Longest increasing subsequence and Schensted shape of some pseudo-random sequences

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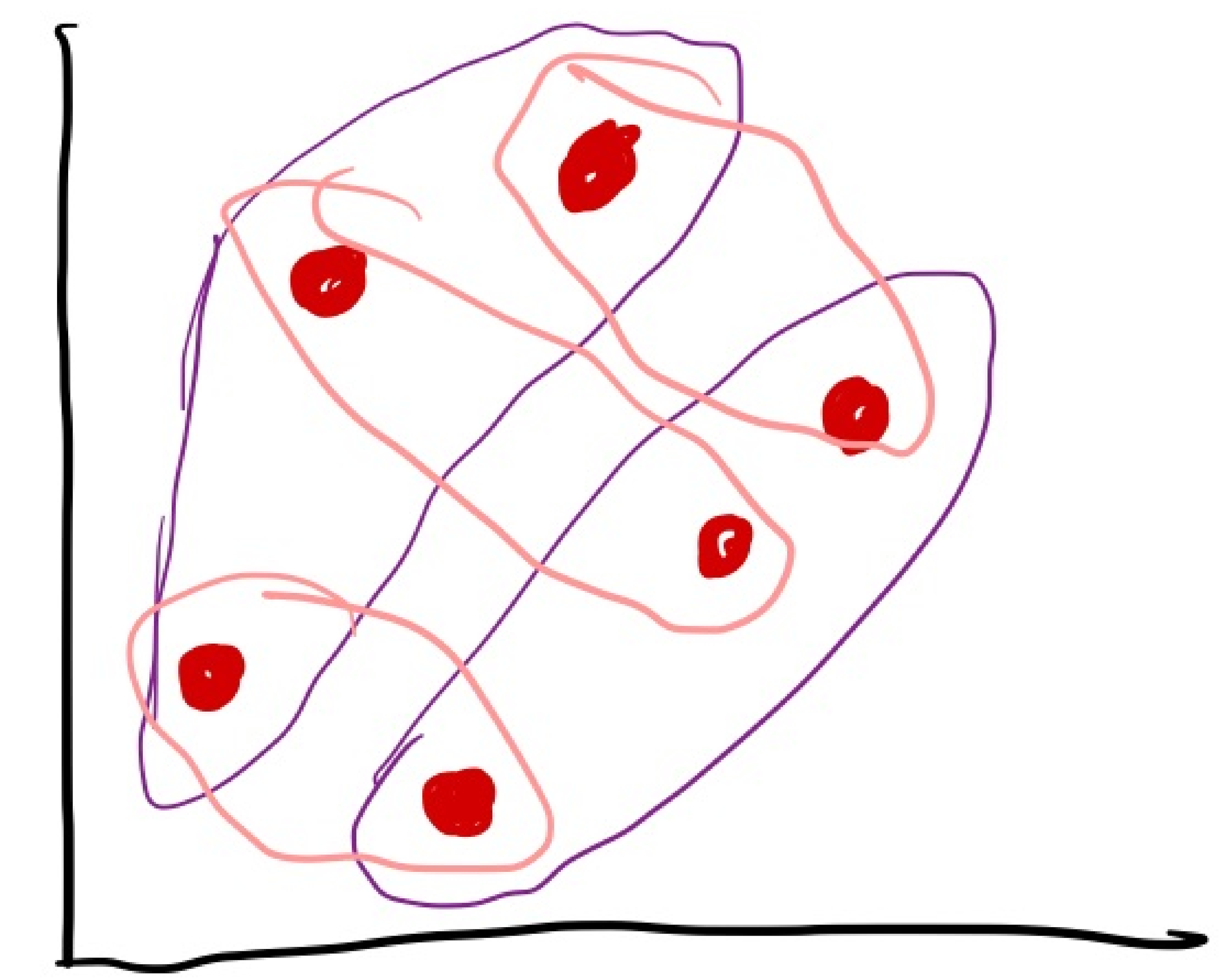
Basic question: Given a permutation $\sigma \in S_n$ (or more generally a sequence of length n) what is the length of the longest increasing/decreasing subsequence??

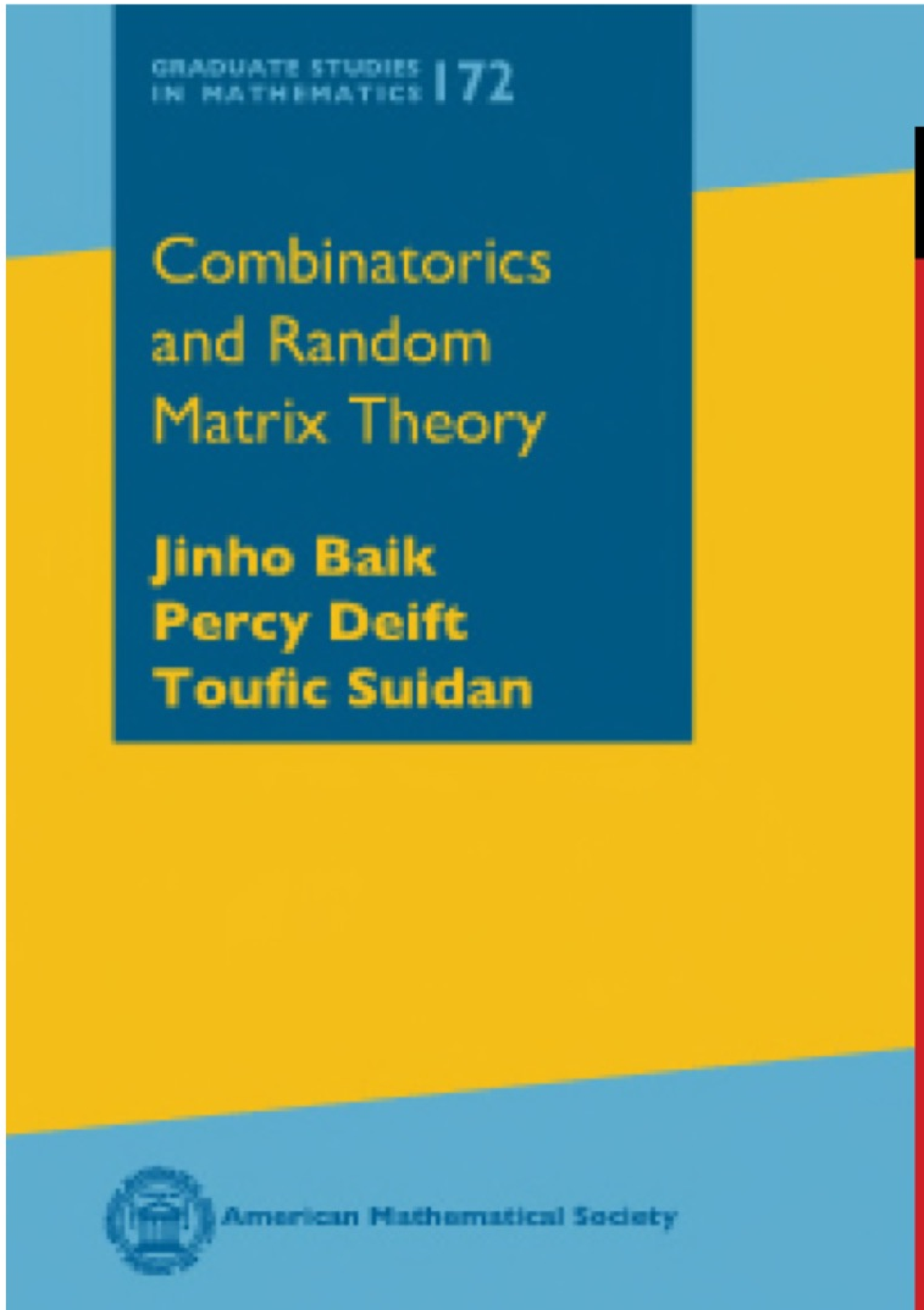
<u>Ex</u>: $n=6$, $\sigma = (2\ 5\ 1\ 6\ 3\ 4)$

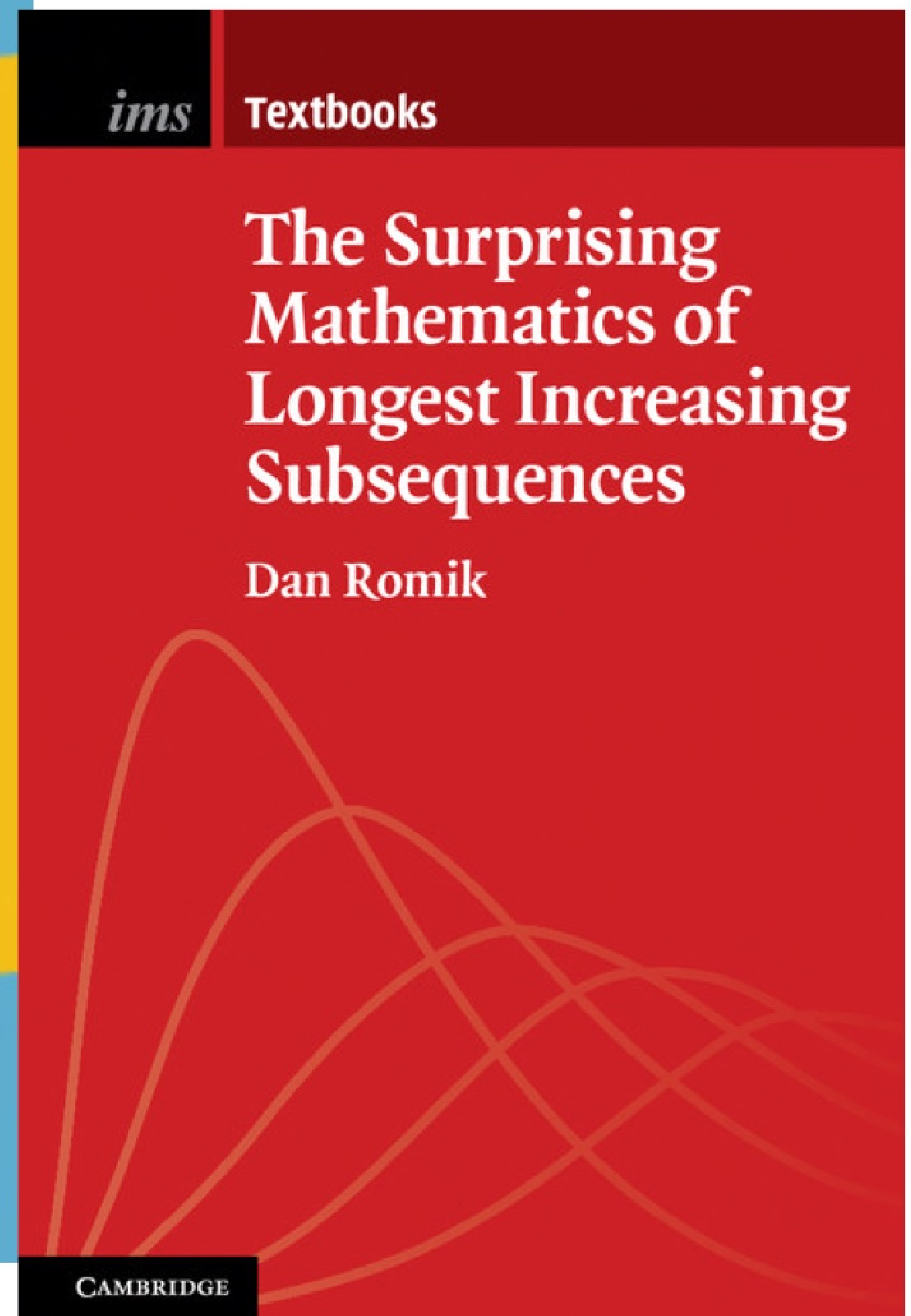
$LIS = 3$; $LDS = 2$



Erdős-Szekeres theorem: If $n > (r-1)(s-1)$ then $\sigma \in S_n$ has either an increasing subsequence of length r or a decreasing subsequence of length s.

Related question: Given a random permutation $\sigma \in S_n$, what is the distribution of the LIS?





Ulam '61

Hammersley '72

Logan-Shepp '77

Vershik-Kerov '77, '85

ℓ_n = LIS of random $\sigma \in S_n$

$$\lim_{n\to\infty} \frac{E(\ell_n)}{\sqrt{n}} = 2$$

Fluctuations: Baik-Deift-Johansson '99 proved

$$\frac{l_n - 2\sqrt{n}}{n^{1/6}} \xrightarrow{\text{dist.}} \chi_{GUE}, \text{ where } \chi_{GUE}$$

follows the Tracy-Widom GUE distribution describing the fluctuations of the largest eigenvalue of a complex Hermitian matrix with independent Gaussian entries (up to Hermitian symmetry).

