KPZ equation with a small noise, deep upper tail and limit shape

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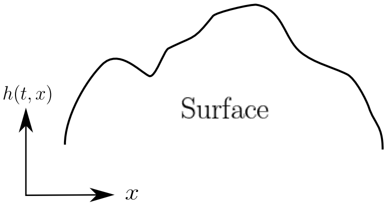
- Introduced by [Kardar-Parisi-Zhang 86]

$$\partial_t h(t,x) = \frac{1}{2}\partial_{xx} h(t,x) + \frac{1}{2}(\partial_x h(t,x))^2 + \xi(t,x),$$

ξ is the space-time white noise, $\mathbb{E}[\xi(t,x)\xi(s,y)] = \delta(t-s)\delta(x-y)$.



- The solution theory is ill posed in the classical way.

The SHE

- Look at the stochastic heat equation (SHE)

$$\partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + Z(t,x)\xi(t,x),$$

We say $h(t,x) := \log Z(t,x)$ is the Hopf-Cole solution to be the KPZ equation.

- Mild solution of the SHE

$$Z(t,x) = \int_{\mathbb{R}} p(t,x-y)Z(0,y)dy + \int_0^t \int_{\mathbb{R}} p(t-s,x-y)Z(s,y)\xi(s,y)dsdy.$$

$p(t,x) := \frac{1}{\sqrt{2\pi t}}e^{-\frac{x^2}{2t}}$ is the heat kernel.

- Iterating above one obtain the chaos expansion of $Z(t,x)$.

- [Mueller 91] positivity of the solution to the SHE.
- Dirac-Delta initial data

$$\partial_t Z(t,x) = \frac{1}{2}\partial_{xx} Z(t,x) + \xi(t,x) Z(t,x), \qquad Z(0,\cdot) = \delta(\cdot).$$

  [Moreno-Flores 14] $Z(t,x)$ is positive for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}$.
- We refer $h = \log Z$ to be the solution to the KPZ equation starting from the **narrow wedge initial data.**

Review of previous results

- Focus on narrow wedge initial data.
- One point fluctuation [Amir-Corwin-Quastel 11]
  - Multi-point results [Quastel-Sarkar 20], [Virag 20].
- Tail behavior, one point large deviations (long time lower tail) [Ghosal-Corwin 18], [Tsai 18], [Cafasso-Claeys 19] (long time upper tail) [Das-Tsai 19] (short time behavior) [L.-Tsai 20].
- Law of iterated logarithm [Das-Ghosal 21]
- Also (more) results in the physics literature by Kamenev, Krajenbrink, Le Doussal, Meerson, Sasorov...

KPZ equation with a small noise

- Consider
$$\partial_t h_\varepsilon(t,x) = \frac{1}{2}\partial_{xx} h_\varepsilon(t,x) + \frac{1}{2}(\partial_x h_\varepsilon(t,x))^2 + \sqrt{\varepsilon}\xi(t,x).$$
with narrow wedge initial data.

- Letting $\varepsilon \to 0$, it is intuitive that $h_\varepsilon \to \mathbf{h} = \log p(t,x)$ which solves
$$\partial_t \mathbf{h}(t,x) = \frac{1}{2}\partial_{xx}\mathbf{h}(t,x) + \frac{1}{2}(\partial_x \mathbf{h}(t,x))^2$$

- **Conditioning:** Force $h_\varepsilon(2,0) > \lambda$, what is the limit shape of h_ε on $[0,2] \times \mathbb{R}$ for large λ?

