Avoiding Local Parametrix Problems

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Roadmap of the Talk

- **Goal**: Show how one can avoid local parametrix problems in R-H theory
- **Example R-H problem**: Fokas, Its, Kitaev characterisation of orthogonal polynomials on $[-1, 1]$
- **New results**: on the Plancherel–Rotach asymptotics (large degree asymptotics)

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- **Example R-H problem**: Fokas, Its, Kitaev characterisation of orthogonal polynomials on $[-1, 1]$
- **New results**: on the Plancherel–Rotach asymptotics (large degree asymptotics)

The timeline will be

Example R-H problem $\Rightarrow$ (**Goal**) Avoiding local parametrix problems $\Rightarrow$ **New result**

Outline of the Talk

1. Orthogonal Polynomials on $[-1, 1]$
2. Avoiding Local Parametrix Problems
3. Orthogonal Polynomials revisited

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Orthogonal Polynomials on $[-1, 1]$

Notation and conventions for this talk:

- $p_n(x)$ will denote **monic** orthogonal polynomials
- orthogonality measure $d\mu(x) = \rho(x)dx$ satisfies the **Szegő condition**:

$$\int_{-1}^{1} \frac{\log \rho(z)}{\sqrt{1 - z^2}} dz > -\infty$$

- $\rho(z)$ will satisfy additional **analyticity conditions** for the R-H analysis (later)

Plancherel–Rotach Asymptotics

Theorem (Szegő)

Assume $d\mu = \rho(z)dz$ *satisfies the* Szegő condition *and let* $D(z)$ *be the* Szegő function

$$D(z) := \exp\left(\frac{\sqrt{z^2-1}}{2\pi}\int_{-1}^{1}\frac{\log\rho(x)}{\sqrt{1-x^2}}\frac{dx}{z-x}\right), \quad D_\infty := \lim_{z\to\infty} D(z),$$

which is holomorphic on $\mathbb{C}\setminus[-1,1]$. *Then*

$$p_n(z) = \frac{D_\infty}{D(z)}\overbrace{\frac{\varphi(z)^{n+1/2}}{2^{n+1/2}(z^2-1)^{1/4}}}^{\approx z^n}(1+\varepsilon_n(z))$$

where $\varphi(z) = z + \sqrt{z^2-1}, \quad z \in \mathbb{C}\setminus[-1,1]$ *and*

$$\lim_{n\to\infty}\varepsilon_n(z) = 0, \quad z \in \mathbb{C}\setminus[-1,1]$$

Simplified Relation

Just think!

$$p_n(z) \sim \frac{z^n D_\infty}{D(z)}(1 + \varepsilon_n(z))$$

where $D(z)$ is explicitly computable from the weight function $\rho(x)$.

We want to bound $\varepsilon_n(z) \Leftarrow$ R-H method

R-H Problem for Orthogonal Polynomials

Theorem (Fokas, Its, Kitaev '91)

Let $\rho(z)$ *be integrable on* $[-1,1]$. *Find a* 2×2 *matrix valued function* $Y^n(z)$ *such that*

- $Y^n(z)$ *has holomorphic entries in* $\mathbb{C} \setminus [-1,1]$.
- $Y^n_+(z) = Y^n_-(z) \begin{pmatrix} 1 & \rho(z) \\ 0 & 1 \end{pmatrix}, \quad z \in (-1,1)$
- $\lim_{z\to\infty} Y^n(z) \begin{pmatrix} z^{-n} & 0 \\ 0 & z^n \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $(Y^n(z))_{11} = p_n(z)$, *where* $p_n(z)$ *is the n-th monic orthogonal polynomial associated to the weight function* $\rho(x)$.

R-H Problem for Orthogonal Polynomials

In fact we have

$$Y^n(z) = \begin{pmatrix} p_n(z) & \mathcal{C}^{(-1,1)}(p_n\rho)(z) \\ \gamma_{n-1}p_{n-1}(z) & \gamma_{n-1}\mathcal{C}^{(-1,1)}(p_{n-1}\rho)(z) \end{pmatrix}$$

where $\gamma_n := -2\pi\mathrm{i}\|p_n\|^{-2}_{L^2((-1,1),\rho(x)dx)}$ is a **normalization constant** and

$$\mathcal{C}^\Sigma : L^2(\Sigma) \to \mathcal{O}(\mathbb{C}\setminus\Sigma), \quad \Phi(k) \mapsto \mathcal{C}^\Sigma\Phi(z) := \frac{1}{2\pi\mathrm{i}}\int_\Sigma \frac{\Phi(k)}{k-z}dk$$

is the **Cauchy operator** (convolution with the Cauchy kernel $(k-z)^{-1}$).

Known Results

The R-H characterization of orthogonal polynomials can be used to compute the convergence rate $\varepsilon_n(z)$ for certain **smooth** $\rho(z)$.

