they have all real simple eigenvalues.
- The $N \times N$ finite truncation $(T_{o,N}^{-})^T$ is an irreducible, diagonally dominant matrix such that $\mathrm{sgn}(\widetilde{\mathrm{S}}_{\mathrm{n}}\widetilde{\mathrm{S}}_{\mathrm{n}-1}) = \mathrm{sgn}(\widetilde{\mathrm{R}}_{\mathrm{n}}\widetilde{\mathrm{P}}_{\mathrm{n}-1})$. Thus, it also has all real simple eigenvalues (see Veselic 1979)

# Real eigenvalues of the infinite Heun matrices

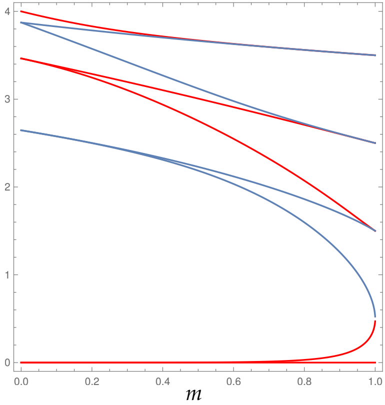
- $T_o^{\pm}$ and $T_\infty^{\pm}$ are closed with compact resolvent.
- The geometric multiplicity of of each eigenvalue is one.
- Importantly, $\lambda_{N,k} \to \lambda_k$ as $N \to \infty$ (Kato 1980)
- Thus, since $\lambda_{N,k} \in \mathbb{R}\ \forall N \in \mathbb{N}$, it follows that $\lambda_k \in \mathbb{R}$.
- Thus, the periodic and antiperiodic eigenvalues of the ZS problem are all real or purely imaginary.
- By previous results (i.e. Theorems 6 and 7), this implies

$$\sigma_{\text{Lax}}(L) \subset \mathbb{R} \cup (-iA, iA)\,, \qquad (33)$$

and the spectrum has at most finitely mand bands.

# Dependence of eigenvalues on the elliptic parameter

- Trajectories of the periodic (red)/antiperiodic (blue) eigenvalues along the Im $z$-axis as $m$ goes from 0 to 1 and $A = 4$.
- Recall that the spectrum is know exactly for $m = 0$ and $m = 1$.




Conjecture: $\mathcal{G} = 2|A| - 1$

# Complex Ince equation

The idea of associating differential operators to three-term recurrence relations dates back at least to the work of Ince and to the so-called Ince's equation:

$$(1 + a\cos 2t)y_{tt} + b(\sin 2t)y_t + (h + d\cos 2t)y = 0\,, \tag{34}$$

where $a$, $b$, and $d$ are real and $|a| < 1$.

It turns out that taking focusing ZS with potential $A\,\mathrm{dn}(x;m)$ can be mapped to a complex Ince equation:

$$(1 + a\cos t)y_{tt} + b(\sin t)y_t + (h + d\cos t + ie\sin t)y = 0\,, \tag{35}$$

where $a$, $b$, $d$, and $e$ are real and $|a| < 1$.

This gives a new class of problems for which three-term recurrence relations are applicable, but the imaginary perturbation complicates the analysis.

# Outline

# Focusing NLS on the circle–semiclassical limit

- Semiclassical focusing NLS with periodic initial data:

$$i\epsilon\partial_t q + \epsilon^2\partial_x^2 q + 2|q|^2 q = 0\,, \tag{36a}$$

$$q(x+l,t;\epsilon) = q(x,t;\epsilon)\,, \quad \forall\, x \in \mathbb{R}\,, t \geq 0\,, \quad 0 < \epsilon \ll 1\,. \tag{36b}$$

- Focusing ZS operator (spatial half of the Lax pair):

$$L^\epsilon \phi = z\phi\,, \qquad L^\epsilon := i\sigma_3(\epsilon\partial_x - Q)\,. \tag{37}$$

  - This is a singular perturbation problem.
  - The spectrum depends on the semiclassical parameter $\epsilon$.
  - When $m = 1$ ($q(x,0) = \operatorname{sech} x$) and $\epsilon = 1/N$ one gets $N$-solitons.

