Hard rods, Poisson line process and Levy Brownian function

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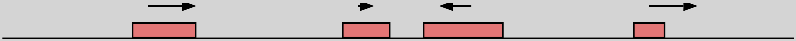
Hard rod space

$\mathsf{x} = (q, v, r)$ is a *rod* $(q, q+r)$ with speed v and length r.



$\mathfrak{Y} :=$ set of *hard rod* configurations $\mathsf{Y} \subset \mathbb{R}^3$ such that

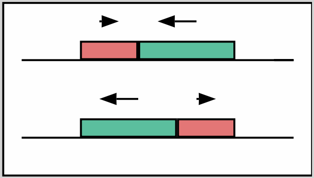
1) No rod intersection: $(q, q+r) \cap (q', q'+r') = \emptyset$; $(q, v, r), (q', v', r') \in \mathsf{Y}$

2) Space locally finite: $\#\{(q, v, r) \in \mathsf{Y} : a \leq q \leq b\} < \infty$.

Hard rod evolution

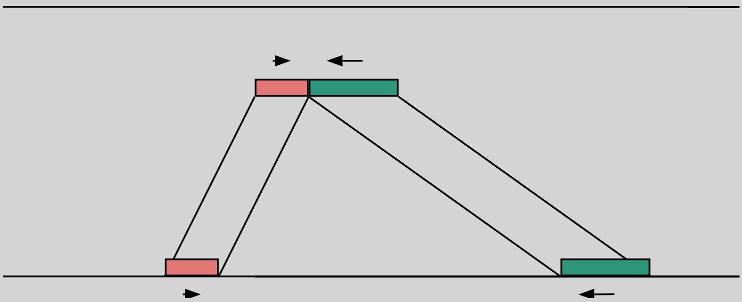
rod (q, v, r) travels at speed v in absence of other rods.

Collision rule: two rods sharing a boundary point with colliding speeds swap positions, keeping their original speeds



Hard rod evolution operator

$$U_t \mathsf{Y} := \text{ configuration of hard rods at time } t.$$



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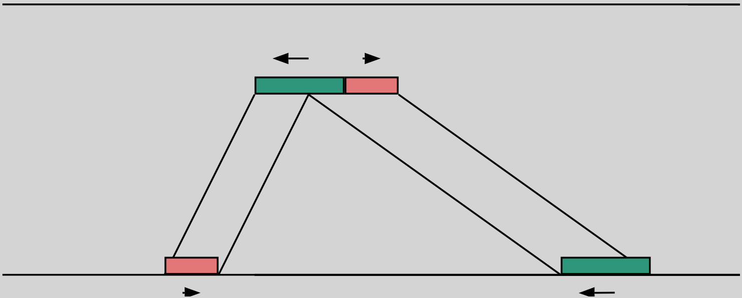

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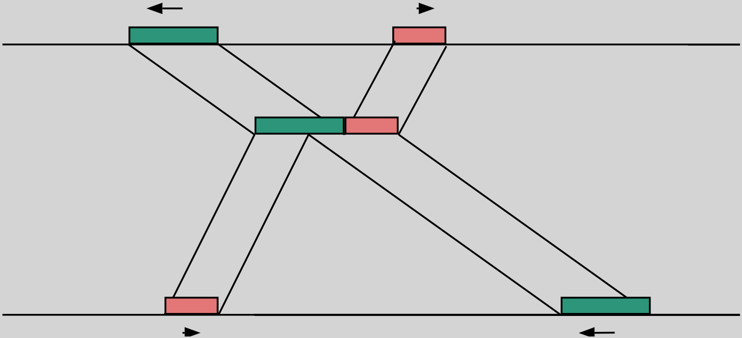
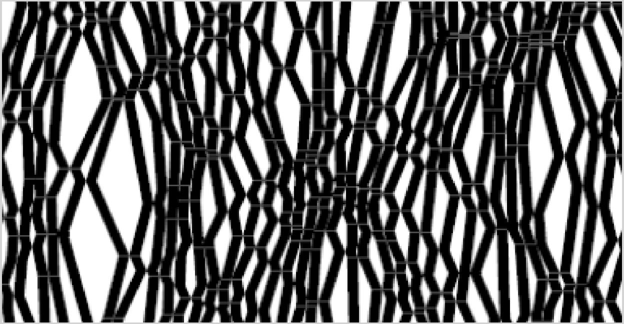


Hard rod evolution operator

$$U_t\mathsf{Y} := \text{ configuration of hard rods at time } t.$$

Figure: Bouchoule-Dubail 2021.



Trajectories of hard rods

Hydrodynamics

Empirical lenght measure

Family of random hard rod configurations Y^ε (inter-rod distances $\approx \varepsilon$)

$$\mathtt{K}_0^\varepsilon \varphi := \varepsilon \sum_{(q,v,r)\in \mathsf{Y}^\varepsilon} r\, \varphi(q,v,r), \qquad \varphi \text{ test function}$$

Macroscopic length measure

Measure $$G_o\varphi := \iiint \varphi(q,v,r)\, r\, g_o(q,v,r)\, dqdvdr.$$

Density g_o satisfying $$\sup_q \int\int (r^2+v^2)\, g_o(q,v,r)\, dv\, dr < \infty$$

Generalized Hydrodynamic theorem. Density fields.