Theorem (Kuijlaars, McLaughlin, Van Assche, Vanlessen '04)

Assume the weight function $\rho_{Jac}(z)$ has the form

$$\rho_{Jac}(z) = (1-z)^{\alpha}(1+z)^{\beta}h(z)$$

where $\alpha, \beta > -1$ and h has an analytic continuation to a neighbourhood of $[-1,1]$. Then

$$\varepsilon_n(z) \sim \sum_{k=1}^{\infty} \frac{\Pi_k(z)}{n^k}$$

where the $\Pi_k(z)$ can be explicitly computed and the expansion is uniform away from $[-1,1]$. In particular $\varepsilon_n(z) = O(n^{-1})$.

A. Kuijlaars, K.T-R McLaughlin, W. Van Assche and M. Vanlessen, *The Riemann–Hilbert approach to strong asymptotics for orthogonal polynomials on* $[-1,1]$, Adv. Math. **188**(2), 337–398 (2004).

Nonlinear steepest descent analysis

Problem: Y^n has not a suitable form for analysis (no partition into n-dependent and n-independent part...)

Define

$$T^n(z) := \begin{pmatrix} 2^n & 0 \\ 0 & 2^{-n} \end{pmatrix} Y^n(z) \begin{pmatrix} \varphi(z)^{-n} & 0 \\ 0 & \varphi(z)^n \end{pmatrix}$$

Then (note $\varphi(z) = z + \sqrt{z^2 - 1}$, $\varphi_+(z)\varphi_-(z) = 1$, $z \in (-1, 1)$)

- $T^n_+(z) = T^n_-(z) \begin{pmatrix} \varphi_+(z)^{-2n} & \rho(z) \\ 0 & \varphi_-(z)^{-2n} \end{pmatrix}, \quad z \in (-1, 1)$
- $\lim_{z\to\infty} T^n(z) = \mathbb{1}$

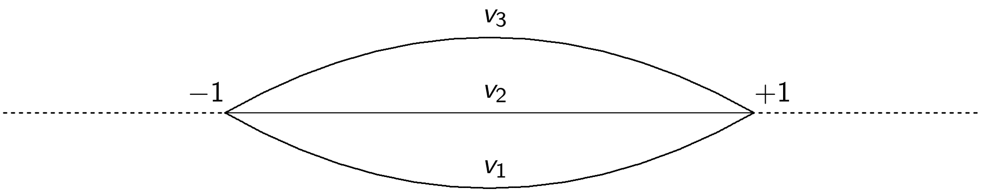
$\Rightarrow$ n-dependence in the jump matrices + normalization at infinity.

Opening of the Lense

Factorize the jump matrix:

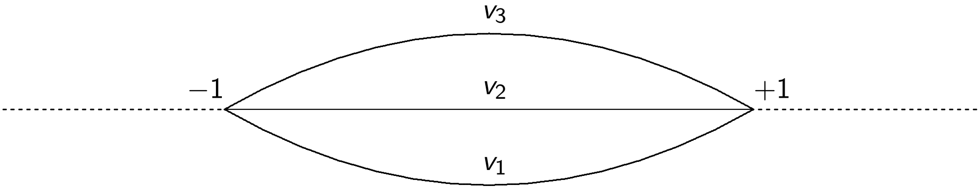
$$\begin{pmatrix} \varphi_+(z)^{-2n} & \rho(z) \\ 0 & \varphi_-(z)^{-2n} \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ \rho(z)^{-1}\varphi_-(z)^{-2n} & 1 \end{pmatrix}}_{=:v_1} \underbrace{\begin{pmatrix} 0 & \rho(z) \\ -\rho(z)^{-1} & 0 \end{pmatrix}}_{=:v_2} \underbrace{\begin{pmatrix} 1 & 0 \\ \rho(z)^{-1}\varphi_+(z)^{-2n} & 1 \end{pmatrix}}_{=:v_3}$$

and **split** the jump contour accordingly:



Note: $\varphi(z)^{-2n} \to 0$ uniformly away ± 1, as $\varphi(z) > 1$ for $z \in \mathbb{C} \setminus [-1, 1]$!

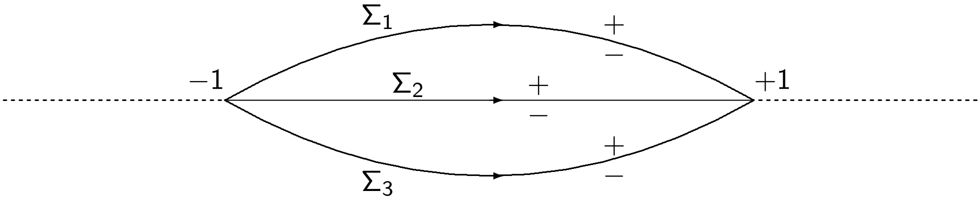
Solution inside the Lense



The new solution will have a different form **inside the lense**:

$$S^n(z) = \begin{cases} T^n(z), & z \text{ outside of the lense} \\ T^n(z)\begin{pmatrix} 1 & 0 \\ -\rho(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, & z \text{ in the upper part of the lense} \\ T^n(z)\begin{pmatrix} 1 & 0 \\ \rho(z)^{-1}\varphi(z)^{-2n} & 1 \end{pmatrix}, & z \text{ in the lower part of the lense} \end{cases}$$