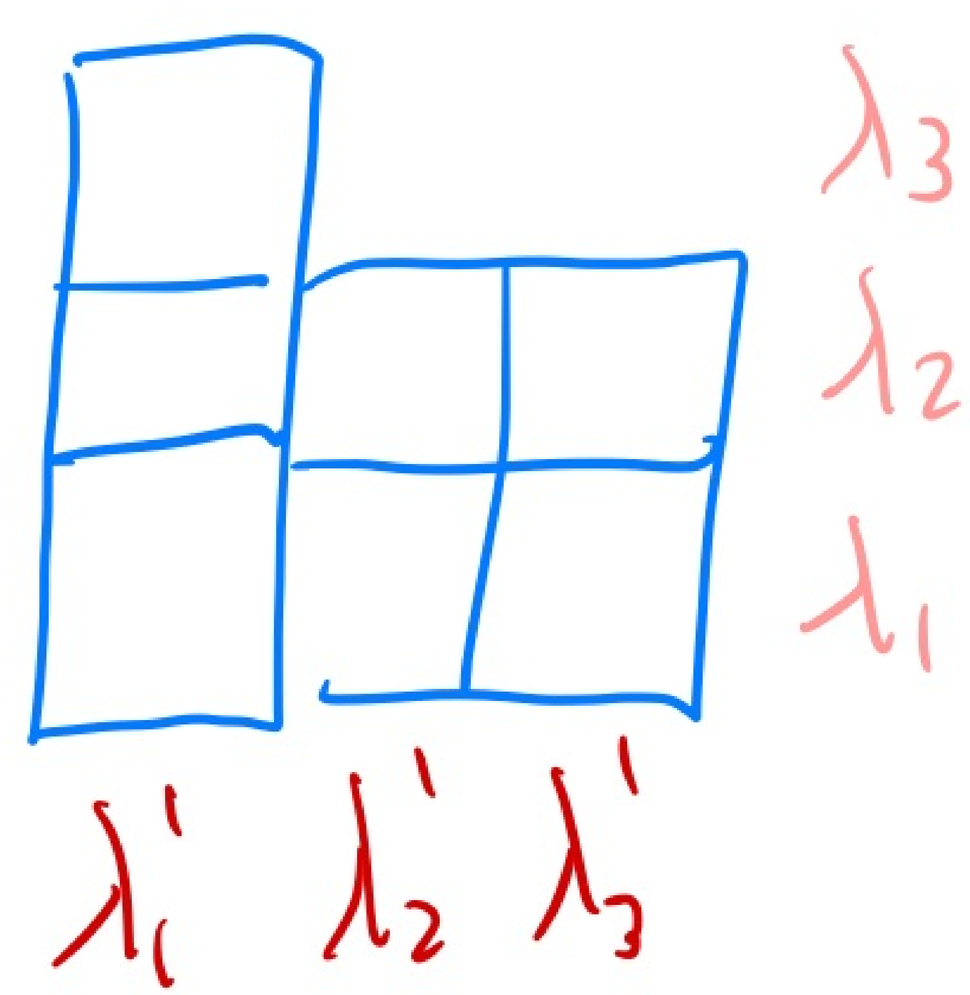
Their proof is based on a formula for a Poissonized version of the distribution of l_n as a Toeplitz determinant (following earlier work of Gessel), and asymptotic analysis of that determinant.

Some finer questions about long increasing/decreasing subsequences accessible by looking at the Schensted shape of a permutation.

The Robinson-Schensted correspondence is a bijection between permutations in S_n and pairs of Standard Young Tableaux of the same shape, of size n.

$$\sigma = 7\,4\,1\,5\,2\,6\,3 \longleftrightarrow \left(\begin{array}{|c|c|c|} \hline 7 & & \\ \hline 4 & 5 & 6 \\ \hline 1 & 2 & 3 \\ \hline \end{array} \;,\; \begin{array}{|c|c|c|} \hline 3 & & \\ \hline 2 & 5 & 7 \\ \hline 1 & 4 & 6 \\ \hline \end{array} \right)$$

Schensted shape is

$sh(\sigma) =$ 

$$= \lambda = (\lambda_1, \lambda_2, \lambda_3)$$

$$(3\ 3\ 1) \vdash n = 7$$

Schensted (1961) proved: $\lambda_1(\sigma) = LIS(\sigma)$ (arm(λ))

$\lambda_1'(\sigma) = LDS(\sigma)$ (leg(λ))

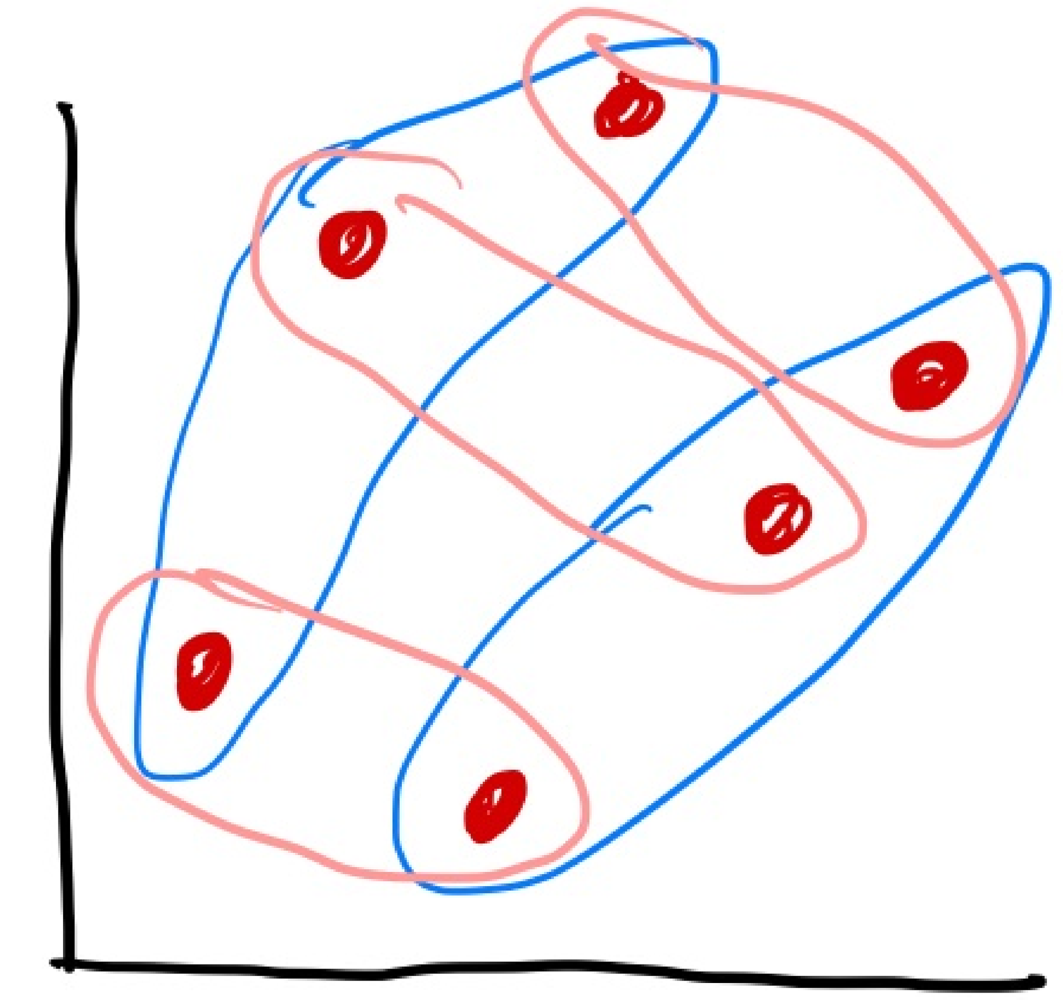
Greene (1974) proved: $\lambda_1 + \lambda_2 + \cdots + \lambda_k = I_k$

$\lambda_1' + \lambda_2' + \cdots + \lambda_k' = D_k$

where I_k = size of largest subsequence which is the union of k disjoint increasing subsequences $\equiv$ largest subsequence with no decreasing subsequence of length $k+1$.

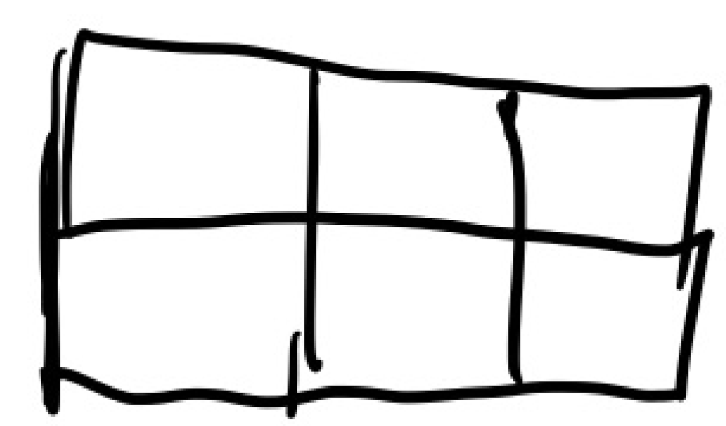
D_k = size of largest subsequence which is the union of k disjoint decreasing subsequences $\equiv$ largest subsequence with no increasing subsequence of length $k+1$.

Ex: $\sigma = (2\ 5\ 1\ 6\ 3\ 4)$



$\lambda_1 = 3$; $\lambda_1 + \lambda_2 = 6$

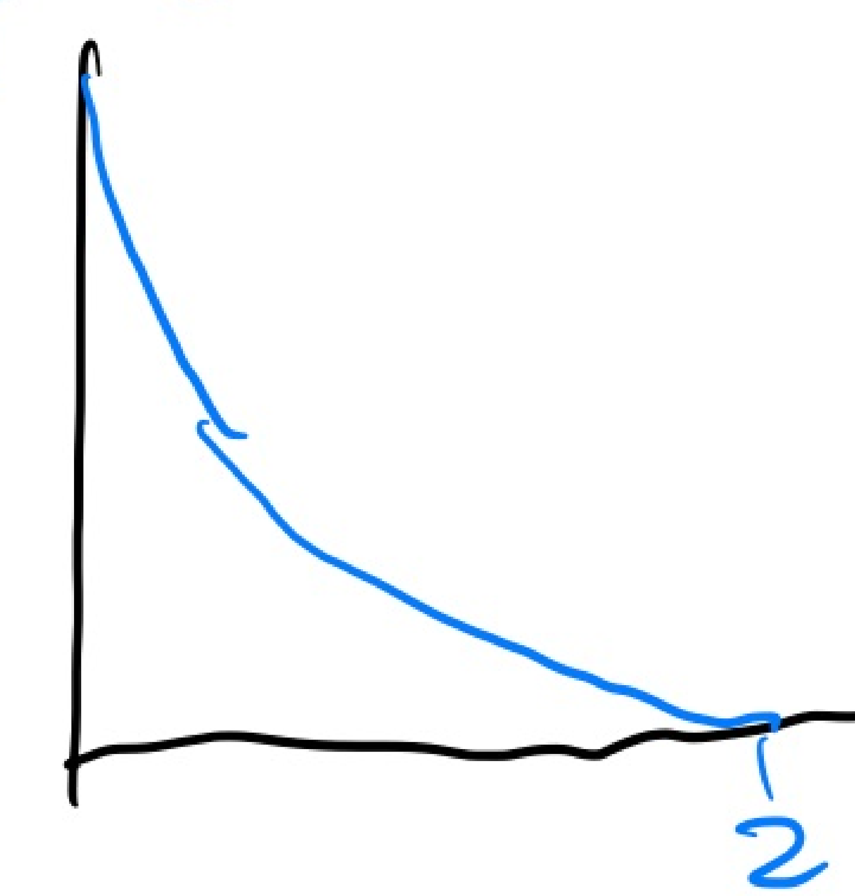
$\lambda_1' = 2$; $\lambda_1' + \lambda_2' = 4$; $\lambda_1' + \lambda_2' + \lambda_3' = 6$

$sh(\sigma) =$ 

Thm (Vershik-Kerov '77, Logan-Shepp '77)

If boxes are of size $\frac{1}{\sqrt{n}}$, the Schensted shape of a random permutation converges as $n \to \infty$ to the curve

$$x+y = \frac{2}{\pi}\left((x-y)\,\mathrm{ArcSin}\left(\frac{x-y}{2}\right) + \sqrt{4-(x-y)^2}\right)$$



Pseudo-random sequences are deterministic sequences which "look random."