Boldrighini, Dobrushin, Sukhov 1982

Let $\mathtt{K}_t^\varepsilon :=$ empirical length measure of $U_t \mathsf{Y}^\varepsilon$. Then

<u>If</u> $\lim_{\varepsilon \to 0} \mathtt{K}_0^\varepsilon = G_o,$ weakly in probability,

<u>then</u> $\lim_{\varepsilon \to 0} \mathtt{K}_t^\varepsilon = G_t,$ weakly in probability, $t \in \mathbb{R}$,

where G_t has density $g(q, v, r; t)$, unique solution of Cauchy problem:

$$\partial_t \, g(q, v, r; t) + \partial_q \left(g(q, v, r; t) \; v^{\mathrm{eff}}_{g_o}(q, v; t) \right) = 0, \quad v \in \mathbb{R}$$

$$v^{\mathrm{eff}}_{g_o}(q, v; t) = v + \frac{\iint \tilde{r} \, (v - \tilde{v}) \, g(q, \tilde{v}, \tilde{r}; t) \, d\tilde{v} \, d\tilde{r}}{1 - \iint \tilde{r} \, g(q, \tilde{v}, \tilde{r}; t) \, d\tilde{v} \, d\tilde{r}}$$

$$g(q, v, r; 0) = g_o(q, v, r)$$

Hydrodynamic theorem. Tagged rod (quasi particle).

$\mathtt{u}_{\mathsf{Y},v;t}(q) :=$ position at time t of a tagged rod initially at $(q, v, 0)$

Theorem Under the previous conditions,

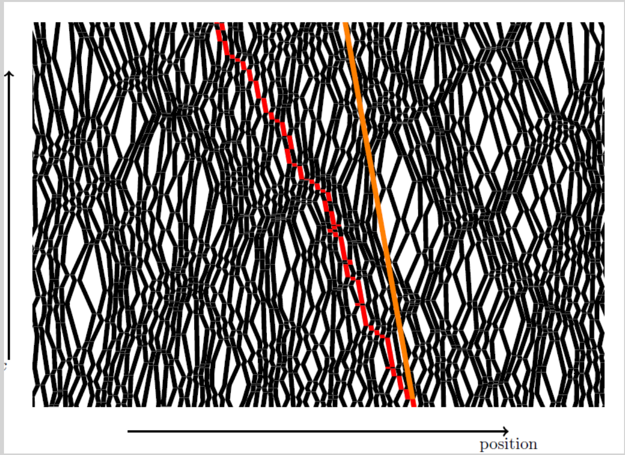
$$\lim_{\varepsilon} \varepsilon\, \mathtt{u}_{\mathsf{Y}^{\varepsilon},v;t}(q) = u_{g_o,v;t}(q), \quad \text{in probability},$$

where $u_{g_o,v;t}(q)$ is the solution of

$$\begin{aligned} \partial_t u_{g_o,v;t}(q) &= v^{\mathsf{eff}}_{g_o}(u_{g_o,v;t}(q), v; t) \\ u_{g_o,v,0}(q) &= q \end{aligned}$$

A tagged rod with bare speed v moves locally at speed $v^{\mathsf{eff}}(\cdot, v; t)$.

Figure: Bouchoule-Dubail 2021.



Tagged rod trajectory and its would be trajectory if isolated.

Collision rate theorem Fix q and v.

$$v_g^{\text{eff}}(q,v;t) = v + \iint g(q,\tilde{v},\tilde{r};t)\,\Phi(v,\tilde{v},\tilde{r})\,\left|v_g^{\mathsf{eff}}(q,v;t) - v_g^{\mathsf{eff}}(q,\tilde{v};t)\right|\,d\tilde{v}\,d\tilde{r}$$

Collision rule:

$$\Phi(v,\tilde{v},\tilde{r}) = \tilde{r}\ \mathsf{sign}(v-\tilde{v})$$

Compare with KdV: Girotti, Grava, Jenkins, McLaughlin 2021 get

$$\Phi(v,\tilde{v}) = \log\left|\frac{\sqrt{v}-\sqrt{\tilde{v}}}{\sqrt{v}+\sqrt{\tilde{v}}}\right|.$$

Incomplete background

Hard rods:

Aizenmann Goldstein Lebowitz 1975

Aizemann Lebowitz Marro 1978, Elastic collision + "pulses"

Boldrighini, Dobrushin, Sukhov 1982, Hydrodynamics $r \equiv d$ constant

Hard rod section on Spohn 1991

Doyon Yoshimura Caux 2017, Simulation hard rod GHD

Generalized Hydrodynamics (GHD):

Cao Bulshadani Spohn 2020, collision rate assumption in Toda chain

Spohn 2020, The collision rate ansatz for the classical Toda lattice

Doyon 2020 Lecture notes on GHD

Box Ball System :

Ferrari Nguyen Rolla Wang 2020, v^{eff} computation for ergodic BBS.

Croydon Sasada 2020, GHD of BBS with finite soliton sizes.

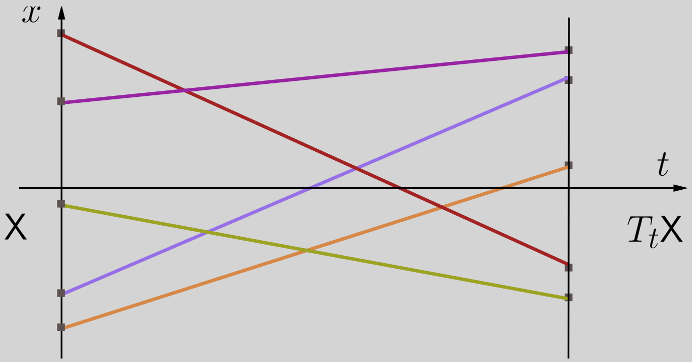
Main tool: Free gas

Harris 1965, Spitzer 1968, 1969 .