- Solutions $q(x,t;\epsilon)$ analyzed in the limit $\epsilon \downarrow 0$ for decaying BCs using Deift-Zhou method.
  - Kamvissis–McLaughlin–Miller 2003 (reflectionless data)
  - Tovbis–Venakides–Zhou 2004 (solitons and radiation)

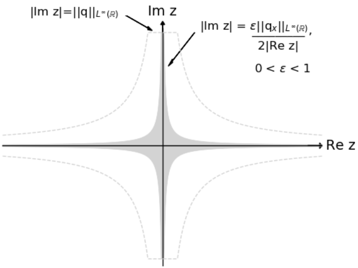
# Semiclassical bounds I

**Lemma 9**

*Let* $z \in \sigma_{\text{Lax}}(L)$. *If* $q \in L^{\infty}(\mathbb{R})$, *then* $|\operatorname{Im} z| \leq \|q\|_{L^{\infty}(\mathbb{R})}$.

**Lemma 10**

*If* $q \in H^1_{\text{loc}}(\mathbb{R})$ *and* $q_x \in L^{\infty}(\mathbb{R})$, *then* $|\operatorname{Im} z||\operatorname{Re} z| \leq \frac{\epsilon}{2}\|q_x\|_{L^{\infty}(\mathbb{R})}$.



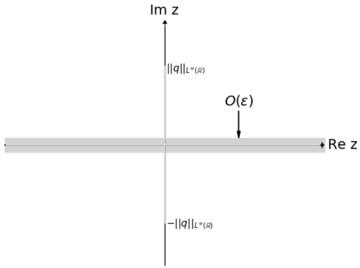
# Semiclassical bounds II

## Lemma 11

*Let* $z \in \sigma_{\mathrm{Lax}}(L)$. *Take* $q \in H^1_{\mathrm{loc}}(\mathbb{R})$ *and* $q_x \in L^\infty(\mathbb{R})$. *Assume* $q(x) > 0$. *If* $\operatorname{Re} z > 0$, *then* $|\operatorname{Im} z| \leq \frac{\epsilon}{2} \|(\ln q)_x\|_{L^\infty(\mathbb{R})}$

## Lemma 12

*Let* $z \in \sigma_{\mathrm{Dir}}(L)$. *Take* $q \in H^1_{\mathrm{loc}}(\mathbb{R})$ *and* $q_x \in L^\infty(\mathbb{R})$. *Assume* $q(x) > 0$. *If* $\operatorname{Re} z > 0$, *then* $|\operatorname{Im} z| \leq \frac{\epsilon}{2} \|(\ln q)_x\|_{L^\infty(\mathbb{R})}$



# Outline

# Recap

- We discussed the spectral theory of nonselfadjoint Dirac operators and their connection to integrable nonlinear PDEs.
- We connected the eigenvalue problem for a nonselfadjoint Dirac operator to the eigenvalue problem for a tridiagonal operator in the case of a two-parameter family of Jacobi elliptic potentials.
- We showed that for each $A \in \mathbb{Z}$ and $m \in (0, 1)$ the spectrum has finitely many bands (resp. gaps).
- Finally, we derived semiclassical bounds on the location of the eigenvalues in the spectral plane.

# Future directions

- One direction of future work is to prove the conjecture:

$$\mathcal{G} = 2|A| - 1\,. \tag{38}$$

  Moreover, is $A \in \mathbb{Z}$ also necessary? That is if $A \notin \mathbb{Z}$ then the potential has infinitely many bands.

- Recently McLaughlin–Nabelek (2019), and Fokas–Lennels (2021) constructed a Riemann-Hilbert problem approach to the inverse scattering problem for general periodic initial data. A very interesting open question is whether one can use the Riemman-Hilbert problem to study semiclassical limits in the case of periodic data.

Thank you!


- G. Biondini, J. Oregero, and A. Tovbis, *On the spectrum of the focusing Zakharov-Shabat operator with periodic potentials* arXiv:2010.04263
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