What does it mean to "look random"? Let's restrict to sequences $(x_1, x_2, \ldots)$; $x_j \in (0,1)$.

Deterministic Simulation of Random Processes 1963

By Joel N. Franklin

1. **Introduction.** For many problems in engineering, economics, mathematics, and the sciences we are required to simulate random processes. The simulation is usually effected by a computer program which generates a non-random deter

Some criteria

① Equidistribution: $\lim_{N\to\infty} \frac{1}{N} \cdot \#\{x_j \mid a < x_j < b \text{ and } 1 \le j \le N\} = b-a$ for all $0 < a < b < 1$.

② Equidistribution by k's: The sequence $(z^{(1)}, z^{(2)}, z^{(3)}, \ldots)$ is equidistributed in the k-dimensional unit cube, where $z^{(n)} = (x_n, x_{n+1}, \ldots, x_{n+k})$.

③ Equipartitioned by k's:

$$\lim_{N\to\infty} \frac{1}{N} \cdot \#\{z^{(j)} \mid 1 \le j \le N \text{ and } z^{(j)} \text{ has the order } \sigma \in S_k\} = \frac{1}{k!}$$

for any $\sigma \in S_k$.

④ If equidistributed, we can also look at correlations:

$$R(\tau) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \left(x_j - \frac{1}{2}\right)\left(x_{j+\tau} - \frac{1}{2}\right) \quad \text{for } \tau = 1, 2, \ldots$$

If $R(\tau) = 0$ for all τ, we say sequence is <u>White</u>.

<u>Open Question</u>: Are any of these criteria necessary and/or sufficient to guarantee that

$$\lim_{n\to\infty} \frac{LIS(x_1, x_2, \ldots, x_n)}{\sqrt{n}} = \text{Constant} \quad ??$$

What about the Schensted shape of $(x_1, \ldots, x_n)$? Does it converge to a limit shape??

Generating pseudo-random sequences:

One easy way is to take the sequence

$(f(1), f(2), f(3), \ldots, f(n))$ where $f(i) = p(i) \bmod 1$

and $p(i)$ is any function you like!

These sequences are explored in Franklin '62 for

$p(i) =$ polynomial in i

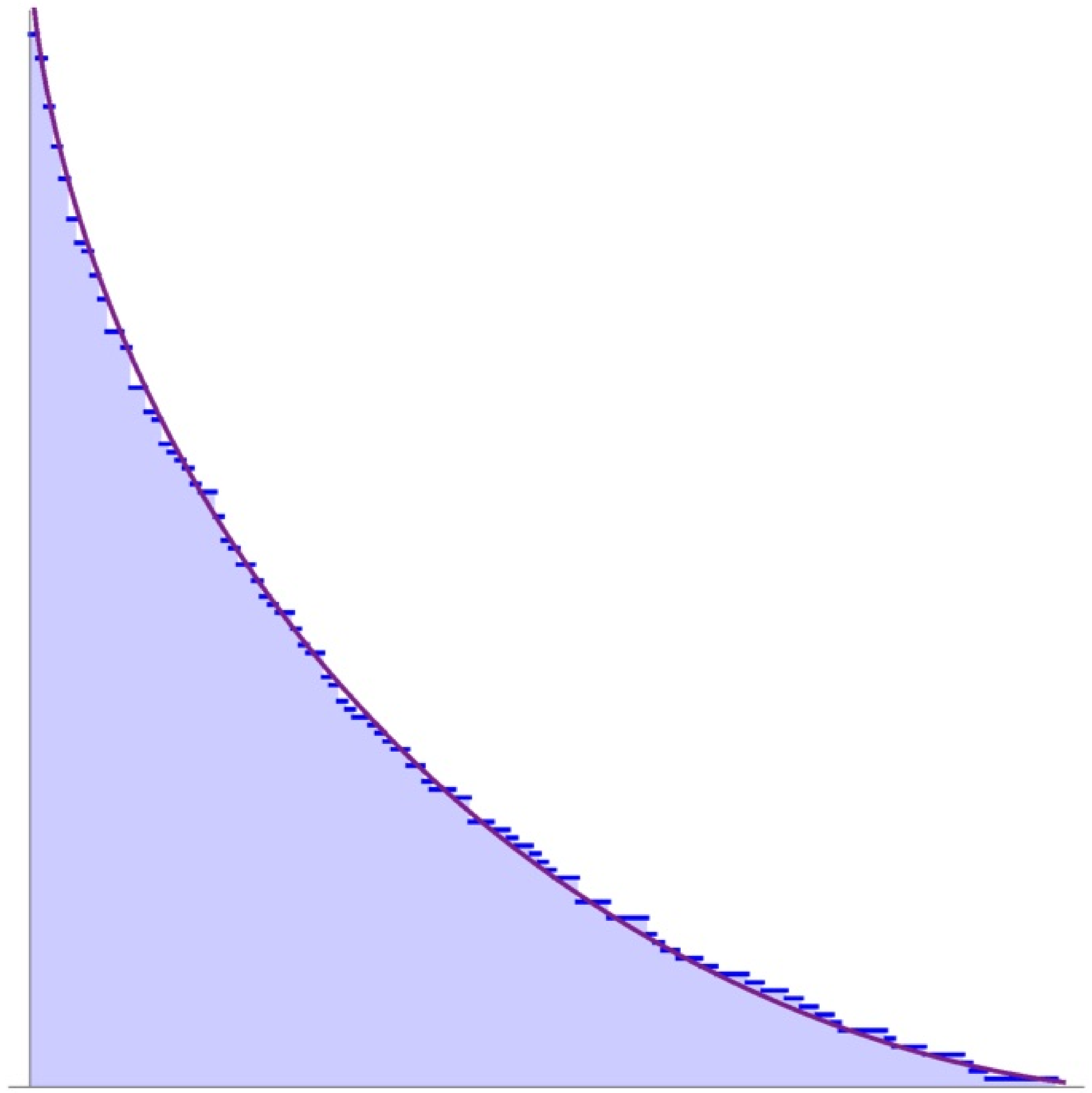
equidistributed by k's for all $k \leq \deg p$

equidistributed by k's $\Longrightarrow$ equipartitioned by k's

white for $\deg p \geq 2$

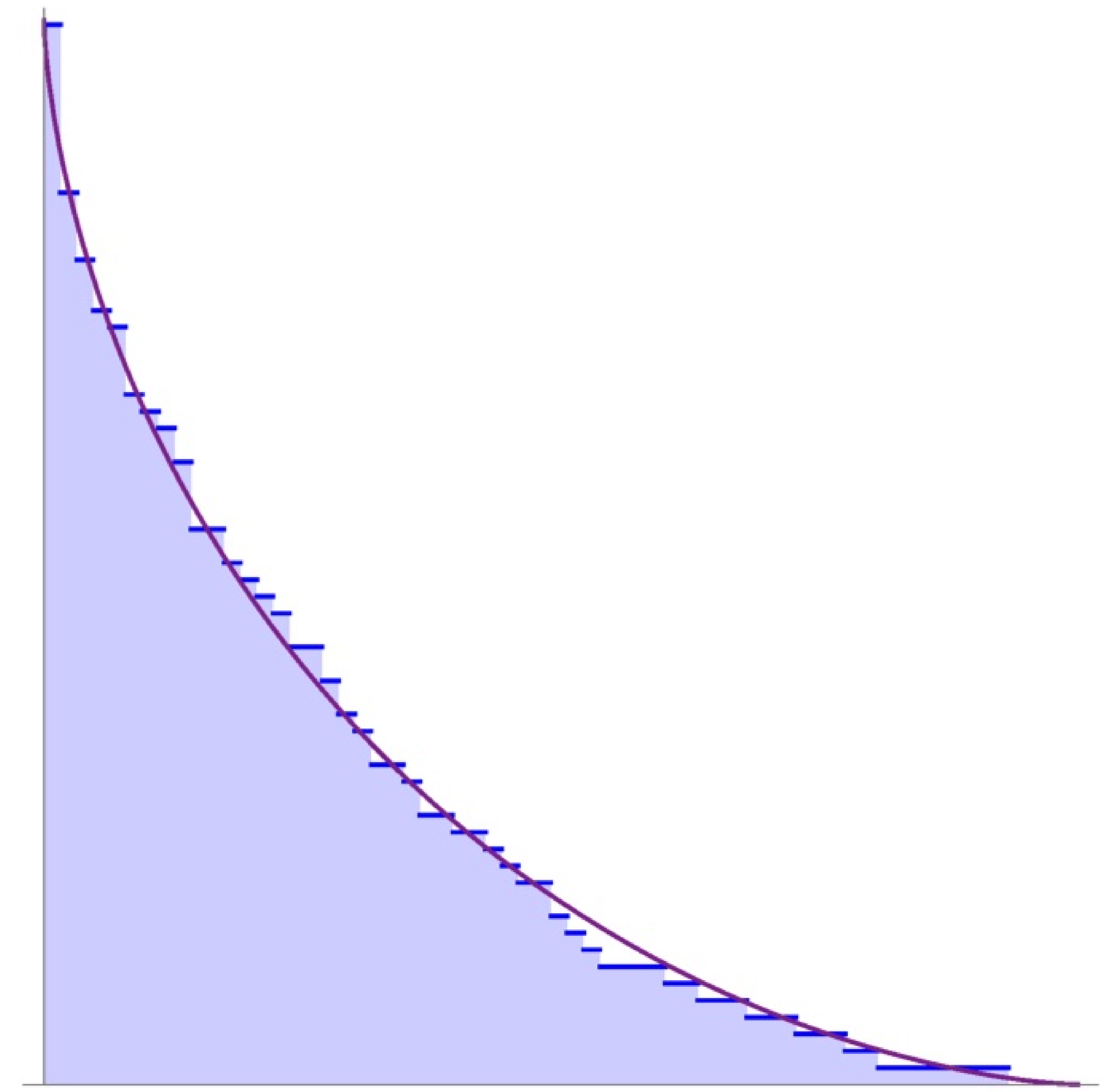
$p(i) = \theta^i$ completely equidistributed for a.e. θ