$\mathfrak{X}$:= space of space locally finite particle configurations X.

$$T_t \mathsf{X} := \{(q+vt, v, r) : (q,v,r) \in \mathsf{X}\}, \qquad \mathsf{X} \in \mathfrak{X}.$$

T_t bijection with inverse T_{-t}. Ballistic particles ignoring each other.



Free gas conserves Poisson

If X is a PP with intensity measure μ, then T_tX is a PP with μT_{-t},

$\mathbb{P}$ is T_t-invariant if X $\sim \mathbb{P}$ implies T_tX $\sim \mathbb{P}$.

Mixing and T_t-invariance implies Poisson

Let $\mathbb{P}$ absolutely continuous with intensity f, shift invariant, mixing and T_t-invariant, then $\mathbb{P}$ is a Poisson process with intensity f.

Proof: n-point correlations at time 0:

$$\rho_n(\mathsf{x}_1, \dots, \mathsf{x}_n) \;=\; \rho_n(S_{v_1 t}\mathsf{x}_1, \dots, S_{v_n t}\mathsf{x}_n) \;\xrightarrow[t\to\infty]{}\; \rho_1(\mathsf{x}_1)\cdots\rho_1(\mathsf{x}_n)$$

because of T_t-invariance and mixing. □

Related results: Stone 1968 and Dobrushin 1956.

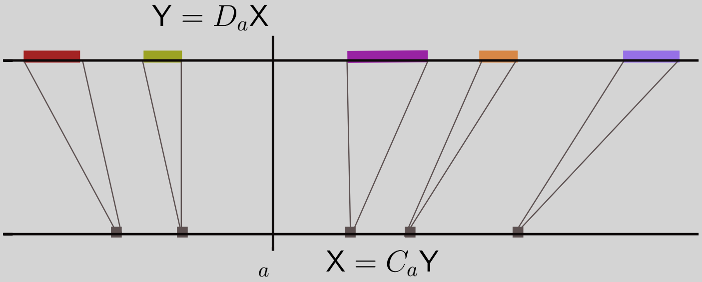
Shift: $S_a(q, v, r) := (q - a, v, r)$.

Dilation and contraction

$\mathfrak{Y}_a :=$ Hard rod configurations with no rod containing $a \in \mathbb{R}$

Dilation: $D_a\mathsf{X} := \big\{\big(q + m_a^q(\mathsf{X}), v, r\big) : (q, v, r) \in \mathsf{X}\big\} \in \mathfrak{Y}_a,$

Contraction: $C_a\mathsf{Y} := \big\{\big(q - m_a^q(\mathsf{Y}), v, r\big) : (q, v, r) \in \mathsf{Y}\big\} \in \mathfrak{X}.$



$m_a^b(\mathsf{X}) :=$ total lenght of rods in the space interval $[a, b]$ (with sign).

Flows

$$\text{Right} \quad J^-_{q,v;t}\mathsf{X} := \{(\tilde{q},\tilde{v},r) \in \mathsf{X} : \tilde{q} < q \text{ and } \tilde{q} + \tilde{v}t > q + vt\}$$

$$\text{Left} \quad J^+_{q,v;t}\mathsf{X} := \{(\tilde{q},\tilde{v},r) \in \mathsf{X} : \tilde{q} > q \text{ and } \tilde{q} + \tilde{v}t < q + vt\}$$

X particles intersecting the trajectory $\{q + vs : 0 < s < t\}$ from right to left and from left to right, respectively.

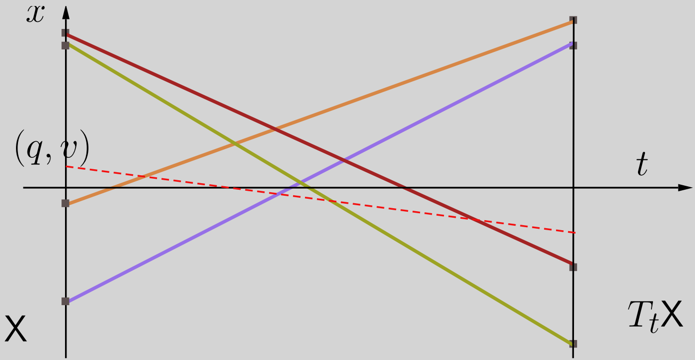
Mass flows:

$$\begin{cases} j^+_{\mathsf{X}}(q,v;t) := m(J^+_{q,v;t}\mathsf{X}), \\ j^-_{\mathsf{X}}(q,v;t) := m(J^-_{q,v;t}\mathsf{X}), \end{cases}$$

Signed mass flows:

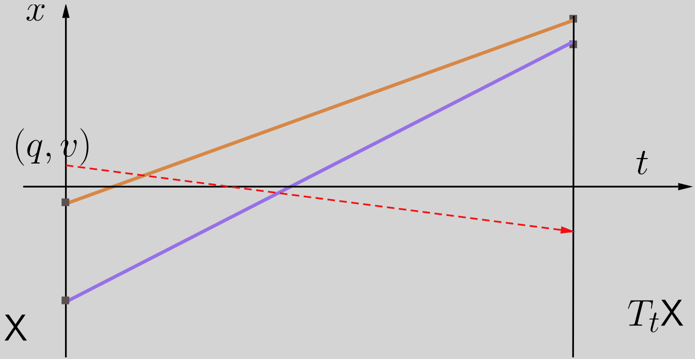
$$j_{\mathsf{X}}(q,v;t) := j^+_{\mathsf{X}}(q,v;t) - j^-_{\mathsf{X}}(q,v;t)$$

Total Flow



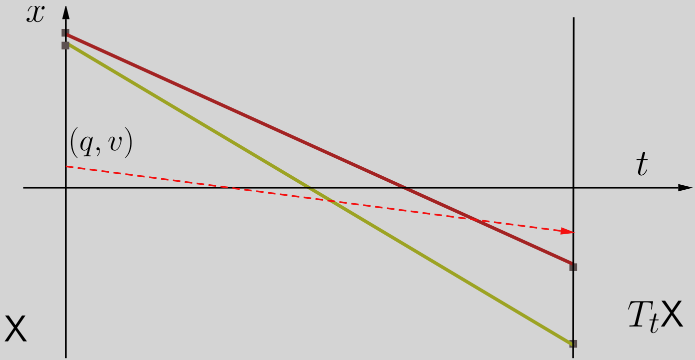
Set of particles in X intersecting (q, v) at times in $[0, t]$

Right Flow



$J^{-}_{q,v;t}$X := particles in X intersecting (q, v) at times in $[0, t]$ from the right.

Left Flow



$J^+_{q,v;t}$X := particles in X intersecting (q, v) at times in $[0, t]$ from the left.

Hard rod dynamics

(1) As seen from tagged rod $o := (0,0,0)$:

$$\hat{U}_t \mathsf{Y} := D_0 T_t C_0 \mathsf{Y}, \qquad \mathsf{Y} \in \mathfrak{Y}_0.$$

(2) As seen from the origin:

$$U_t \mathsf{Y} := S_{-o_t(\mathsf{Y})} \hat{U}_t \mathsf{Y}, \qquad \mathsf{Y} \in \mathfrak{Y}_0,$$

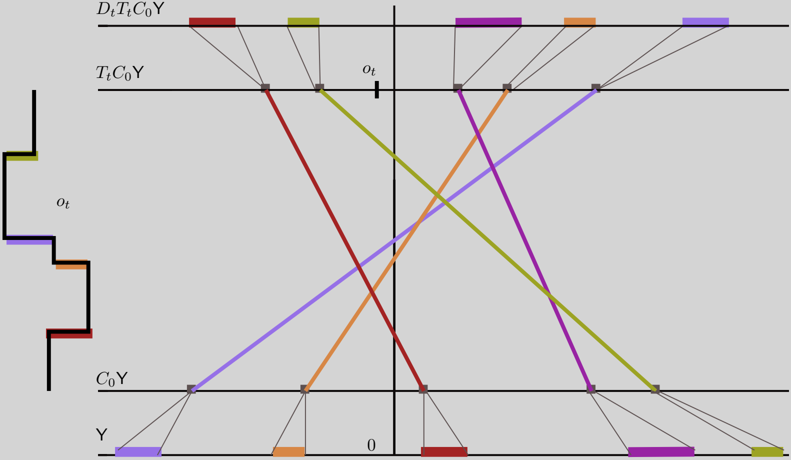
$$o_t(\mathsf{Y}) := m(J^+_{o;t} C_0 \mathsf{Y}) - m(J^-_{o;t} C_0 \mathsf{Y}),$$

position at time t of o added to Y at time 0.

(3) Let $(q,v,r) \in \mathsf{Y}$ with $q < 0 < q + r$. Since $S_q \mathsf{Y} \in \mathfrak{Y}_0$, we define

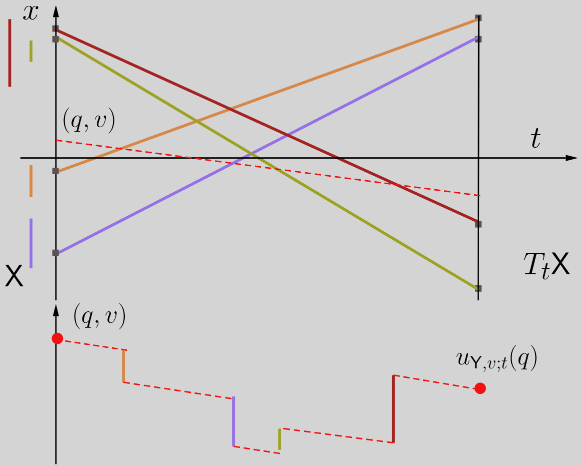
$$U_t \mathsf{Y} := S_{-q} U_t S_q \mathsf{Y}, \qquad \mathsf{Y} \in \mathfrak{Y} \setminus \mathfrak{Y}_0.$$

Hard rod dynamics



Tagged rod motion – quasi particle

$u_{\mathsf{Y},v;t}(q) :=$ position of Y particle $(q, v, 0)$ at time t.

