Community Detection in Sparse Random Hypergraphs

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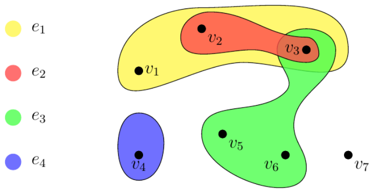
MSRI Seminar

Joint work with Soumik Pal (University of Washington)

Hypergraph

- $H = (V, E)$, V: vertex set, E: hyperedge set.



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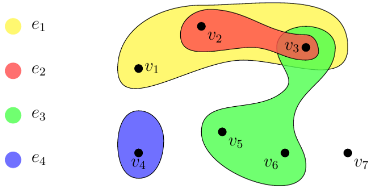

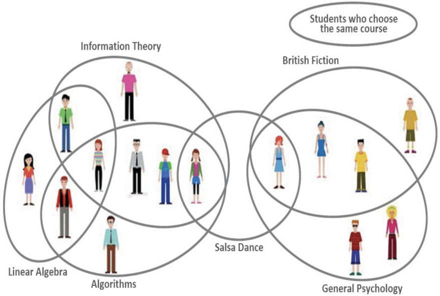

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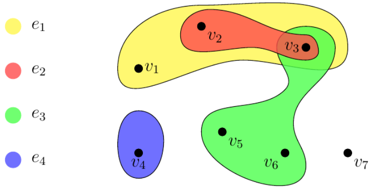

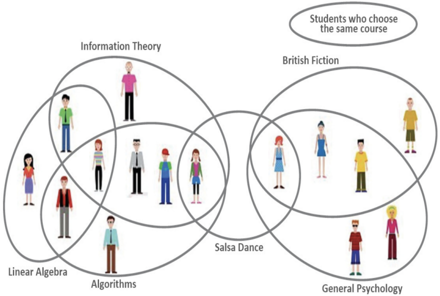

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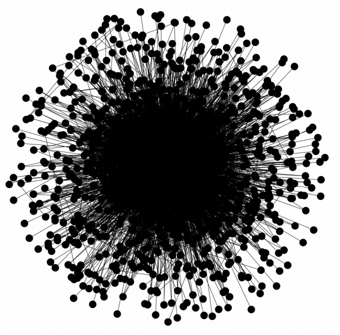
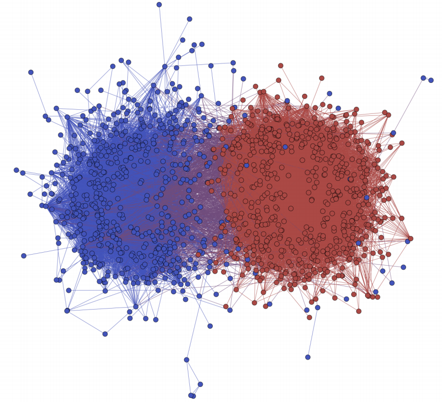
- co-authorship network
- chat group in social network
- Protein interaction network

Community detection





Political blogs data from Adamic-Glance (05). Figure from Abbe (18)

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- Consider a (unknown) partition of n vertices into two *communities* of size $n/2$. Generate edges within each community with probability p. Generate edges across communities with probability $q < p$.
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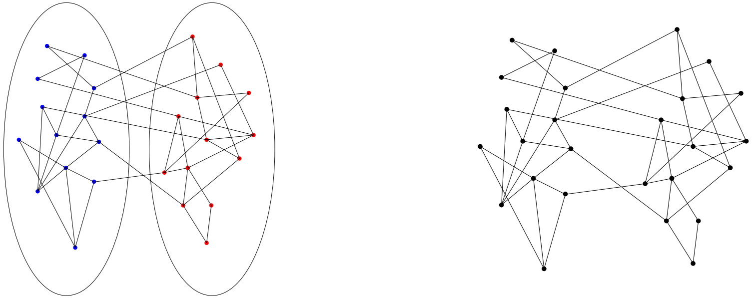
- Task: observe a graph $G \sim \mathcal{G}(n, p, q)$, find the unknown partition with high probability (efficiently and accurately).



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- $\|A - \mathbb{E}A\| = O(\sqrt{np})$ when $\frac{(p+q)n}{2} = \Omega(\log n)$. $o(n)$ vertices are mis-classified.