not completely equidistributed for any algebraic θ



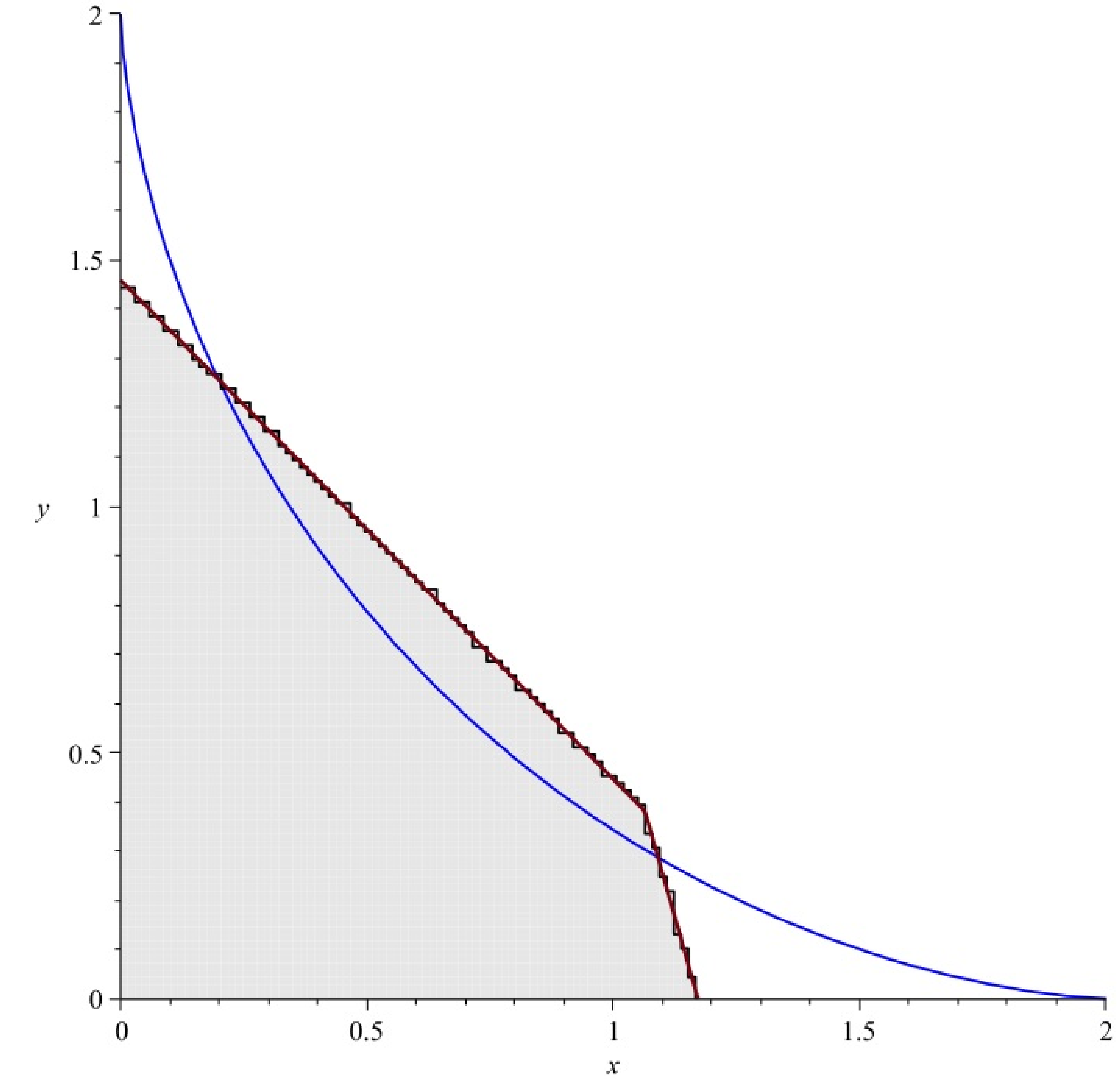
$n = 5000$

$p(i) = e \cdot i^2$



$n = 1000$

$p(i) = (\pi - 2.12)^i$



$n = 4700$

$p(i) = e \cdot i$

n	arm$/\sqrt{n}$	leg$/\sqrt{n}$
2000	1.8025	1.83358
4000	1.85594	1.91318
5000	1.8809	1.85262
10000	1.93	1.86

Weyl sequences and Sós permutations

For a fixed irrational number $\alpha \in (0,1)$ we can look at the sequence $(\alpha i \bmod 1)_{i=1}^{n}$, as well as the induced permutation, $w(n,\alpha)$

For $n=7$, $\alpha=0.3$, $(0.3, 0.6, 0.9, 0.2, 0.5, 0.8, 0.1) \longrightarrow 3\,5\,7\,2\,4\,6\,1 = w(n,\alpha)$

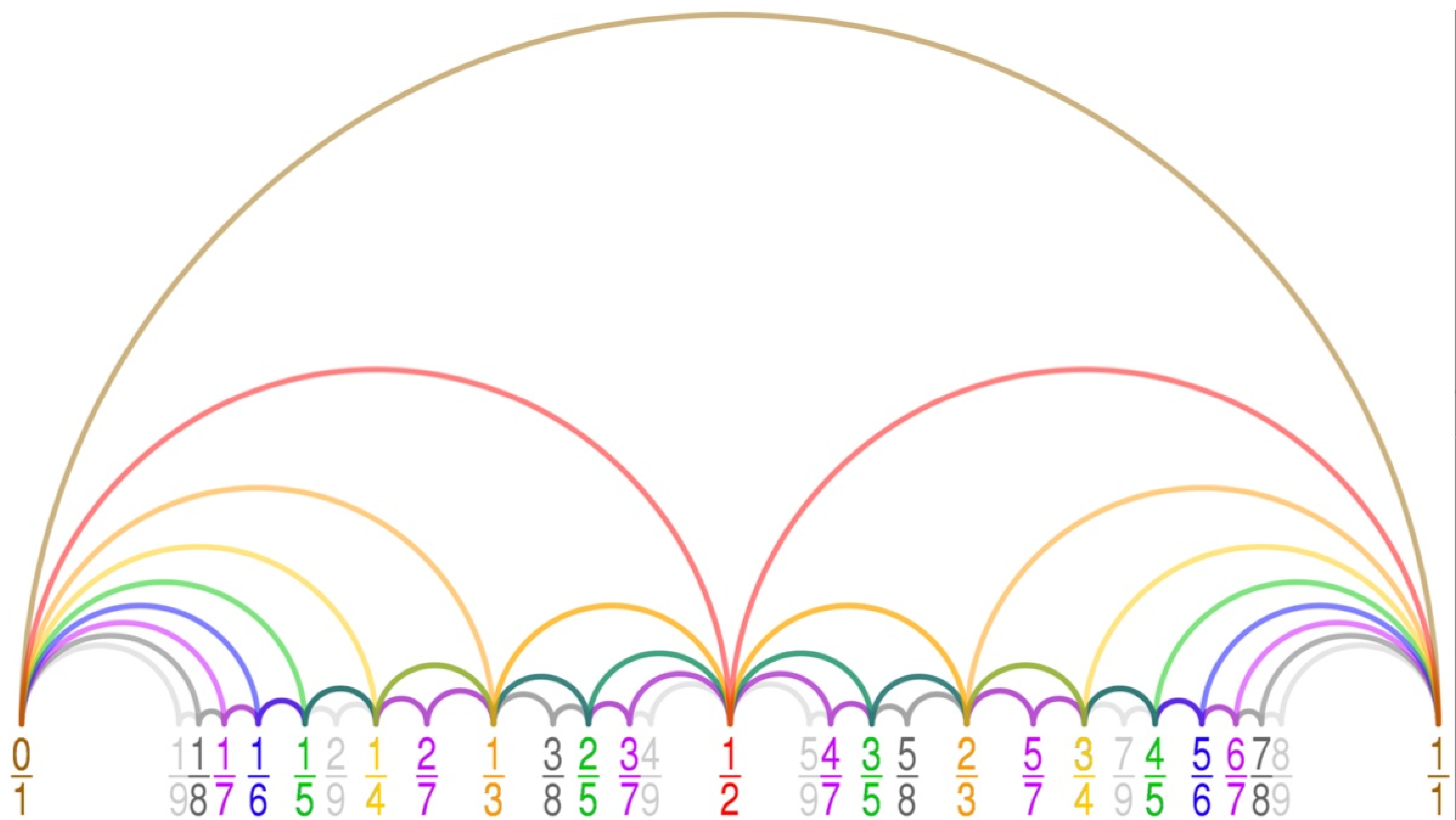
These sequences (sometimes called Weyl sequences) are the subject of a the famous 3-gap theorem which states $\left\{e^{2\pi i f(j)}\right\}_{j=1}^{n}$; $f(j)=\alpha j \bmod 1$

always divides the unit circle into $(n-1)$ pieces of at most 3 distinct lengths. (Steinhaus conjecture)

Proved by Świerczkowski, Surányi, and Sós independently (then many others)

Surányi's Bijection: $w(n,\alpha) = w(n,\alpha')$ if and only if α and α' are in the same Farey interval of order n

Farey intervals divide $(0,1)$ into subintervals whose endpoints are reduced fractions with denominator at most n.



The number of Farey intervals grows like $\sim \frac{3n^2}{\pi^2}$, so

$$\#\left\{ w(n,\alpha) \;\middle|\; 0 < \alpha < 1 \right\} \sim \frac{3n^2}{\pi^2}$$

Sós structure: Sós showed if α is in the Farey interval $\left[\frac{a}{b}, \frac{c}{d}\right)$, then for $\pi^{-1} = w(n,\alpha)$

$$\pi(1) = b \quad \text{and}$$

$$\pi(i+1) - \pi(i) = \begin{cases} b & \text{if } \pi(i) \leq n-b \\ b-d & \text{if } n-b < \pi(i) \\ -d & \text{if } d \leq \pi(i) \end{cases}$$

i.e. only 3 possible "jumps" are allowed.

We call $\{w(n,\alpha)\}$ the Sós permutations and denote them $Sós_n$.

Bockting-Conrad, Sarah (1-DPL); Kashina, Yevgenia (1-DPL); Petersen, T. Kyle (1-DPL); Tenner, Bridget Eileen (1-DPL)
Sós permutations. (English summary)
Amer. Math. Monthly 128 (2021), no. 5, 407–422.
05A05 (11B05 11B57 11K06)

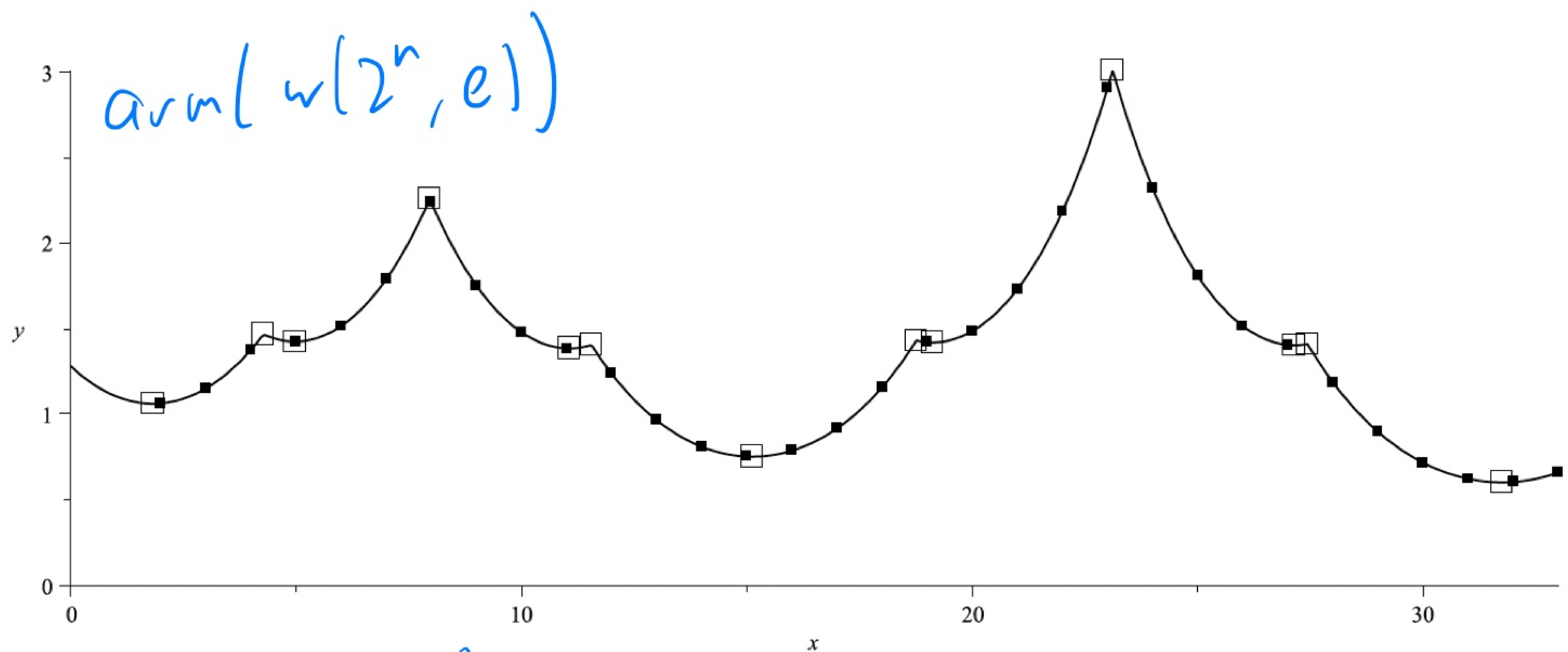
The question of the longest increasing/decreasing subsequence of a SSS permutation was addressed by Boyd and Steele (1979).