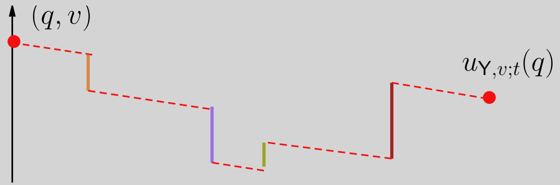


Tagged rod motion – quasi particle

Hard rod configuration $\mathsf{Y} \in \mathfrak{Y}_q$

$u_{\mathsf{Y},v;t}(q)$:= position of particle $(q, v, 0)$ at time t:

$$u_{\mathsf{Y},v;t}(q) := q + vt + j_{\mathsf{Y}}(q, v; t).$$



Recall mass flow: $j_{\mathsf{Y}}(q, v; t) := m(J^{+}_{q,v;t} C_0 \mathsf{Y}) - m(J^{-}_{q,v;t} C_0 \mathsf{Y})$.

Invariant measures for hard rod dynamics

$\mathbb{P}$ on $\mathfrak{Y}$ is U_t-invariant if $\mathsf{Y} \sim \mathbb{P}$, then $U_t\mathsf{Y} \sim \mathbb{P}$

Spacial mixing and time invariance imply Poisson

Theorem. *Let $\mathbb{P}$ on $\mathfrak{Y}$ be shift invariant and U_t-invariant.*

Let $\mathsf{Y} \sim \mathrm{Palm}(\mathbb{P})$ and assume that the law of $C_0\mathsf{Y}$ on $\mathfrak{X}$ is mixing.

Then $C_0\mathsf{Y}$ is a Poisson process.

Proof.

$\mathbb{P}$ shift-invariant and U_t-invariant implies $\hat{\mathbb{P}}$ is $\hat{U}_t$-invariant. (Harris)

By definition, this is equivalent to $\hat{\mathbb{P}}D_0$ on $\mathfrak{X}$ is T_t-invariant.

Free gas result and hypothesis $\hat{\mathbb{P}}D_0$ is mixing, implies $\hat{\mathbb{P}}D_0$ is Poisson. $\square$

Macroscopic setup

Mass of f at q:

$$\sigma_f(q) := \iint r\, f(q,v,r) dv dr$$

Dilation and contraction

Point:

$$\begin{aligned} D_{f,a}(q) &:= q + \int_a^q \sigma_f(x)\, dx \\ C_{g,a}(q) &:= q - \int_a^q \sigma_g(x) dx \end{aligned}$$

Configuration:

$$\begin{aligned} D_a f(q,v,r) &:= \frac{f(D_{f,a}^{-1}(q),v,r)}{1+\sigma_f(D_{f,a}^{-1}(q))} \\ C_a g(q,v,r) &:= \frac{g(C_{g,a}^{-1}(q),v,r)}{1-\sigma_g(C_{g,a}^{-1}(q))} \end{aligned}$$

Mass conservation

$$\int_{D_{f,a}(b)}^{D_{f,a}(c)} \sigma_{D_a f}(x)\, dx \;=\; \int_b^c \sigma_f(x)\, dx$$

$$\int_{C_{g,a}(b)}^{C_{g,a}(c)} \sigma_{C_a g}(x)\, dx \;=\; \int_b^c \sigma_g(x)\, dx$$

Growth rate

$$\frac{d}{dq} D_{f,a}(q) \;=\; 1 + \sigma_f(q) \;\in [1, +\infty)]$$

$$\frac{d}{dq} C_{g,a}(q) \;=\; 1 - \sigma_g(q) \;\in (0, 1]$$

Macroscopic evolutions

Free gas

$$T_t f(q,v,r) = f(q-vt,v,r)$$

Signed mass flow

$$j_g(q,v;t) \;:=\; \iint \int_q^{q+(v-w)t} r\, C_q g(x,w,r)\, dx\, dw\, dr$$

Hard rods

$$\mathcal{U}_t g(q,v,r) \;:=\; S_c\, D_{q+vt}\, T_t\, C_q\, g; \qquad c = j_g(q,v;t)$$

Tagged hard rod

$$u_{g,v;t}(q) \;:=\; q + vt + j_g(q,v;t)$$

Lemma: $\mathcal{U}_t g$ is the unique solution of the Cauchy problem:

$$\mathcal{U}_t g(q,v,r) \coloneqq g\big(u^{-1}_{g,v;t}(q), v, r\big)\,\frac{d}{dq}\,u^{-1}_{g,v;t}(q)$$

Hard rods and line processes

Free gas representation of lines

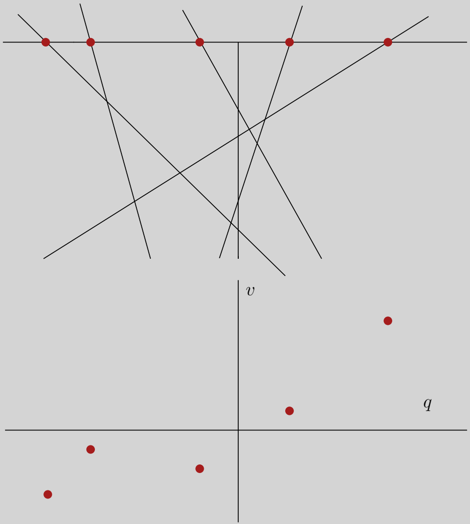
The map

$$(x, v) \mapsto \{(t, x + vt) : t \in \mathbb{R}\}.$$

is a bijection:

$\mathbb{R}^2 \setminus \{\mathsf{o}\} \longrightarrow$ space of lines contained in $\mathbb{R}^2$, excluding parallels to t-axis.