Feige–Ofek 05, Lei–Rinaldo 13, Le–Levina–Vershynin 16, Benaych-Georges–Bordenave–Knowles 17, Latala–van Handel–Youssef 17, Alt–Ducatez–Knowles 19, Tikhomirov–Youssef 19

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A huge body of work for more general cases and different settings: survey by Abbe (18).

Bounded expected degrees

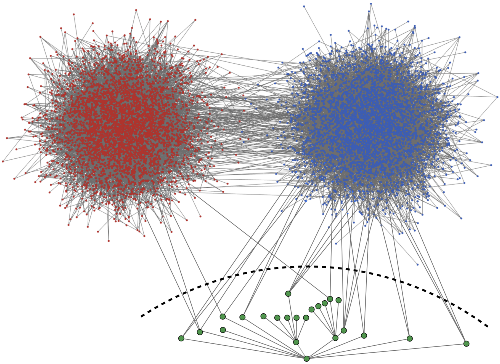

Figure: Abbe et al. (2018), $a = 2.2$, $b = 0.06$, $n = 100000$, apply spectral method directly on A

When $p = \frac{a}{n}, q = \frac{b}{n}$, top eigenvectors are localized on high degree vertices.

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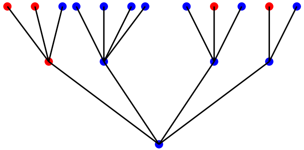

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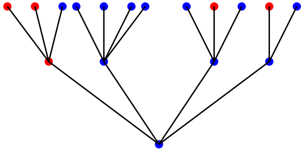

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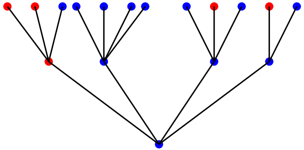

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- The second eigenvector of $B^{(\ell)}$ can be used to estimate $\sigma = (\sigma_1, \dots, \sigma_n)$, better than random guess.

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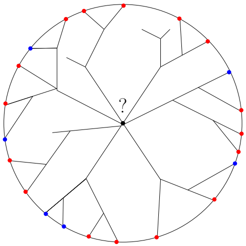

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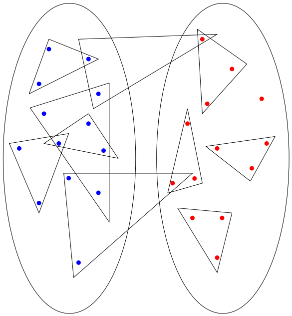

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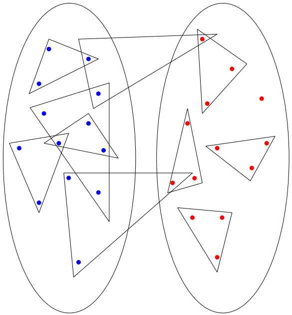

Task: observe H, construct a label estimator $\hat{\sigma} \in \{-1, +1\}^n$ correlated with the true σ.

Ghoshdastidar-Dukkipati (14, 15)

Community detection on HSBM

- Exact recovery: Chien-Lin-Wang (18), Kim-Bandeira-Goemans (18)
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- Spectral method in the bounded expected degree regime?

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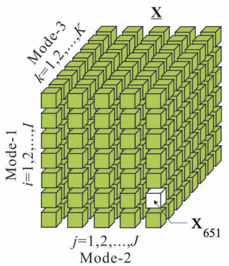

Figure: an order-3 tensor

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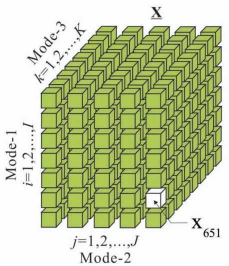

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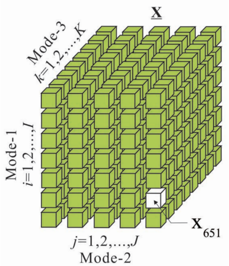

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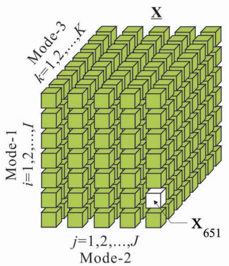

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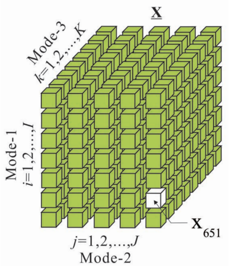

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Ke-Shi-Xia (20): Tensor unfolding and power iteration, $o(n)$ mis-classified vertices when the average degree $\gg \log^2(n)$.