R-H Problem for S^n



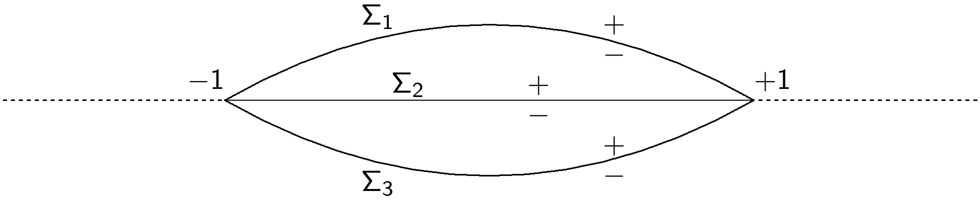
Jump condition:

$$S_+^n(k) = S_-^n(k) v_{\mathcal{S}}^n(k)$$

where

$$v_{\mathcal{S}}^n(k) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \rho(k)^{-1}\varphi(k)^{-2n} & 1 \end{pmatrix}, & \text{for } k \in \Sigma_1 \cup \Sigma_3, \\ \begin{pmatrix} 0 & \rho(k) \\ -\rho(k)^{-1} & 0 \end{pmatrix}, & \text{for } k \in \Sigma_2 = (-1, 1), \end{cases}$$

Behaviour of $v_{\mathcal{S}}^n$



$v_{\mathcal{S}}^n(k)$ is n-independent for $k \in [-1, 1]$.

$$v_{\mathcal{S}}^n(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{O(\rho(k)^{-1} \varphi(k)^{-2n})}_{\to 0 \text{ uniformly } \textbf{away from } \pm 1}, \quad k \in \Sigma_1 \cup \Sigma_3$$

Because,

$$|\varphi(z)| > 1, \quad z \in \mathbb{C} \setminus [-1, 1]$$

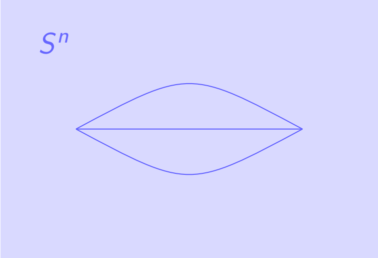
$$\lim_{z \to k} |\varphi(z)| = 1, \quad k \in [-1, 1]$$

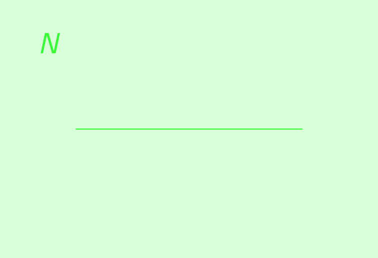
R-H Solutions S^n and N

Partition of R-H problems into **constant** and **decaying** parts

We now ignore the jump matrix on $\Sigma_1 \cup \Sigma_3$:

$\Rightarrow$ We get a R-H problem for N with jump **only on** $[-1, 1]$:





The Model R-H problem for N

We get the **model problem** (outer parametrix problem)

$$N_+(k) = N_-(k) \underbrace{\begin{pmatrix} 0 & \rho(k) \\ -\rho(k)^{-1} & 0 \end{pmatrix}}_{=:v_{\mathcal{N}}(k)}, \quad k \in (-1,1)$$

$$\lim_{z\to\infty} N(z) = \mathbb{1}$$

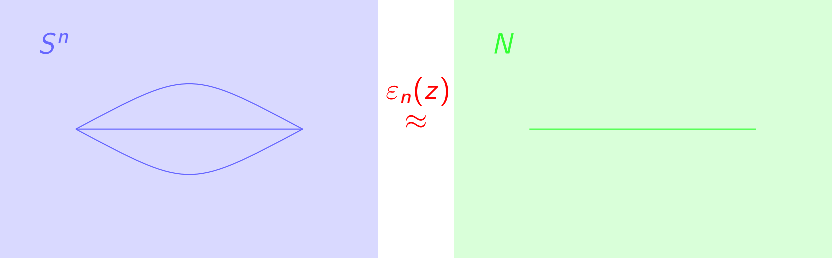
This R-H problem is explicitly solvable:

$$N(z) = \begin{pmatrix} D_\infty & 0 \\ 0 & D_\infty^{-1} \end{pmatrix} \begin{pmatrix} \frac{a(z)+a(z)^{-1}}{2} & \frac{a(z)-a(z)^{-1}}{2\mathrm{i}} \\ \frac{a(z)-a(z)^{-1}}{-2\mathrm{i}} & \frac{a(z)+a(z)^{-1}}{2} \end{pmatrix} \begin{pmatrix} D(z) & 0 \\ 0 & D(z)^{-1} \end{pmatrix}.$$

Here $a(z) = \left(\frac{z-1}{z+1}\right)^{1/4}$.