Free gas representation of lines



Lines intersecting a segment

$$[\mathsf{a},\mathsf{b}] := \{(1-u)\mathsf{a} + u\mathsf{b} : u \in [0,1]\}, \quad \mathsf{a},\mathsf{b} \in \mathbb{R}^2$$

$$\overline{\mathsf{ab}} := \text{Set of lines intersecting } [\mathsf{a},\mathsf{b}].$$

Let $[\mathsf{a},\mathsf{b}]$ be oriented $\mathsf{a} \to \mathsf{b}$, and define

$$v_{\mathsf{ab}} := \frac{\mathsf{b}_x - \mathsf{a}_x}{\mathsf{b}_t - \mathsf{a}_t}$$

For $\mathsf{a} < \mathsf{b}$, define

$$\overline{\mathsf{ab}}_- := \{(x,v) \in \overline{\mathsf{ab}} : v > v_{\mathsf{ab}}\}, \quad \text{intersecting from right}$$
$$\overline{\mathsf{ab}}_+ := \{(x,v) \in \overline{\mathsf{ab}} : v < v_{\mathsf{ab}}\}, \quad \text{intersecting from left}$$

Consider a space locally finite measure μ on $\mathbb{X}$.

Poisson marked line process

View free gas configuration as line process with marks:

$$\text{Free gas particle } (q,v,r) = \text{line } (q,v) \text{ with mark } r.$$

and denote

$$\mathsf{X} := \text{Poisson process with intensity measure } \mu.$$

Marked line white noise

Random measure ω on $\mathbb{X}$ satisfying:

a) $\omega\varphi := \int \varphi d\omega \sim \mathsf{Normal}(0, \int \varphi^2 d\mu), \qquad \varphi \in L_2(\mu).$

b) $\mathsf{Cov}(\omega\varphi_1, \omega\varphi_2) = \int \varphi_1\varphi_2 \, d\mu.$

μ is called *control measure*.

Levy Brownian surface

Consider a *distance* d in $\mathbb{R}^2$ and define

$\eta : \mathbb{R}^2 \to \mathbb{R}$ as a Gaussian process with $\eta(\mathsf{o}) = 0$ and

$$\mathsf{Cov}\big(\eta(\mathsf{a}), \eta(\mathsf{b})\big) = \tfrac{1}{2}\big(\mathrm{d}(\mathsf{o}, \mathsf{a}) + \mathrm{d}(\mathsf{o}, \mathsf{b}) - \mathrm{d}(\mathsf{a}, \mathsf{b})\big), \quad \mathsf{a}, \mathsf{b} \in \mathbb{R}^2$$

Levy 1948, see Lifshits 2012, Example 3.6.

Levy Brownian surface as a function of line white noise

Chentsov 1957: Let ω be line white noise in $\mathbb{X}$ with control μ and

$$\eta(\mathsf{a}) := \omega\big(\overline{\mathsf{oa}}\big).$$

By definition, $\eta(\mathsf{a}) \overset{\text{law}}{=} N\big(0, \mu(\overline{\mathsf{oa}})\big)$.

Covariances

$$\begin{aligned}
&\mathsf{Cov}\big(\eta(\mathsf{a}),\eta(\mathsf{b})\big)\\
&\quad = \int \mu(d\omega)\big(\omega(\overline{\mathsf{oa}})\omega(\overline{\mathsf{ob}})\big)\\
&\quad = \int \mu(d\omega)\big([\omega(\overline{\mathsf{oa}}\cap\overline{\mathsf{ab}})+\omega(\overline{\mathsf{oa}}\cap\overline{\mathsf{ob}})]\,[\omega(\overline{\mathsf{ob}}\cap\overline{\mathsf{ab}})+\omega(\overline{\mathsf{oa}}\cap\overline{\mathsf{ob}})]\big)\\
&\quad = \int \mu(d\omega)\,\omega(\overline{\mathsf{oa}}\cap\overline{\mathsf{ob}})^2 \;=\; \mu(\overline{\mathsf{oa}}\cap\overline{\mathsf{ob}}) \;=\; \tfrac{1}{2}\big(\mu(\overline{\mathsf{oa}})+\mu(\overline{\mathsf{ob}})-\mu(\overline{\mathsf{ab}})\big).
\end{aligned}$$

Hence, η is Levy Brownian surface for the distance

$$\mathrm{d}(\mathsf{a},\mathsf{b}) := \mu(\overline{\mathsf{ab}}).$$

Lantuéjoul-Chentsov surfaces

A marked line $\mathsf{x} = (x, v, r)$ induces the surface

$$\xi_{(x,v,r)}(\mathsf{a}) := r\left(1\{(x,v) \in \overline{\mathsf{oa}}_+\} - 1\{(x,v) \in \overline{\mathsf{oa}}_-\}\right);$$