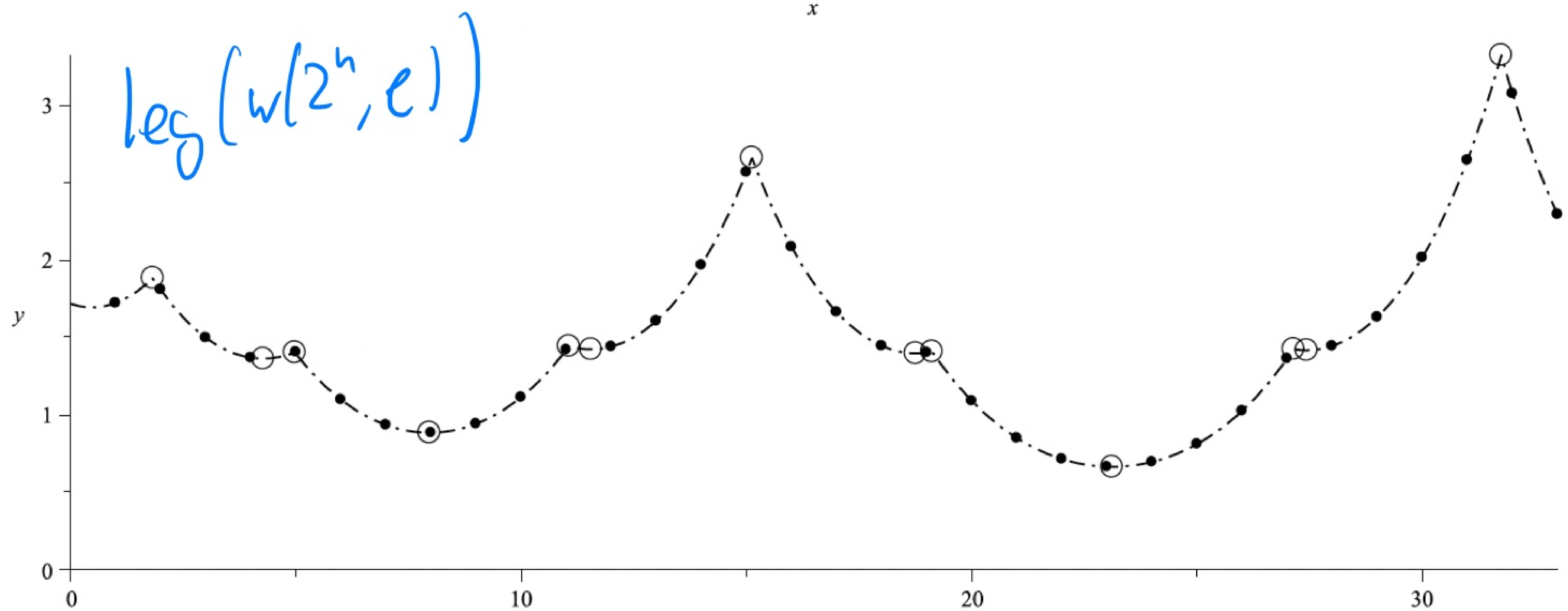
They proved $\lim_{n\to\infty} \frac{\operatorname{arm}(n,\alpha)}{\sqrt{n}}$ and $\lim_{n\to\infty} \frac{\operatorname{leg}(n,\alpha)}{\sqrt{n}}$

do not exist for any α. Instead these quantities oscillate in n such that

① A local min of $\frac{\operatorname{arm}(n,\alpha)}{\sqrt{n}}$ occurs exactly at a local max of $\frac{\operatorname{leg}(n,\alpha)}{\sqrt{n}}$, and vice versa

② For all n, α, $\frac{\operatorname{arm}(n,\alpha)}{\sqrt{n}} \cdot \frac{\operatorname{leg}(n,\alpha)}{\sqrt{n}} \leq 2$, with equality occurring exactly at the local extrema from ①.





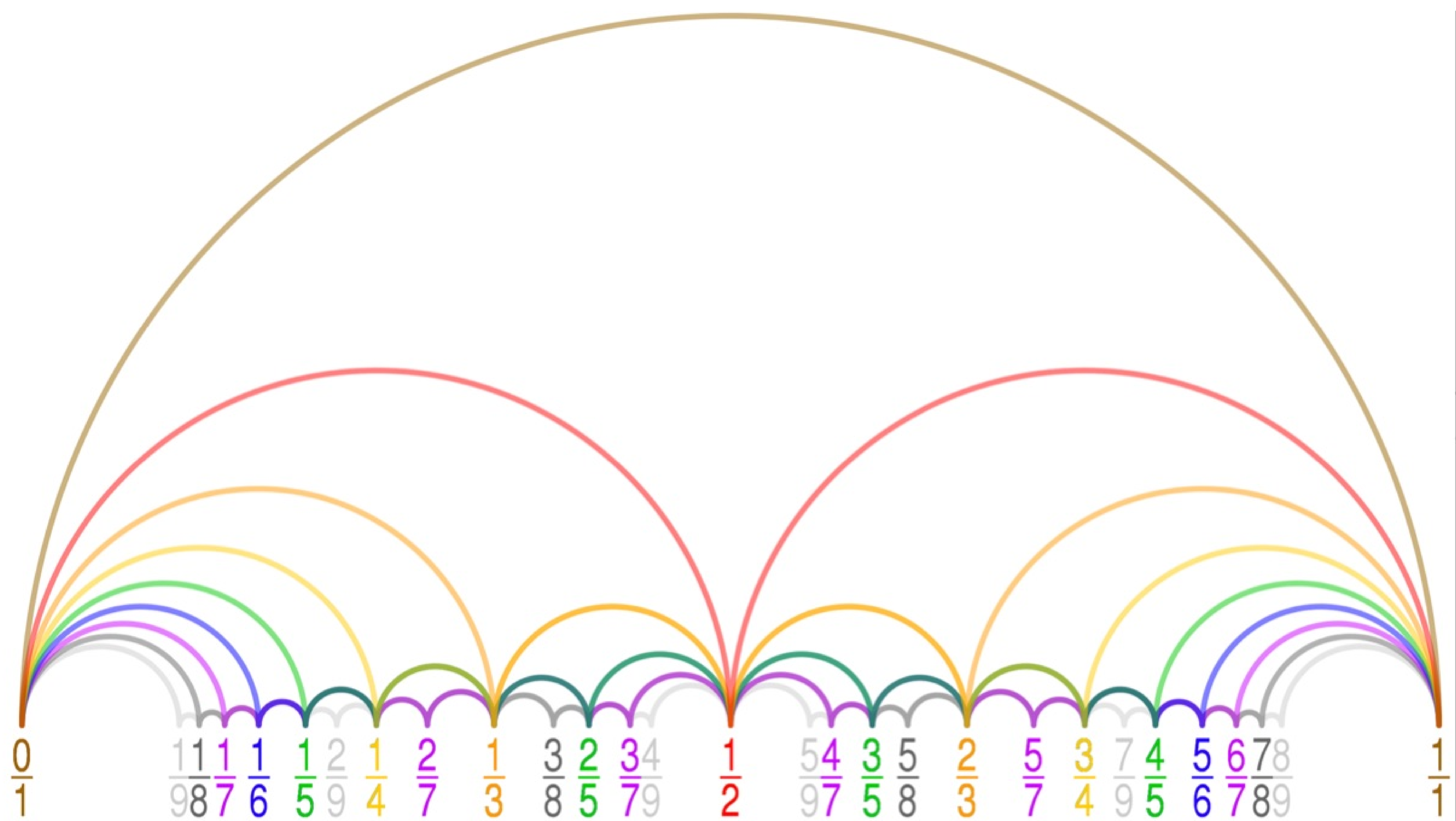
Very precise estimates are given in terms of the continued fraction expansion of α.

$$\alpha = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_1, a_2, \ldots]$$

The k^{th} principal convergent of α is

$$[a_1, a_2, \ldots, a_k] = \frac{p_k}{q_k}$$

Recursion: $p_i = p_{i-2} + a_i p_{i-1}$; $q_i = q_{i-2} + a_i q_{i-1}$



Intermediate convergents:

$$\frac{p_{i,j}}{q_{i,j}} = [a_1, a_2, \ldots, a_{i-1}, j]$$

for $1 \leq j < a_i$

Walks on the Farey tree:

$$[a_1, \ldots, a_j + 1] = [a_1, \ldots, a_j, 1]$$

$\downarrow$ Right $\qquad \downarrow$ Left

$[a_1, \ldots, a_j + 1, 1] \qquad [a_1, \ldots, a_j, 1, 1]$

Boyd and Steele's result (1979) Let $\alpha \in (0,1)$ have principal convergents $\left\{ \frac{p_k}{q_k} \right\}_{k=1}^{\infty}$. Denote $\delta_k = \left| \alpha - \frac{p_k}{q_k} \right|$.

For $\delta_{2h}^{-1} \leq n < \delta_{2h+2}^{-1}$, we have

$$q_{2h+1}(1 + n\delta_{2h+1}) - 2 < \operatorname{arm}(w(n,\alpha)) \leq q_{2h+1}(1 + n\delta_{2h+1})$$

and for $\delta_{2h-1}^{-1} \leq n < \delta_{2h+1}^{-1}$, we have

$$q_{2h}(1 + n\delta_{2h}) - 2 < \operatorname{leg}(w(n,\alpha)) \leq q_{2h}(1 + n\delta_{2h}).$$

Some implications: (1) $\frac{\mathrm{arm}(w(n,\alpha))}{\sqrt{n}}$ and $\frac{\mathrm{leg}(w(n,\alpha))}{\sqrt{n}}$ remain bounded if and only if α has bounded continued fraction expansion.

(2) They are periodic in n if and only if α has periodic continued fraction expansion, i.e., α solves a quadratic equation.

What about the full Schensted shape of $w(n,\alpha)$?

When $\frac{arm}{\sqrt{n}}$, $\frac{leg}{\sqrt{n}}$ are at local extrema,

a good guess is $sh(w(n,\alpha))$ is close to a triangle with side

lengths given by Boyd/Steele, and

recall $\frac{arm}{\sqrt{n}} \cdot \frac{leg}{\sqrt{n}} = 2$

at local extrema.

Theorem: L-Petersen (2021) (2-slope theorem)

For any $n \in \mathbb{N}$, let $\alpha \in (0,1)$ have principal convergents $\left\{ \frac{p_k}{q_k} \right\}_{k=1}^{\infty}$.

Again, $\delta_k = \left| \alpha - \frac{p_k}{q_k} \right|$. Choose h such that $\delta_{2h}^{-1} \le n \le \delta_{2h+2}^{-1}$.

and define the lines

$$L_1 : \; y = q_{2h+*}\left(1 + n\,\delta_{2h+*}\right) - 2 q_{2h+*}^2\, \delta_{2h+*}\, x$$

$$L_2 : \; x = q_{2h+1}\left(1 + n\,\delta_{2h+1}\right) - 2 q_{2h+1}^2\, \delta_{2h+1}\, y$$

where
$$* = \begin{cases} 0 & \text{if } \delta_{2h}^{-1} \le n \le \delta_{2h+1}^{-1} \\ 2 & \text{if } \delta_{2h+1}^{-1} \le n \le \delta_{2h+2}^{-1} \end{cases}$$

For $0 \le x \le q_{2h+1}(1+n\delta_{2h+1})$ let

$$L(x;n,\alpha) = \min(L_1(x), L_2(x))$$

Then $\operatorname{dist}(\partial(sh(w(n,\alpha))), L(x;n,\alpha)) \le 8.$

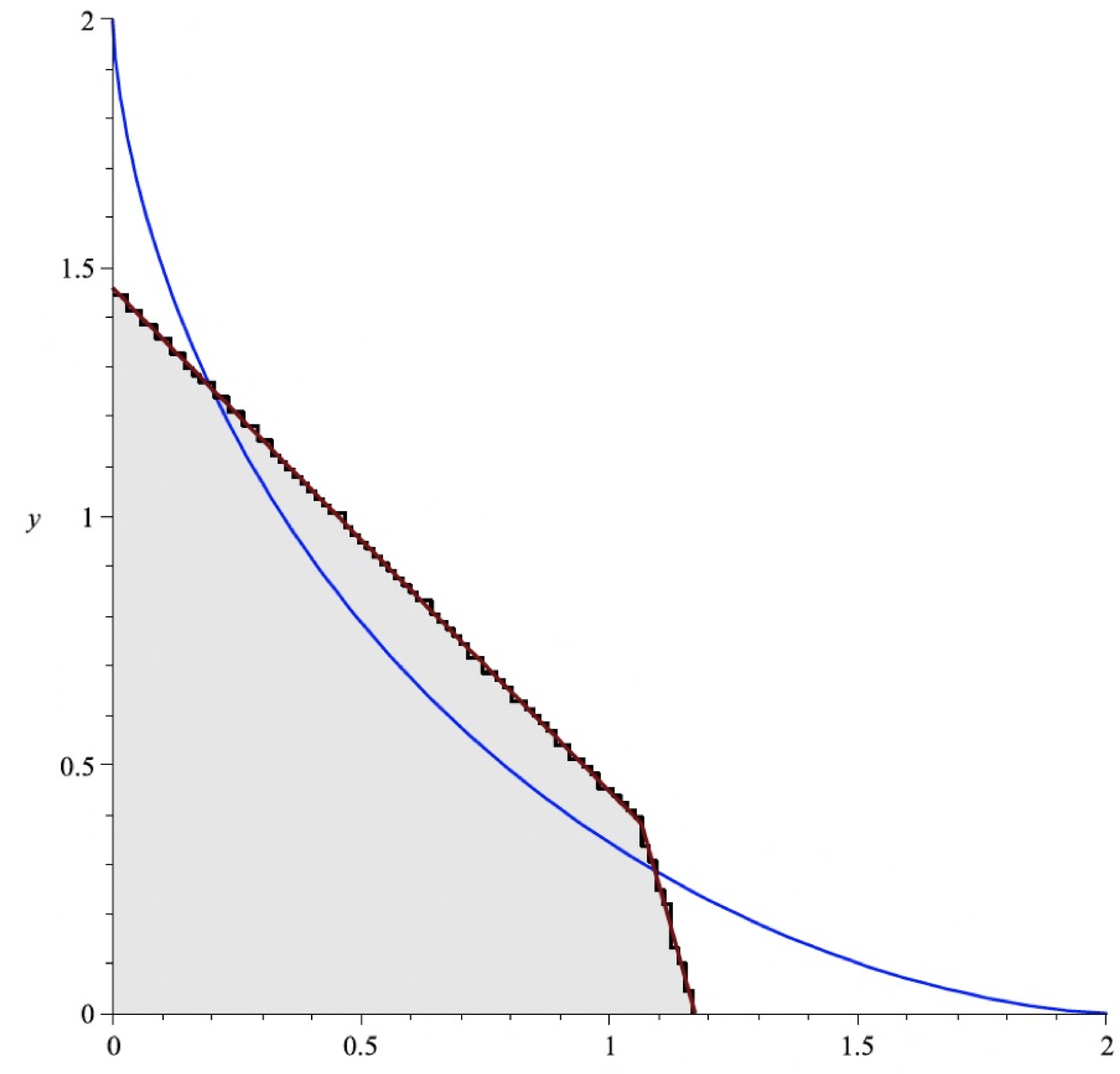
Note: slopes are piecewise constant in n.