Solutions S^n and N cont.

Recall: $p_n(z) \sim D(z)^{-1}(1 + \varepsilon_n(z))$



$$\Downarrow \qquad\qquad\qquad\qquad \Uparrow$$

$$p_n(z) \qquad\qquad\qquad\qquad D(z)$$

Problem: Jump matrices are not close in the ∞-norm to each other!

Recall:

$$v_{\mathcal{S}}^n(k) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \underbrace{O(\rho(k)^{-1}\varphi(k)^{-2n})}_{\to 0 \ \text{ uniformly } \textbf{away from } \pm 1} \ , \quad k \in \Sigma_1 \cup \Sigma_3$$

Local Parametrix Solution

Remedy: Solve two **local parametrix problems**

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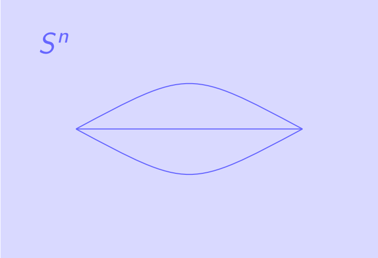
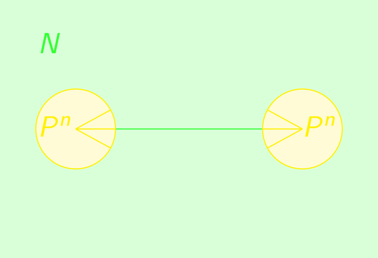




such that

- jumps of S^n and P^n match inside the discs
- $\|N - P^n\|_\infty \to 0$ as $n \to \infty$ on the boundary of the discs

Local Parametrix Problem

All P^n, $n \in \mathbb{N}$ are derived from an $\widehat{P}^\infty$.

$\Rightarrow$ need to solve a local parametrix (Bessel, Airy, ...) R-H problem for $\widehat{P}^\infty$

Deriving the $\widehat{P}^\infty$ R-H Problem

The local parametrix problem $\widehat{P}^\infty$ is obtained from P^n through

- An n-dependent change of variables $z \to \zeta$,
- Conjugation to make the jump matrices ζ-independent
- A limit $n \to \infty$.

Bessel Parametrix Problem

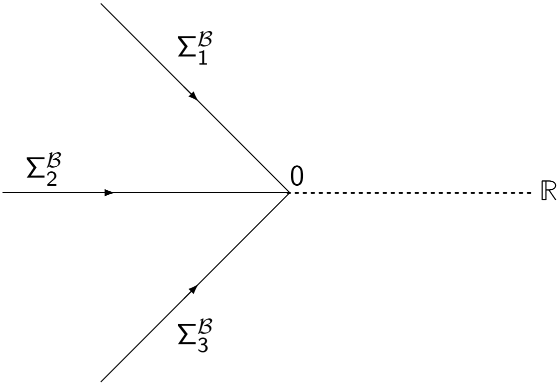


Figure: Contour for the Bessel R-H problem

Bessel Parametrix Problem

The choice $\rho_{\mathsf{Jac}}(z) = (1-z)^{\alpha}(1+z)^{\beta}h(z)$ leads to the following local parametrix problem around $z = +1$:

$$\widehat{P}^{\infty}_{+}(\zeta) = \widehat{P}^{\infty}_{-}(\zeta) v_{\mathcal{B}}(\zeta) \quad \text{with} \quad v_{\mathcal{B}}(\zeta) = \begin{cases} \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{\alpha\pi\mathrm{i}} & 1 \end{pmatrix}, & \text{for } \zeta \in \Sigma_1^{\mathcal{B}}, \\ \begin{pmatrix} 1 & 0 \\ \mathrm{e}^{-\alpha\pi\mathrm{i}} & 1 \end{pmatrix}, & \text{for } \zeta \in \Sigma_3^{\mathcal{B}}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \zeta \in \Sigma_2^{\mathcal{B}}. \end{cases}$$

$$\widehat{P}^{\infty}(\zeta) \to (2\pi\zeta^{1/2})^{-\sigma_3/2} \frac{1}{\sqrt{2}} \left(\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix} + o(1) \right) \mathrm{e}^{2\zeta^{1/2}\sigma_3}$$

uniformly as $\zeta \to \infty$. **Note**: Jump matrices do not depend on ζ

Analytic Continuation

Problem: requires analytic continuation properties of $\rho(z)$ in the yellow disc!

Recall: Kuijlaars et al considered the weight function

$$\rho_{\mathsf{Jac}}(z) = (1-z)^{\alpha}(1+z)^{\beta}h(z)$$

where $h(x)$ has an **analytic continuation** in a neighbourhood of $[-1,1]$.