Adjacency matrix of a hypergraph

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  which counts the number of closed walks of length k in H: $(i_0, e_1, i_1, \ldots, i_{k-1}, e_k, i_0)$.

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Detection by self-avoiding walks on hypergraphs

- A walk of length ℓ: $(v_0, e_1, v_1, \cdots e_\ell, v_\ell)$ such that $v_i \neq v_{i+1}$ and $\{v_{i-1}, v_i\} \subset e_i$.

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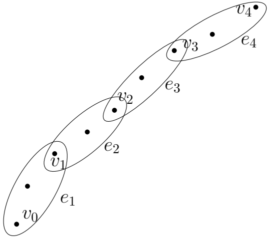

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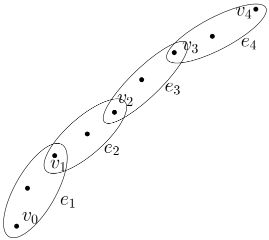

- **self-avoiding walk matrix** $B^{(\ell)}$: $B_{ij}^{(\ell)}$ counts the number of self-avoiding walks of length ℓ from i to j.

Model parameters

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- Angelini et al. (15): conjectured $\beta^2 = \alpha$ is the detection threshold for all $d \geq 2$.

Theorem (Pal-Z., 21)

Assume $\beta^2 > \alpha$. *Set* $\ell = c\log(n)$ *for a proper constant* c. *Let* x *be a unit second eigenvector of* $B^{(\ell)}$. *There exists a constant* t *such that, defining the label estimate* $\hat{\sigma}_i$ *as*

$$\hat{\sigma}_i = \begin{cases} +1 & \text{if } x_i\sqrt{n} \geq t, \\ -1 & \text{otherwise}, \end{cases}$$

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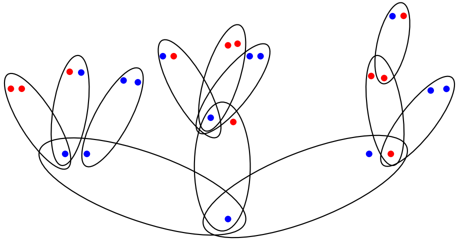
- Dimension reduction: construct $B^{(\ell)}$ of n^2 entries from the adjacency tensor T of n^d entries.
- Spectral clustering: detect the community according to the second eigenvector.

Local structure: multi-type Poisson hypertrees



- Start with a root ρ with label $\tau(\rho)$, generate Pois $\left(\frac{\alpha}{d-1}\right)$ many hyperedges that pairwise intersects at ρ.
- Assign a *type* (the number of $+$ labels) to each hyperedge independently.
- Keep constructing subsequent generations by induction.

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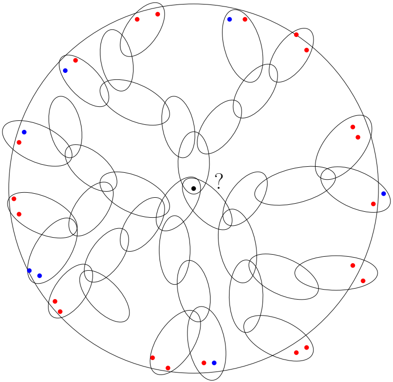
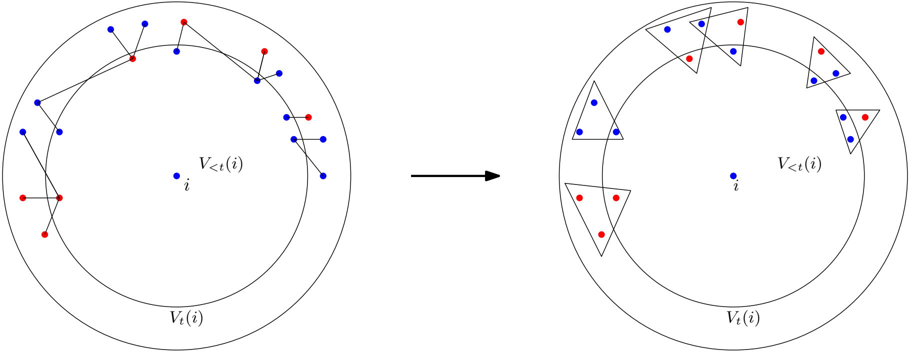