Movies for $\alpha =$

$\pi = [3; 7, 15, 1, 292, 1, 1, 1, 2, \dots]$

$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \dots]$

$\sqrt{2} = [1; 2, 2, 2, 2, 2, 2, 2, \dots]$



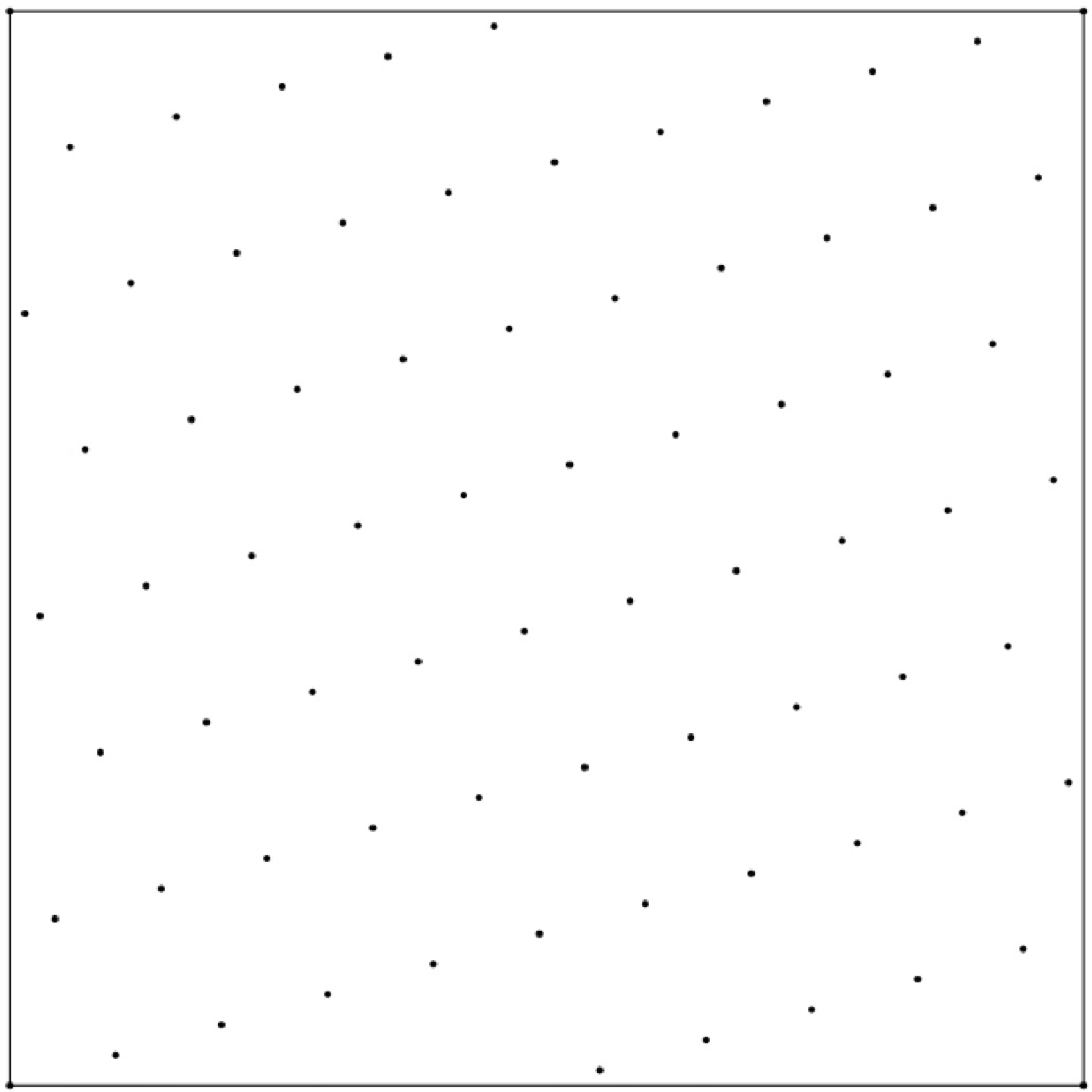
Sketch of the proof of the 2-slope theorem

Step 1: Choose $N > n$, and approximate α by a rational number a/N. A good choice is $N = q_m$ for some convergent p_m/q_m.

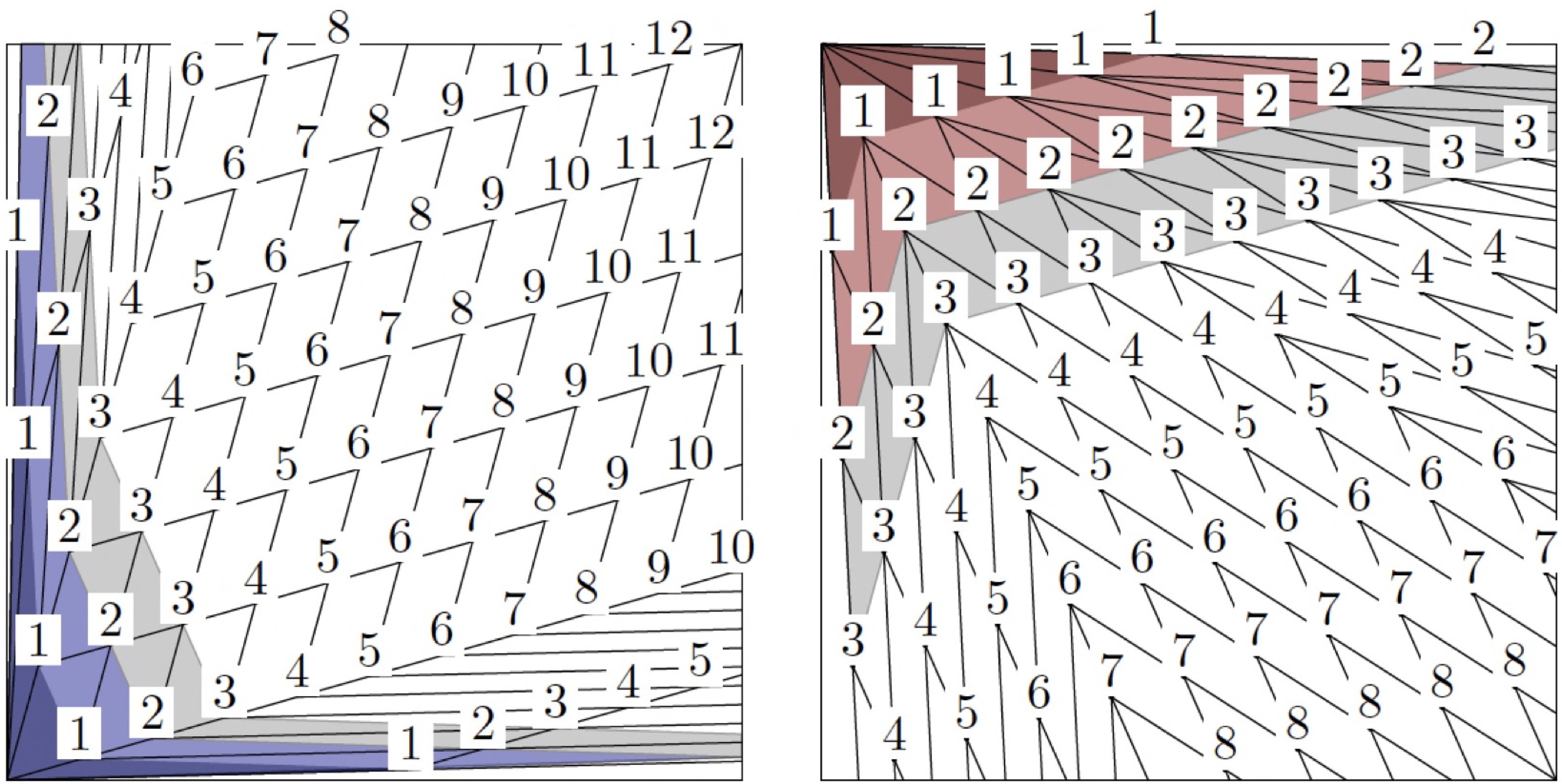
Notice that $w(n,\alpha) = w(n, \frac{a}{N})$, and that the order structure of $w(n, \frac{a}{N})$ is the same as that of $\left(a \cdot i \mod N\right)_{i=1}^{n}$.

Step 2: Note that $\{(i, a \cdot i \mod N)\}$ forms a lattice on $\mathbb{T}^2$. Exploit lattice structure to find long increasing and decreasing paths.

Ex: The permutation $w(n, \frac{51}{71})$ is described for $n<71$ by the lattice $L_{51,71} = \{(i, 51 \cdot i \bmod 71)\}$. Increasing subsequences of $w(n, \frac{51}{71})$ correspond to up-right lattice paths, and decreasing subsequences to down-right lattice paths.



The *lattice length* of an up-right vector $\vec{x}$ is the maximum number of up-right vectors that can be summed to $\vec{x}$.
For down-right vectors there is a similar definition in terms of decomposition into down-right vectors.



Clearly
LIS = max lattice length in $[0,n] \times [0,N]$

LDS = max lattice length in $[0,n] \times [0,-N]$.

The most important vectors for constructing long increasing/decreasing paths are the ones with lattice length 1. Call them **unit increasing/decreasing lattice vectors.** Order them according to slope:

$\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_d$ unit increasing vectors

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_e$ unit decreasing vectors

$$m_{\vec{u}_1} > m_{\vec{u}_2} > \cdots > m_{\vec{u}_d} > 0 > m_{\vec{v}_e} > \cdots > m_{\vec{v}_2} > m_{\vec{v}_1}$$

$$\vec{u}_1 = \langle 1, a \rangle \quad ; \quad \vec{v}_1 = \langle 1, -(N-a) \rangle$$

Note: $\vec{u}_{i+1}-\vec{u}_i=\vec{v}_j$ for some j, and similarly for $\vec{v}_{i+1}-\vec{v}_i=\vec{u}_j$.