- $(1-z)^{\alpha}$, $(1+z)^{\beta}$ introduce **multiplicative** jump conditions.
- $\Rightarrow$ Local parametrix problem and solution $\widehat{P}^{\infty}$ can be computed ☺

Logarithmic Weight Function

O. Conway, P. Deift considered recently the logarithmic weight

$$\rho_{\log}(z) = \log \frac{2c}{1-z}, \quad z \in (-1,1),\ c > 1.$$

The logarithm introduces an additional jump conditions.

T. O. Conway and P. Deift, *Asymptotics of Polynomials Orthogonal with respect to a Logarithmic Weight*, SIGMA **14**, 056, 66 pages (2018).

Logarithmic parametrix problem

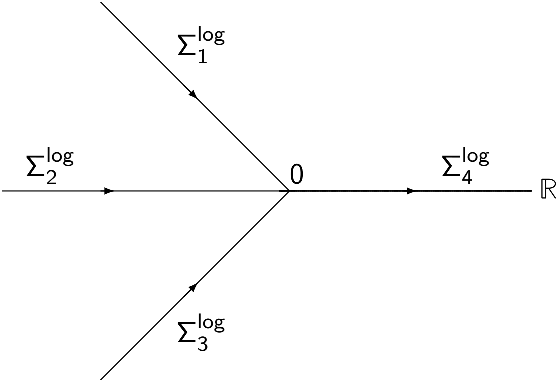


Figure: Contour for logarithmic local parametrix problem

Note: There is an additional jump contour $\Sigma_4^{\log}$

Logarithmic parametrix problem cont.

$$\widehat{P}^{\infty}_{+}(\zeta) = \widehat{P}^{\infty}_{-}(\zeta) v_{\log}(\zeta) \quad \text{with} \quad v_{\log}(\zeta) = \begin{cases} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta \in \Sigma_1^{\log}, \\ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, & \text{for } \zeta \in \Sigma_3^{\log}, \\ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \zeta \in \Sigma_2^{\log}. \\ \left(\sqrt{\frac{\log|\frac{2k}{\zeta}| + \mathrm{i}\pi}{\log|\frac{2k}{\zeta}| - \mathrm{i}\pi}} \right)^{\sigma_3}, & \text{for } z \in \Sigma_4^{\log} \end{cases}$$

No solution to the above R-H problem is known ☹

The authors avoided this issue through a comparison to the Legendre problem ($\rho_{Leg}(z) \equiv 1$).

Outline of the talk

1. Orthogonal Polynomials on $[-1, 1]$
2. Avoiding Local Parametrix Problems
3. Orthogonal Polynomials revisited

The Cauchy Integral Operator

Let $\Phi(k) \in L^2(\Sigma)$. Define the **Cauchy Operator** $\mathcal{C}^{\Sigma}$ by

$$\mathcal{C}^{\Sigma} : L^2(\Sigma) \to \mathcal{O}(\mathbb{C} \setminus \Sigma),$$

$$\Phi(k) \mapsto \mathcal{C}^{\Sigma}\Phi(z) := \frac{1}{2\pi \mathrm{i}} \int_{\Sigma} \frac{\Phi(k)}{k - z} dk$$

Define the **Cauchy Boundary Operators** $\mathcal{C}^{\Sigma}_{\pm}$ by

$$\mathcal{C}^{\Sigma}_{\pm}\Phi(k) := \lim_{z \to k\pm} \mathcal{C}^{\Sigma}\Phi(z)$$

$\mathcal{C}^{\Sigma}_{\pm} : L^2(\Sigma) \to L^2(\Sigma)$ are bounded linear operators.

Another Cauchy-type Integral Operator

For a Riemann–Hilbert problem with jump matrix $v(k)$ define

$$w(k) := v(k) - \mathbb{1} \in L^{\infty}(\Sigma)$$

and the bounded operator

$$\mathcal{C}_w^{\Sigma} : L^2(\Sigma) \to L^2(\Sigma), \quad \Phi(k) \mapsto \mathcal{C}_-^{\Sigma}(\Phi \cdot w)(k)$$

Note: $\|\mathcal{C}_w^{\Sigma}\|_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim \|w\|_{L^\infty(\Sigma)}$

Singular Integral Formulation

Theorem (Zhou '89)

There is a bijective correspondence between normalized solutions $M(z)$ of a Riemann–Hilbert problem with $M_-(k) - \mathbb{1} \in L^2(\Sigma)$ and solutions of the associated singular integral equation

$$(\mathbb{1} - \mathcal{C}_w^\Sigma)\Phi = \mathcal{C}_-^\Sigma(w)$$

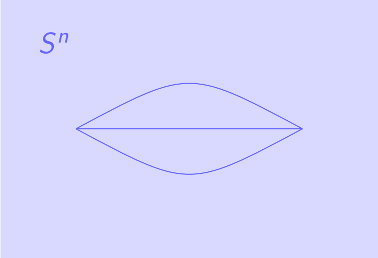
given by

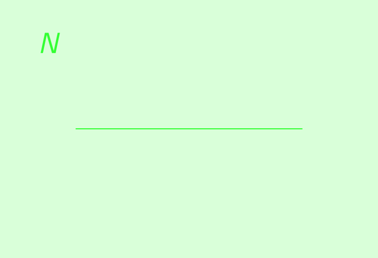
$$\Phi(k) \longrightarrow M(z) := \mathbb{1} + \mathcal{C}^\Sigma((\Phi + \mathbb{1})w)(z),$$

$$M(z) \longrightarrow \Phi(k) := M_-(k) - \mathbb{1}.$$

A Reminder

- The following arguments hold in various application of R-H problems.
- For illustration we will keep referring back to the orthogonal polynomials R-H problems:





Two Singular Integral Equations

Consider

$$(\mathbb{1} - \mathcal{C}_{w_{\mathcal{N}}})\Phi_{\mathcal{N}} = \mathcal{C}_-^{\Sigma}(w_{\mathcal{N}}), \qquad (\mathbb{1} - \mathcal{C}_{w_{\mathcal{S}}^n})\Phi_{\mathcal{S}}^n = \mathcal{C}_-^{\Sigma}(w_{\mathcal{S}}^n)$$

corresponding to the R-H problems for S^n and N.

To understand the convergence rate of $\varepsilon_n(z) \to 0$, we need to understand the convergence rate of $S^n \to N$ or equivalently:

$$\Phi_{\mathcal{S}}^n \to \Phi_{\mathcal{N}} \quad \text{in } L^2(\Sigma), \qquad n \to \infty$$

Two Singular Integral Equations

$$(\mathbb{1} - \mathcal{C}_{w_\mathcal{N}})\Phi_\mathcal{N} = \mathcal{C}_-^\Sigma(w_\mathcal{N}), \qquad (\mathbb{1} - \mathcal{C}_{w_\mathcal{S}^n})\Phi_\mathcal{S}^n = \mathcal{C}_-^\Sigma(w_\mathcal{S}^n)$$

We have

$$w_\mathcal{S}^n \to w_\mathcal{N} \ \text{ in } L^2(\Sigma) \quad \Rightarrow \quad \mathcal{C}_-^\Sigma(w_\mathcal{S}^n) \to \mathcal{C}_-^\Sigma(w_\mathcal{N}) \ \text{ in } L^2(\Sigma)$$

but

$$\|\mathcal{C}^\Sigma_{w_\mathcal{S}^n} - \mathcal{C}^\Sigma_{w_\mathcal{N}}\|_{L^2(\Sigma)\to L^2(\Sigma)} = \|\mathcal{C}^\Sigma_{w_\mathcal{S}^n - w_\mathcal{N}}\|_{L^2(\Sigma)\to L^2(\Sigma)} \lesssim \underbrace{\|w_\mathcal{S}^n - w_\mathcal{N}\|_{L^\infty(\Sigma)}}_{\not\to 0}$$

The Plan

Note:

- $\|w_{\mathcal{S}}^n - w_{\mathcal{N}}\|_{L^p(\Sigma)} \to 0$ for $p \in [1, \infty)$.
- $w_{\mathcal{S}}^n - w_{\mathcal{N}}$ is concentrated around ± 1 (stationary points of the phase function)

Idea:

1. Show a priori estimates of the form:

$$\Phi_{\mathcal{S}}^n = S_-^n - \mathbb{1},\ \Phi_{\mathcal{N}} = N - \mathbb{1} \textit{ do not "blow up" around } \pm 1,$$

2. Show that $\mathcal{C}_{w_{\mathcal{S}}^n} \to \mathcal{C}_{w_{\mathcal{N}}}$ on the space of functions that *do not "blow up" near* ± 1,
3. Conclude: $\Phi_{\mathcal{S}}^n \to \Phi_{\mathcal{N}}$ away from ± 1.

Some Standard Assumptions

Partition $\Sigma = \Sigma^{mod} \dot{\cup} \Sigma^{exp}$

$$v_{\mathcal{N}}(k) = \begin{cases} v_{\mathcal{S}}^{n}(k), & k \in \Sigma^{mod}, \\ \mathbb{1}, & k \in \Sigma^{exp}. \end{cases}$$

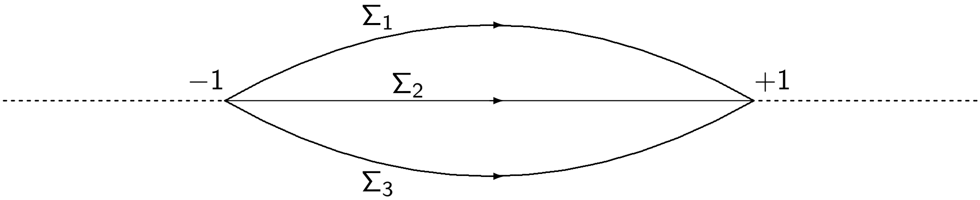
and

$$v_{\mathcal{S}}^{n}(k) \to \mathbb{1}, \qquad k \in \Sigma^{exp}$$

exponentially fast away from a finite number of points (e.g. ± 1).