Define the surface $\xi = \xi[\mathsf{X}]$ as

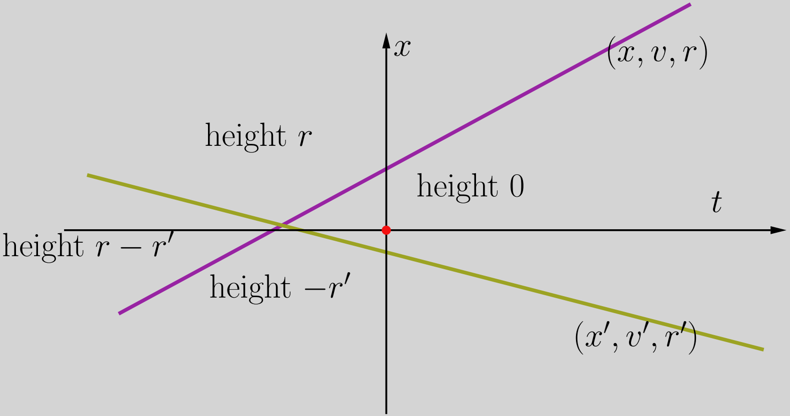
$$\xi(\mathsf{a}) := \sum_{\mathsf{x} \in \mathsf{X}} \xi_{\mathsf{x}}(\mathsf{a}).$$

In particular, $\xi(\mathsf{o}) = 0$ and

$$\xi(\mathsf{b}) - \xi(\mathsf{a}) = \sum_{(x,v,r) \in \mathsf{X} \cap \overline{\mathsf{ab}}_-} r - \sum_{(x,v,r) \in \mathsf{X} \cap \overline{\mathsf{ab}}_+} r.$$

(Notation abuse: $\overline{\mathsf{ab}}$ instead of $\{(x, v, r) \in \mathbb{R}^3 : (x, v) \in \overline{\mathsf{ab}}\}$).

Lantuéjoul-Chentsov surfaces



Law of large numbers for LCS

Let X^ε be Poisson marked line process with intensity measure $\varepsilon^{-1}\mu$.

Define the empirical surface associated to X^ε:

$$\xi^\varepsilon(\mathsf{a}) := \varepsilon \sum_{\mathsf{x}\in\mathsf{X}^\varepsilon} \xi_{\mathsf{x}}(\mathsf{a}), \quad \mathsf{a} \in \mathbb{R}^2.$$

Define the mass of μ measure by

$$m_\mu\varphi := \int \mu(dq dv dr)\, r\, \varphi(q, v, r).$$

Proposition (LLN for LCS).

$$\lim_{\varepsilon\to 0}(\xi^\varepsilon(\mathsf{b}) - \xi^\varepsilon(\mathsf{a})) = m_\mu(\overline{\mathsf{ab}}_-) - m_\mu(\overline{\mathsf{ab}}_+). \qquad \text{a.s.}$$

Proof. Sum of functions of independent Poisson processes $\mathsf{X} \cap \overline{\mathsf{ab}}_-$ and $\mathsf{X} \cap \overline{\mathsf{ab}}_+$, with intensities $\mu(\cdot \cap \overline{\mathsf{ab}}_-)$ and $\mu(\cdot \cap \overline{\mathsf{ab}}_+)$, respectively. □

Lantuéjoul-Chentsov surface fluctuations

Proposition LCS fluctuations converge to Levy Brownian surface:

$$\eta^\varepsilon(\mathsf{b}) := \frac{\xi^\varepsilon(\mathsf{b}) - \int \xi^\varepsilon(\mathsf{b})\, d\mu}{\varepsilon^{1/2}} \xrightarrow{\text{law}} \eta(\mathsf{b}), \qquad \mathsf{b} \in \mathbb{R}^2$$

where η is Levy Brownian surface with control distance

$$\mathrm{d}(\mathsf{a},\mathsf{b}) := \int \mu(dxdvdr)\, r^2\, \mathbb{1}\{(x,v) \in \overline{\mathsf{ab}}\}.$$

Proof. Variance of height differences:

$$\mathsf{Var}\big(\eta^\varepsilon(\mathsf{b}) - \eta^\varepsilon(\mathsf{a})\big) = \tfrac{1}{\varepsilon}\,\mathsf{Var}\big(\xi^\varepsilon(\mathsf{b}) - \xi^\varepsilon(\mathsf{a})\big)$$

$$= \varepsilon\,\mathsf{Var} \sum_{(x,v,r)\in \mathsf{X}^\varepsilon} r\ \mathbb{1}\{(x,v) \in \overline{\mathsf{ab}}_-\} + \varepsilon\,\mathsf{Var} \sum_{(x,v,r)\in \mathsf{X}^\varepsilon} r\ \mathbb{1}\{(x,v) \in \overline{\mathsf{ab}}_+\}$$

$$= \int \mu(dqdvdr)\ r^2\ \mathbb{1}\{(x,v) \in \overline{\mathsf{ab}}\}.$$

□

Hard rod dynamics and Lantuéjoul-Chentsov surfaces

Dilation Mass of X = height difference of LCS:

$$m_a^{\mathsf{b}}(\mathsf{X}) = \xi(\mathsf{b}) - \xi(\mathsf{a}), \qquad \mathsf{a} = (0,a),\ \mathsf{b} = (0,b).$$

Dilated particle position: distance to the origin plus LCS height difference:

$$D_a(\mathsf{X}) = \big\{(q + \xi(0,q) - \xi(0,a), v, r) : (q,v,r) \in \mathsf{X}\big\}.$$

Fluxes

$$J^+_{q,v;t}\mathsf{X} = \mathsf{X} \cap \overline{\mathsf{ab}}_+, \qquad \mathsf{a} = (0,q),\ \mathsf{b} = (t, q+vt).$$
$$J^-_{q,v;t}\mathsf{X} = \mathsf{X} \cap \overline{\mathsf{ab}}_-, \qquad \mathsf{a} = (0,q),\ \mathsf{b} = (t, q+vt).$$

Displacement of an initial particle (q,v) with respect to its free trajectory:

$$j_{\mathsf{X}}(q,v;t) = \xi(\mathsf{b}) - \xi(\mathsf{a}), \qquad \mathsf{a} = (0,q),\ \mathsf{b} = (t, q+vt).$$

Corollary: Law of large numbers for hard rods

Let X^ε be a Poisson process with intensity $f \in \mathcal{F}$
(finite second moment of velocity and lenght marginals)

Let $\mathsf{Y}^\varepsilon = D_0\mathsf{X}^\varepsilon$ and $\mathtt{K}_t^\varepsilon :=$ empirical length measure of $U_t\mathsf{Y}^\varepsilon$.