Local Analysis



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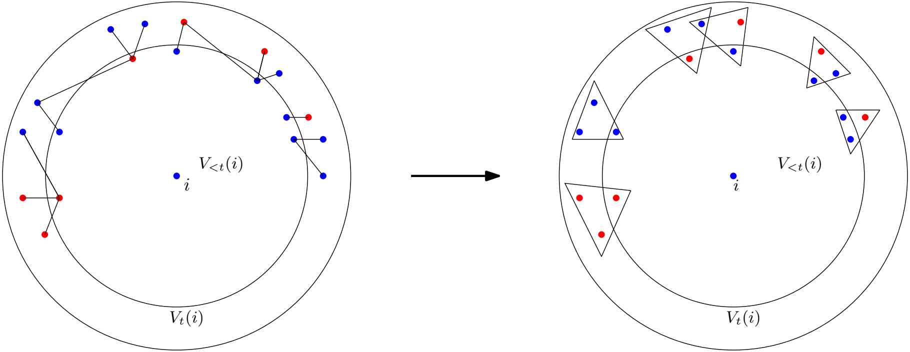

Exploration process on hypergraphs. Control the boundary size and number of $\pm$ labels at distance t.

The moment method

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- Counting centered SAWs : $\Delta_{ij}^{(\ell)} := \sum_{w \in \mathrm{SAW}_{ij}} \prod_{t=1}^{\ell} (A_{i_{t-1} i_t}^{e_{i_t}} - \overline{A}_{i_{t-1} i_t}^{e_{i_t}})$.

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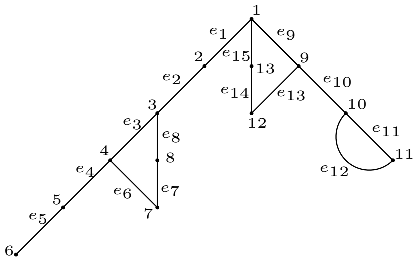

Figure: concatenations of 4 SAWs of length 5

Spectral gap for $B^{(\ell)}$

When $\beta^2 > \alpha$, $B^{(\ell)}$ has a spectral gap asymptotically almost surely:

- $\lambda_1(B^{(\ell)}) = \Theta(\alpha^\ell)$ up to a $\log n$ factor.
- $\lambda_2(B^{(\ell)}) = \Omega(\beta^\ell)$, and $\lambda_2(B^{(\ell)}) = O(n^{-\gamma}\alpha^\ell)$ for some $\gamma > 0$.
- $\lambda_3(B^{(\ell)}) = O(n^\epsilon \alpha^{\ell/2})$ for any $\epsilon > 0$.

Further Problems

- Non-backtracking operator for random hypergraphs with k blocks (work in progress with Ludovic Stephan)
- Non-uniform hypergraphs (with Ioana Dumitriu and Haixiao Wang)
- Impossibility for detection below the threshold
- Applications in tensor completion

Tensor Analog of Matrix Problems

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Statistical and computational gap

- Tensor PCA: $X = \lambda v^{\otimes k} + Z$
  Montanari-Richard (14), Chen (18), Ben Arous-Mei-Montanari-Nica (17), Ben Arous-Gheissari-Jagannath (18), Wein-Alaoui-Moore (19), Huang-Huang-Yang-Cheng (20), Ding-Hopkins-Steurer (20), Ben Arous-Huang-Huang (21),...
- Tensor completion
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No such gap in many hypergraph community detection problems:

Exact recovery: Kim-Bandeira-Goemans (17, 18), Ahn-Lee-Suh (18), Chien-Lin-Wang (18), Zhang-Tan (21).

Conclusion

- Community detection on random hypergraphs can be analyzed by spectral methods on sparse random matrices.
- Moment methods can be applied to random hypergraphs.
- Sparse random tensors are not well understood.

Thank You!