Our Example

In our example $\Sigma^{\mathsf{mod}} = (-1, 1)$, $\Sigma^{\mathsf{exp}} = \Sigma_1 \cup \Sigma_3$:



Residual R-H problem

Define $R^n := S^n N^{-1}$. Then:

$$R^n_+(k) = R^n_-(k) v^n_{\mathcal{R}}(k), \qquad k \in \Sigma^{exp}$$

where

$$v^n_{\mathcal{R}}(k) = N(k) v^n_{\mathcal{S}}(k) N^{-1}(k), \qquad k \in \Sigma^{exp}$$

and

$$\lim_{z\to\infty} R^n(z) = \mathbb{1}.$$

The Bijection again

Recall the bijection from Zhou's Theorem:

$$\Phi(k) \longrightarrow R^n(z) := \mathbb{1} + \mathcal{C}^{\Sigma^{\exp}}((\Phi + \mathbb{1})w_{\mathcal{R}}^n)(z),$$

$$R^n(z) \longrightarrow \Phi(k) := R_-^n(k) - \mathbb{1}.$$

Note:

$$R^n(z) = \mathbb{1} + \mathcal{C}^{\Sigma^{\exp}}((\Phi + \mathbb{1})w_{\mathcal{R}}^n)(z) = \mathbb{1} + \mathcal{C}^{\Sigma^{\exp}}(R_-^n w_{\mathcal{R}}^n)(z)$$

$$= \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{\exp}} \frac{R_-^n(k) w_{\mathcal{R}}^n(k)}{k - z}\, dk$$

A Priori Estimates

So we have:

$$R^n(z) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{exp}} \frac{R_-^n(k) w_{\mathcal{R}}^n(k)}{k - z}\, dk$$
$$= \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \underbrace{\int_{\Sigma^{exp}} \frac{S_-^n(k) w_{\mathcal{S}}^n(k) N^{-1}(k)}{k - z}\, dk.}_{\leq \|S_-^n\|_{L^p(\Sigma^{exp})} \|w_{\mathcal{S}}^n N^{-1}\|_{L^q(\Sigma^{exp})} \operatorname{dist}(z, \Sigma^{exp})^{-1}}$$

A Priori Estimates

So we have:

$$R^n(z) = \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \int_{\Sigma^{exp}} \frac{R_-^n(k) w_{\mathcal{R}}^n(k)}{k - z}\, dk$$
$$= \mathbb{1} + \frac{1}{2\pi \mathrm{i}} \underbrace{\int_{\Sigma^{exp}} \frac{S_-^n(k) w_{\mathcal{S}}^n(k) N^{-1}(k)}{k - z}\, dk.}_{\leq \|S_-^n\|_{L^p(\Sigma^{exp})} \|w_{\mathcal{S}}^n N^{-1}\|_{L^q(\Sigma^{exp})} \operatorname{dist}(z, \Sigma^{exp})^{-1}}$$

We want **a priori estimates** on $\|S_-^n\|_{L^p(\Sigma^{exp})}$ and $\|w_{\mathcal{S}}^n N^{-1}\|_{L^q(\Sigma^{exp})}$ such that

$$\|S_-^n\|_{L^p(\Sigma^{exp})} \|w_{\mathcal{S}}^n N^{-1}\|_{L^q(\Sigma^{exp})} \to 0$$

A Priori Estimates cont.

Having an a priori estimates such that

$$\|S_-^n\|_{L^p(\Sigma^{exp})}\|w_{\mathcal{S}}^n N^{-1}\|_{L^q(\Sigma^{exp})} \leq \delta(n) \to 0$$

we get

$$R^n(z) = \mathbb{1} + O(\delta(n)\,\mathrm{dist}(z,\Sigma^{exp})^{-1})$$

or equivalently ($R^n = S^n N^{-1}$)

$$S^n(z) = (\mathbb{1} + O(\delta(n)\,\mathrm{dist}(z,\Sigma^{exp})^{-1}))N(z)$$

A Priori Estimates cont.

Having an a priori estimates such that

$$\|S_-^n\|_{L^p(\Sigma^{exp})}\|w_{\mathcal{S}}^n N^{-1}\|_{L^q(\Sigma^{exp})} \le \delta(n) \to 0$$

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$$R^n(z) = \mathbb{1} + O(\delta(n)\,\mathrm{dist}(z, \Sigma^{exp})^{-1})$$

or equivalently ($R^n = S^n N^{-1}$)

$$S^n(z) = (\mathbb{1} + O(\delta(n)\,\mathrm{dist}(z, \Sigma^{exp})^{-1}))N(z)$$

Recall: $p_n(z) \sim D(z)^{-1}(1 + \varepsilon_n(z))$

Outline of the talk

1. Orthogonal Polynomials on $[-1, 1]$

2. Avoiding Local Parametrix Problems

3. Orthogonal Polynomials revisited

A Priori Estimates for S^n

Recall: S^n is built from p_n, p_{n-1}, $\mathcal{C}^{(-1,1)}(p_n\rho)$ and $\mathcal{C}^{(-1,1)}(p_{n-1}\rho)$