Computing unit lattice vectors for $L_{51,71}$

$\vec{u}_0=\langle 0,71\rangle$; $\vec{u}_1=\langle 1,51\rangle$

$\vec{u}_1-\vec{u}_0=\langle 1,-20\rangle=\vec{v}_1$

$\vec{v}_1+\vec{u}_1=\langle 2,31\rangle=\vec{u}_2$

$\vec{u}_2+\vec{v}_1=\langle 3,11\rangle=\vec{u}_3$

$u_3+\vec{v}_1=\langle 4,-9\rangle=\vec{v}_2$

$\vec{v}_2+\vec{u}_3=\langle 7,2\rangle=\vec{u}_4$

$\vec{u}_4+\vec{v}_2=\langle 11,-7\rangle=\vec{v}_3$

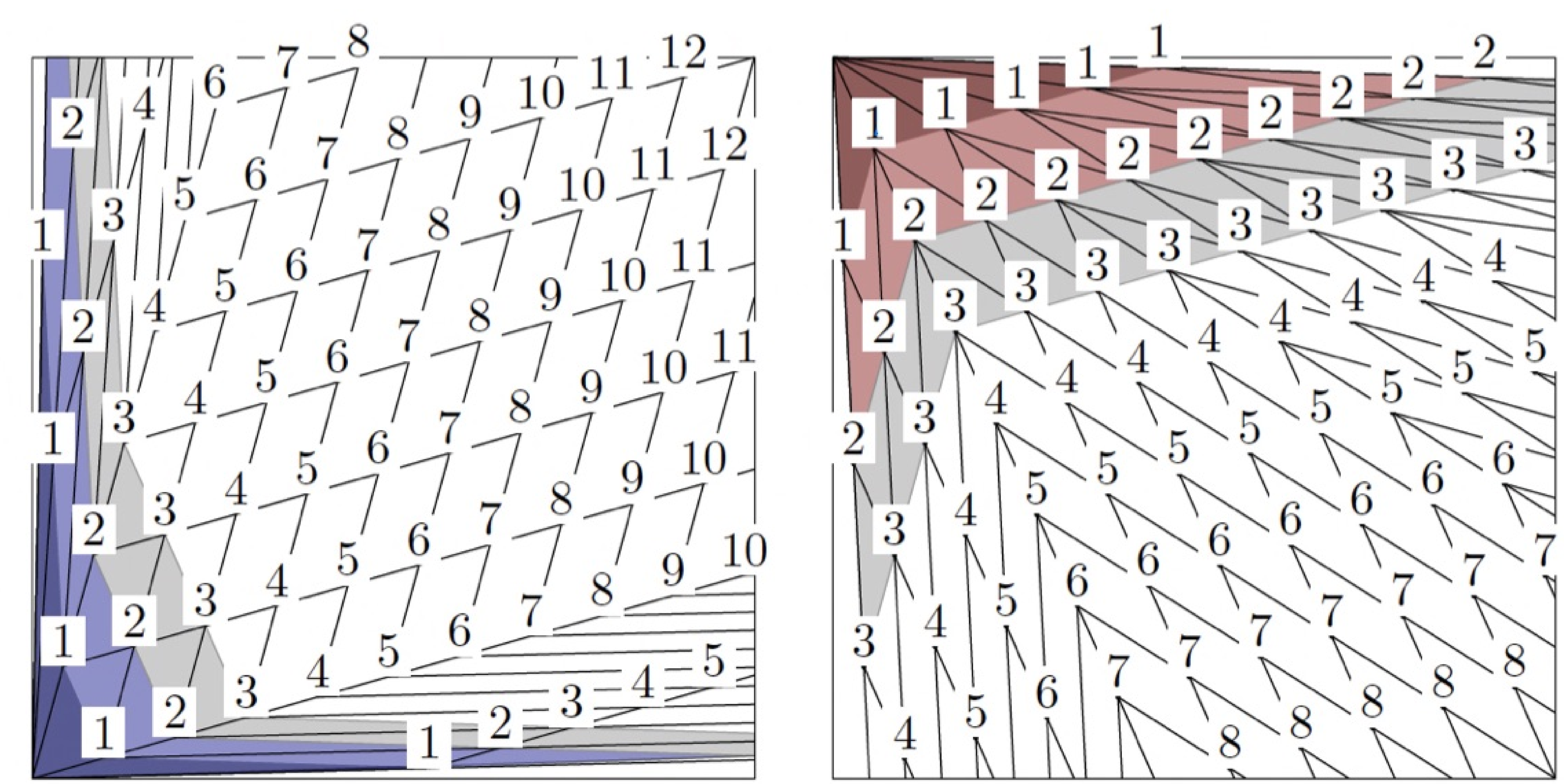
$\vec{v}_3+\vec{u}_4=\langle 18,-5\rangle=\vec{v}_4$

$\vec{v}_4+\vec{u}_4=\langle 25,-3\rangle=\vec{v}_5$

$\vec{v}_5+\vec{u}_4=\langle 32,-1\rangle=\vec{v}_6$

$\vec{v}_6+\vec{u}_4=\langle 39,1\rangle=\vec{u}_7$

$\vec{u}_7+\vec{v}_6=\langle 71,0\rangle$



Looks like Euclidean algorithm, except we subtract rather than divide at each step. Call it the slow Euclidean algorithm.

i	j	a_i	$r_{i,j}$	$s_{i,j}$	$t_{i,j}$	$\mathbf{u}$	$\mathbf{v}$
-1	1		71	1	0		
0	1		**51**	0	**1**	$\mathbf{u}_1 = (1, 51)$	
1	1	1	**20**	1	$\mathbf{-1}$		$\mathbf{v}_1 = (1, -20)$
2	1		**31**	-1	**2**	$\mathbf{u}_2 = (2, 31)$	
	2	2	**11**	-2	**3**	$\mathbf{u}_3 = (3, 11)$	
3	1	1	**9**	3	$\mathbf{-4}$		$\mathbf{v}_2 = (4, -9)$
4	1	1	**2**	-5	**7**	$\mathbf{u}_4 = (7, 2)$	
5	1		**7**	8	$\mathbf{-11}$		$\mathbf{v}_3 = (11, -7)$
	2		**5**	13	$\mathbf{-18}$		$\mathbf{v}_4 = (18, -5)$
	3		**3**	18	$\mathbf{-25}$		$\mathbf{v}_5 = (25, -3)$
	4	4	**1**	23	$\mathbf{-32}$		$\mathbf{v}_6 = (32, -1)$
6	1		**1**	-28	**39**	$\mathbf{u}_5 = (39, 1)$	
	2	2	0	-51	71		

At each step

$$r_{i,j} = s_{i,j} N + t_{i,j}\, a$$

Closely related to the intermediate convergents for $\frac{a}{N} = [a_1, a_2, \dots, a_k]$

$$\frac{p_i}{q_i} = [a_1, \dots, a_i] \quad (i \le k)$$

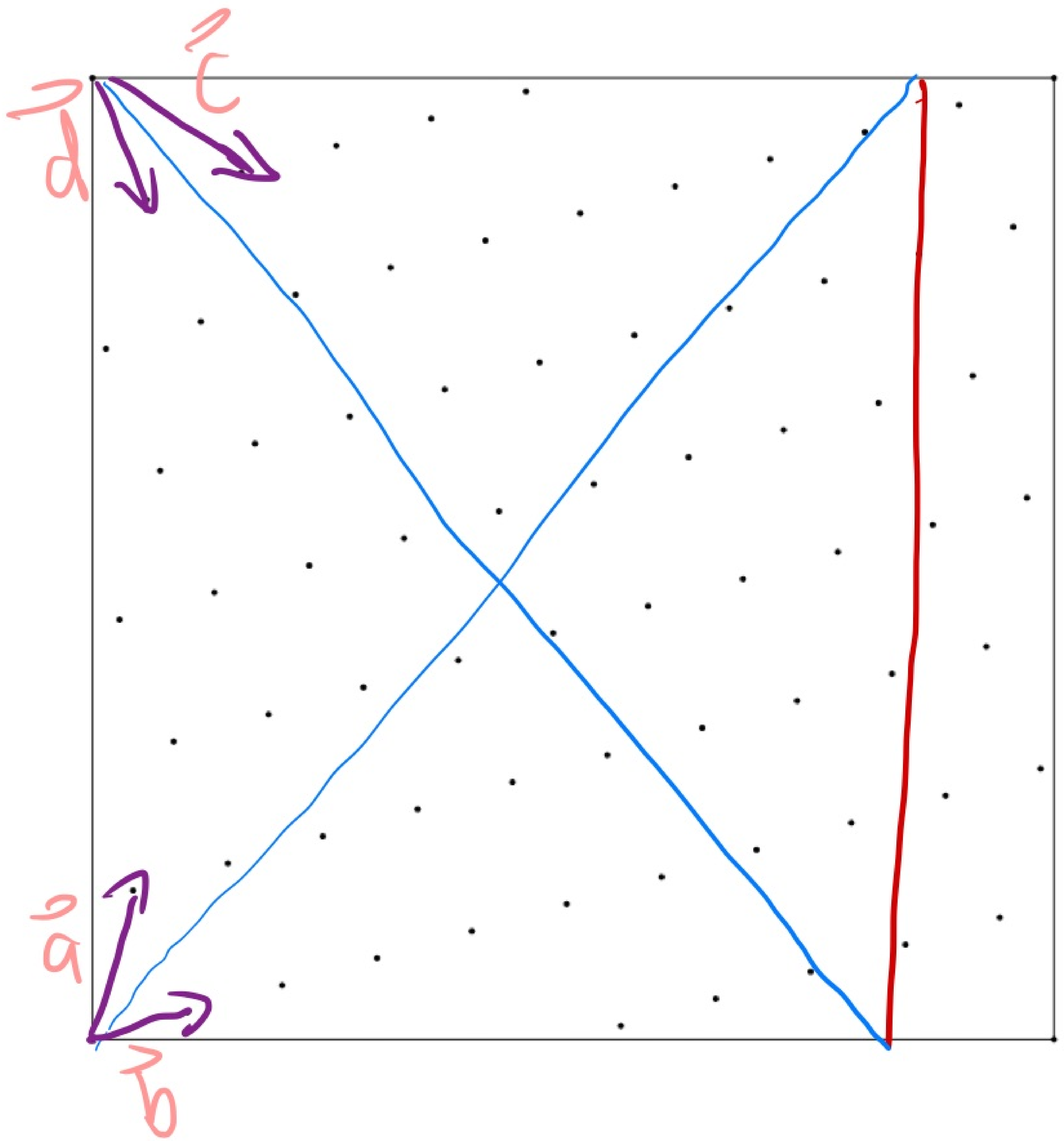
$$\frac{p_{i,j}}{q_{i,j}} = [a_1, \dots, a_{i-1}, j] \quad \text{for } j < a_i.$$

Fact:

$$r_{i,j} = N q_{i,j} \left| \frac{a}{N} - \frac{p_{i,j}}{q_{i,j}} \right|$$

$$t_{i,j} = \pm q_{i,j} \quad \text{for } i \text{ even (odd)}$$

Best unit lattice vectors



A believable fact: Long increasing/decreasing paths in $[0,n]\times[0,N]$ and $[0,n]\times[0,-N]$ can be constructed using the unit lattice vectors with slope closest to the aspect ratio