<u>then</u> $\lim_{\varepsilon\to 0} \mathtt{K}_t^\varepsilon \varphi = G_t\varphi,$ almost surely, $t \in \mathbb{R}$,

where G_t has density

$$g(q, v, r; t) := r\ (\mathcal{S}_{o_t} D_0 T_t f)(q, v, r),$$

and $g(q, v, r; t)$, is the unique solution of the Cauchy problem.

Corollary: Fluctuations for hard rods

$$\lim_{\varepsilon\to 0} \frac{\mathtt{K}_t^\varepsilon \varphi - G_t\varphi}{\varepsilon^{1/2}} \overset{\text{law}}{=} \text{Normal}(0, \tilde{G}_t\varphi^2)$$

where $\tilde{G}_t$ has density

$$\tilde{g}(q,v,r;t) := r^2(\mathcal{S}_{o_t} D_0 T_t f)(q,v,r),$$

Many open problems

KPZ ?

Brownian motion with drift instead of straight lines

Periodic curves

Relation with KdV

Larger dimensions:

Poisson Hyperplane processes and Levy Brownian hypersurface.

References

[1] Michael Aizenman, Sheldon Goldstein, and Joel L Lebowitz, *Ergodic properties of an infinite one dimensional hard rod system*, Communications in Mathematical Physics **39** (1975), no. 4, 289–301.

[2] Michael Aizenman, Joel Lebowitz, and Joaquin Marro, *Time-displaced correlation functions in an infinite one-dimensional mixture of hard rods with different diameters*, Journal of Statistical Physics **18** (1978), no. 2, 179–190.

[3] C. Boldrighini, R. L. Dobrushin, and Yu. M. Sukhov, *One-dimensional hard rod caricature of hydrodynamics*, J. Stat. Phys. **31** (1983), no. 3, 577–616.

[4] Xiangyu Cao, Vir B. Bulchandani, and Herbert Spohn, *The GGE averaged currents of the classical Toda chain*, J. Phys. A **52** (2019), no. 49, 495003, 12. MR4051734

[5] Nikolai Nikolaevich Chentsov, *Lévy brownian motion for several parameters and generalized white noise*, Theory of Probability & Its Applications **2** (1957), no. 2, 265–266.

[6] David A. Croydon and Makiko Sasada, *Generalized hydrodynamic limit for the box-ball system*, Comm. Math. Phys. **383** (2021), no. 1, 427–463. MR4236070

[7] Benjamin Doyon, *Generalized hydrodynamics of the classical Toda system*, J. Math. Phys. **60** (2019), no. 7, 073302, 21. MR3980061

[8] Benjamin Doyon, Takato Yoshimura, and Jean-Sébastien Caux, *Soliton gases and generalized hydrodynamics*, Physical review letters **120** (2018), no. 4, 045301.

[9] Pablo A. Ferrari, Chi Nguyen, Leonardo T. Rolla, and Minmin Wang, *Soliton decomposition of the box-ball system*, Forum Math. Sigma **9** (2021), Paper No. e60, 37. MR4308823

[10] M. Girotti, T. Grava, R. Jenkins, and K. D. T.-R. McLaughlin, *Rigorous asymptotics of a KdV soliton gas*, Comm. Math. Phys. **384** (2021), no. 2, 733–784. MR4259375

[11] T. E. Harris, *Diffusion with "collisions" between particles*, J. Appl. Probability **2** (1965), 323–338. MR184277

[12] Christian Lantuéjoul, *Geostatistical simulation: models and algorithms*, Springer Science & Business Media, 2013.

[13] Paul Lévy, *Processus Stochastiques et Mouvement Brownien. Suivi d'une note de M. Loève*, Gauthier-Villars, Paris, 1948. MR0029120

[14] Mikhail Lifshits, *Lectures on Gaussian processes*, SpringerBriefs in Mathematics, Springer, Heidelberg, 2012. MR3024389

[15] Frank Spitzer, *Uniform motion with elastic collision of an infinite particle system*, J. Math. Mech. **18** (1968/1969), 973–989. MR0243646

[16] ______, *Random processes defined through the interaction of an infinite particle system*, Probability and Information Theory (Proc. Internat. Sympos., McMaster Univ., Hamilton, Ont., 1968), 1969, pp. 201–223. MR0268964

[17] Herbert Spohn, *Large scale dynamics of interacting particles*, Berlin etc.: Springer-Verlag, 1991 (English).

[18] Herbert Spohn, *Collision rate ansatz for the classical toda lattice*, Physical Review E **101** (2020), no. 6, 060103.

[19] Charles Stone, *On a theorem by Dobrushin*, Ann. Math. Statist. **39** (1968), 1391–1401. MR231441