$\Rightarrow$ The **a priori estimate** for $S^n(z)$ is based on the following theorem:

Theorem

Assume the weight $\rho(z)$, $z \in (-1,1)$ satisfies the Szegő condition. Then

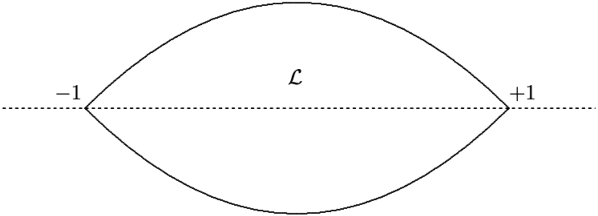
$$\lim_{n\to\infty} 2^{-n}\|p_n\|_{L^2([-1,1],\rho(z)dz)} = \sqrt{\pi}\exp\left(\frac{1}{2\pi}\int_{-1}^{1}\frac{\log\rho(z)}{\sqrt{1-z^2}}\,dz\right).$$

A Priori Estimate for N

Note: N is built from the Szegö function $D(z)$, which depends only on $\rho(z)$ (so does $w_{\mathcal{S}}^n$).

$\Rightarrow$ The **a priori estimate** for $w_{\mathcal{S}}^n(z)N(z)$ is derived from the following assumptions on the weight functions $\rho(z)$, $z \in (-1, 1)$:

- $\rho(z)$ has an analytic continuation to a lense-shaped neighbourhood $\mathcal{L}$
- $\rho(z)$, $\rho(z)^{-1} \in O((z \pm 1)^{-1/\nu + \delta})$ for some $\nu > 8$, $\delta > 0$



Lense shaped Neighbourhood $\mathcal{L}$

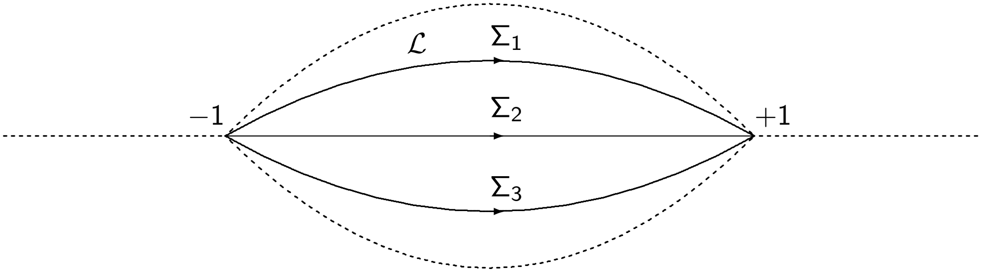


Figure: Jump contour for S.

Main Result

Theorem (P. '20)

Assume the weight function $\rho(z)$ has an analytic continuation to a lense-shaped neighbourhood $\mathcal{L}$, and satisfies:

$$|\rho(z)| = O(|z \pm 1|^{-1/\nu+\delta}), \quad |\rho(z)^{-1}| = O(|z \pm 1|^{-1/\nu+\delta}), \quad z \in \mathcal{L}$$

for some $\nu > 8$ and $\delta > 0$. Then the corresponding orthogonal polynomials satisfy

$$p_n(z) = D(z)^{-1} \frac{D_\infty \varphi(z)^{n+1/2}}{2^{n+1/2}(z^2-1)^{1/4}}(1 + \underbrace{O(n^{-\lambda} z^{-1})}_{=\varepsilon_n(z)})$$

locally uniformly in $\mathbb{C} \setminus [-1,1]$ and $\lambda = 1/2 - 4/\nu > 0$.

M. P., *Riemann–Hilbert Theory without local Parametrix Problems: Applications to Orthogonal Polynomials*, to appear in J. Math. Anal. Appl.

Recap

- We have shown how one can avoid local parametrix problems in R-H theory, provided one has certain a priori esimates.
- We have derived the a priori estimates for the R-H problem for orthogonal polynomials on $[-1, 1]$
- We obtained an improved error estimate in the Plancherel–Rotach asymptotics for the orthogonal polynomials.
- More generally, we compared two R-H problems in the coarser L^p-**norm** of the jump matrices, instead of the standard L^∞-**norm**.

Outlook

- Find further applications for R-H problems with no know local parametrix solutions $\widehat{P}^{\infty}$,
- Try to understand the behaviour of $\widehat{P}^{\infty}$ for the logarithmic weight (see also Van Assche '99 on irrationality of $\zeta(2n+1)$ and Magnus '18),
- Derive "better" a priori estimates to improve the error terms

W. Van Assche, *Multiple orthogonal polynomials, irrationality and transcendence* , Contemp. Math. **236**, 325–342 (1999).

A. Magnus, *Gaussian integration formulas for logarithmic weights and application to 2-dimensional solid-state lattices*, J. Approx. Theory **228**, 21–57 (2018).

Thank You!