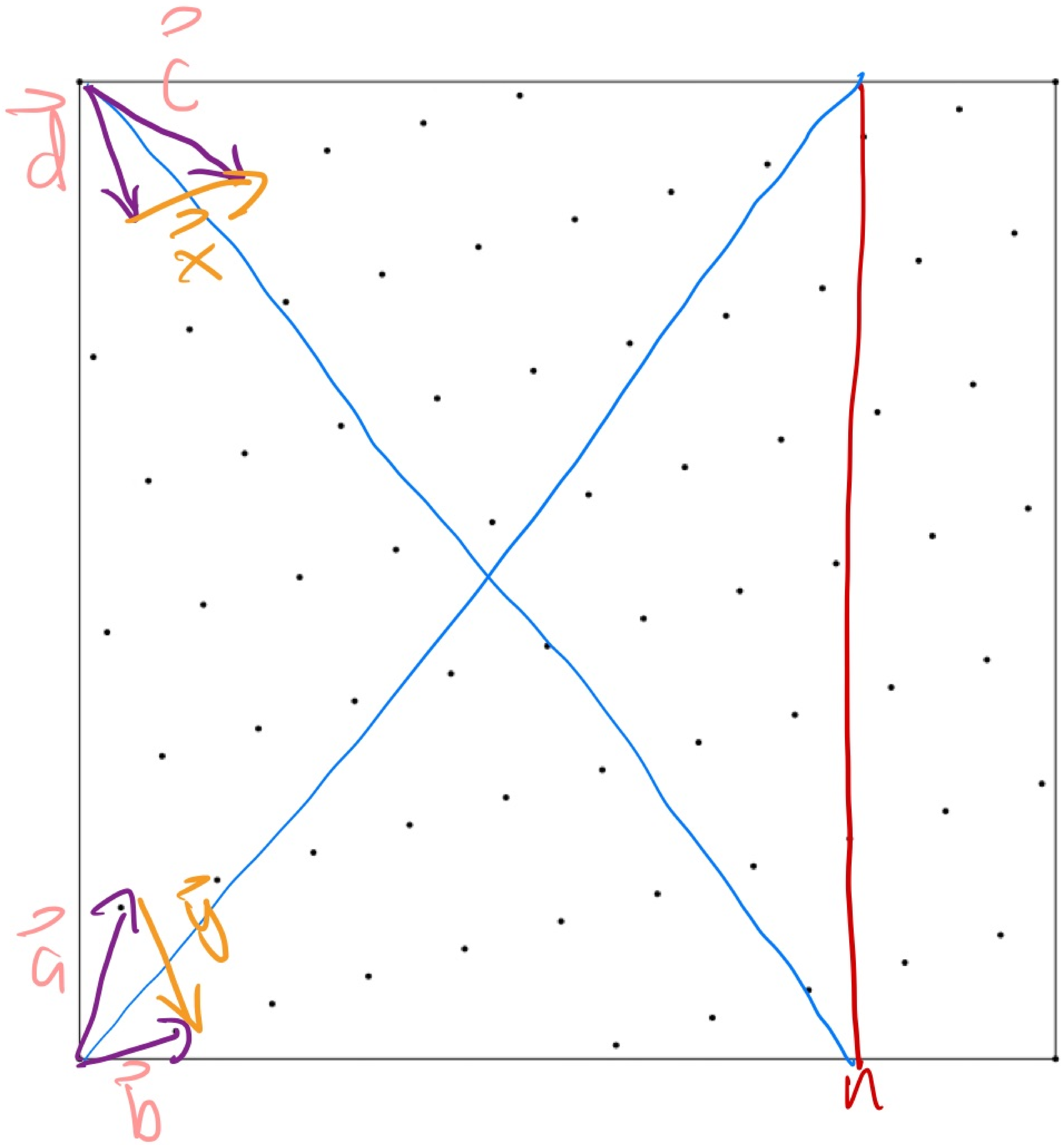
$$\tau = \frac{n}{N}.$$

From previous slide, slope is

$$\frac{r_{i,j}}{t_{i,j}} = N\left|\frac{a}{N} - \frac{p_{i,j}}{q_{i,j}}\right| = N\delta_{i,j}$$

So we look for the first time

$$N\delta_{i,j} < \frac{n}{N} \implies \delta_{i,j} < n$$



Letting $\vec{x} = \vec{c} - \vec{d}$; $\vec{y} = \vec{b} - \vec{a}$, we can prove $(\vec{x}, \vec{y})$ form a basis for $L_{a,N}$, and $\vec{x}$ and $\vec{y}$ are described by principal (not intermediate) convergents.

Proposition: LIS of $w(n, \frac{a}{N})$ is # lines of slope $m_{\vec{y}}$ that intersect $[0,n] \times [0,N] \cap L_{a,N}$.

LDS is # lines of slope $m_{\vec{x}}$ that intersect $[0,n] \times [0,N] \cap L_{a,N}$.

This gives Boyd/Steele result.

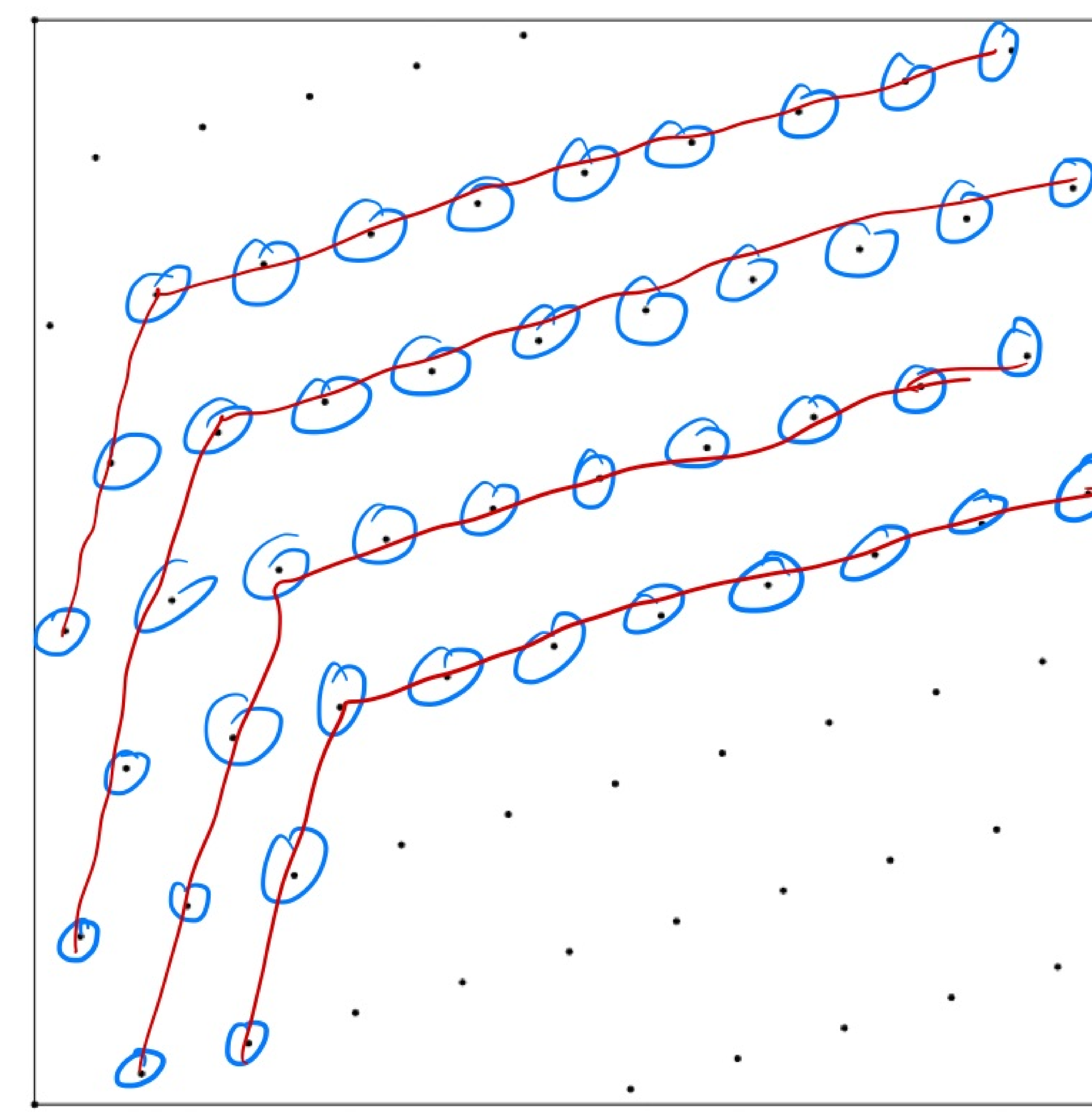
For the rest of the Schensted shape, recall

I_k = size of largest subset of $[0,n]\times[0,N]\cap L_{a,N}$ which is the union of k increasing paths.

An easy bound is $I_k \le \sum_{i\ge 0} \min(l_i, k)$ where

l_j = # lattice points in $[0,n]\times[0,N]\cap L_{a,N}$ on the line of slope $m_{\vec{y}}$ passing through $j\vec{x}$.

$k=4$



A larger set would necessarily have a decreasing set of size $k+1$.

Let $\mathrm{lines}(k) = \#\{ j \mid l_j \le k \}$

For some $k_0 > 0$, for all $k < k_0$ we prove

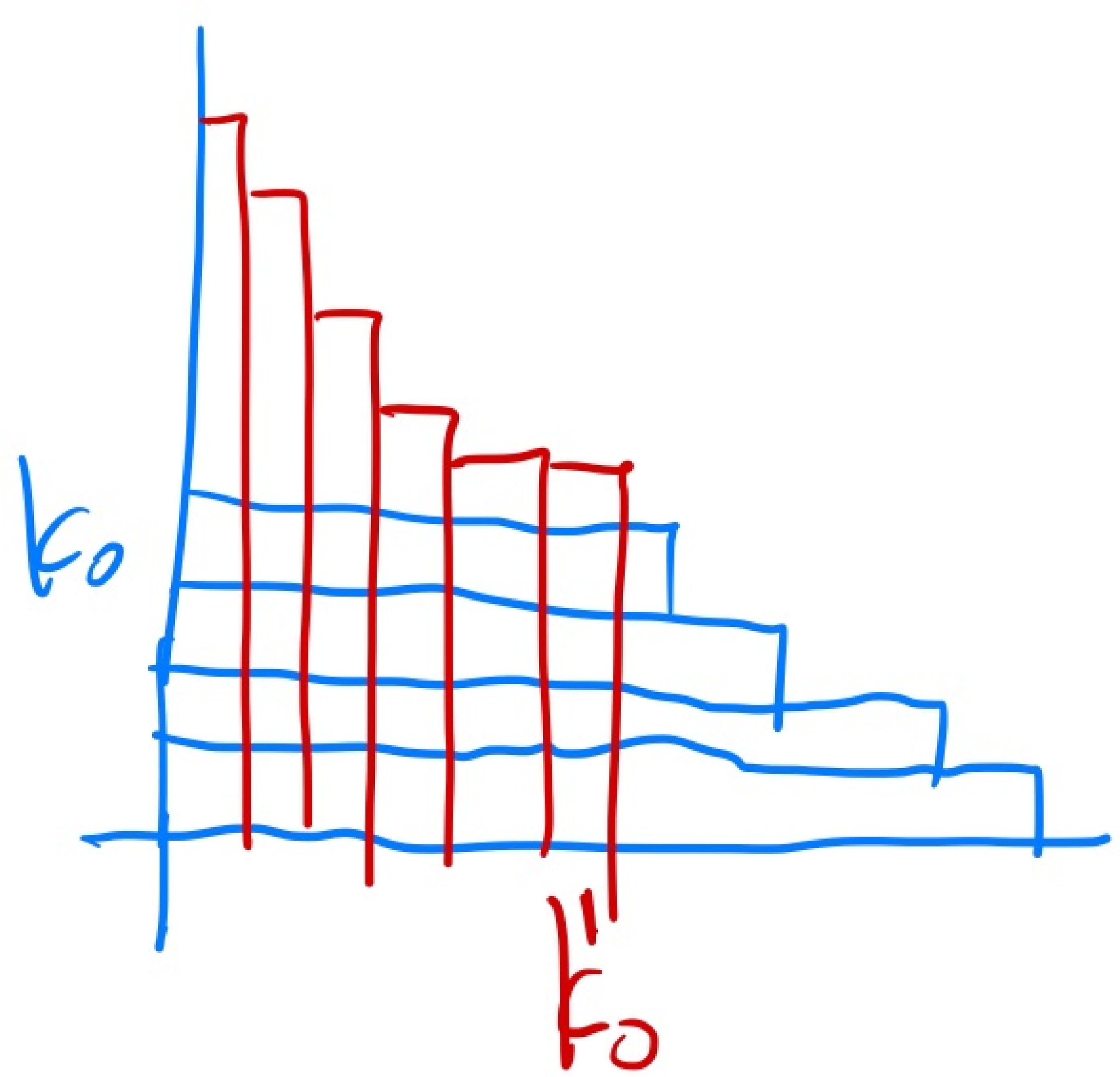
$$-3+\sum_{i>0}\min(l_i,k) \le I_k = \lambda_1+\lambda_2+\cdots+\lambda_k \le \sum_{i\ge 0}\min(l_i,k)$$

Thus $\lambda_j = I_{j+1} - I_j$ satisfies

$$|\lambda_j - lmos[j]| \le 3.$$

This gives a tight bound on λ_j for $j=1,2,\dots,k_0$

A similar bound on λ'_j describes the rest of the shape.



Further questions

1. We have described the Schensted shape of a particular $w(n,\alpha)$. What about a random α for fixed n? If α is chosen uniformly in $(0,1)$ $P(\text{identity}) = P(\text{reverse identity}) = \frac{1}{n}$.
   Does $\frac{1}{\sqrt{n}} E(LIS(n))$ have a limit? What about the average Schensted shape?

2. Better pseudo-random sequences. Can we characterize LIS, LDS, Schensted shape of
$$\left(\alpha \cdot i^2 \bmod 1\right)_{i=1}^{n}?$$
This is a subpermutation of $w(n^2,\alpha)$, so the collection of such permutations is contained in S_{n^2}.

③ Other structured permutations? Pattern avoidance?

For the collection of alternating (zig-zag) permutations there is a formula for the distribution of the length of the LIS which involves a Toeplitz determinant, reminiscent of the seminal Baik-Deift-Johansson work. Worth a look.

(Stanley, A Survey of Alternating Permutations)

Thank You!!