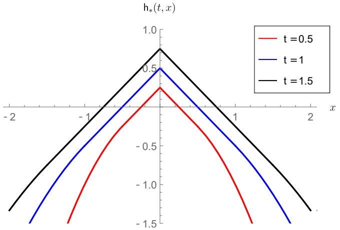
Theorem (Gaudreau Lamarre-L.-Tsai 21)

Fix arbitrary $\delta > 0$. *Define* $h_{\varepsilon,\lambda}(t,x) = \lambda^{-1} h_\varepsilon(t, \lambda^{\frac{1}{2}} x)$, *we have*

$$\lim_{\lambda\to\infty} \lim_{\varepsilon\to 0} \mathbb{P}\Big(dist_\delta(h_{\varepsilon,\lambda}, \mathsf{h}_*) < \delta \,\Big|\, h_\varepsilon(2,0) \geq \lambda \Big) = 1,$$

where $dist_\delta(f,g) = \|f-g\|_{L^\infty([\delta,2]\times[-\delta^{-1},\delta^{-1}])}$ *and*

$$\mathsf{h}_*(t,x) = \begin{cases} \frac{t}{2} - |x|, & |x| \leq t, \\ -\frac{x^2}{2t} & |x| \geq t. \end{cases}$$



This result confirms the prediction by [Kamenev-Meerson-Sasarov 16].

- The theorem is equivalent to show that

$$\lim_{\lambda\to\infty}\lim_{\varepsilon\to 0}\mathbb{P}\Big(\mathrm{dist}_\delta(\lambda^{-1}\log Z_{\varepsilon,\lambda},\mathsf{h}_*)\,\Big|\,Z_\varepsilon(2,0)\geq e^\lambda\Big)=1$$

where $Z_{\varepsilon,\lambda}(\bullet,\bullet)=Z_\varepsilon(\bullet,\lambda^{\frac{1}{2}}\bullet)$ solves the SHE

$$\partial_t Z_\varepsilon(t,x)=\frac{1}{2}\partial_{xx}Z_\varepsilon(t,x)+\sqrt{\varepsilon}\xi(t,x)Z_\varepsilon(t,x),$$
$$Z_\varepsilon(0,\bullet)=\delta(\bullet).$$

- We start with a result of functional large deviation principle with Z_ε.

Theorem (L.-Tsai 20)

Fix $\delta > 0$. $\{Z_\varepsilon(\bullet, \bullet)\}_{\varepsilon\in(0,1)} \subseteq C([\delta, 2] \times \mathbb{R})$ satisfies a LDP with speed ε^{-1} and a good rate function

$$I(f) = \inf\Big\{\frac{1}{2}\|\rho\|^2_{L^2([0,2]\times\mathbb{R})} : \mathsf{Z}(\rho) = f, \rho \in L^2([0, 2] \times \mathbb{R})\Big\}.$$

where $\mathsf{Z}(\rho) = \mathsf{Z}(\rho; t, x)$ is defined as the mild solution of

$$\partial_t \mathsf{Z}(\rho; t, x) = \frac{1}{2}\partial_{xx}\mathsf{Z}(\rho; t, x) + \rho(t, x)\mathsf{Z}(\rho; t, x), \qquad \mathsf{Z}(\rho; 0, \bullet) = \delta(\bullet).$$

More explicitly,

$$\mathsf{Z}(\rho; t, x) := p(t, x)\mathbb{E}\Big[\exp\Big(\int_0^2 \rho(s, B_b(s))ds\Big)\Big].$$

$p(t, x)$ is the heat kernel and B_b is a Brownian bridge from $(0, 0)$ to $(2, 0)$.

An important observable

- Let
$$\widetilde{\mathcal{K}}_\lambda = \arg\inf\left\{ \tfrac{1}{2}\|\rho\|^2_{L^2([0,2]\times\mathbb{R})} : \mathsf{Z}(\rho;2,0) \geq e^\lambda \right\}.$$
- By scaling, $\widetilde{\mathcal{K}}_\lambda = \{\lambda\rho(\lambda\bullet, \lambda^{\frac{1}{2}}\bullet) : \rho \in \mathcal{K}_\lambda\}$, where
$$\mathcal{K}_\lambda = \arg\inf\left\{ \tfrac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} : \mathsf{Z}(\rho;2\lambda,0) \geq \frac{1}{\sqrt{\lambda}} e^\lambda \right\}.$$

Proposition (Gaudreau Lamarre-L.-Tsai 21)

We have
$$\lim_{\varepsilon\to 0} \mathbb{P}\Big(dist_\delta(\lambda^{-1}\log Z_{\varepsilon,\lambda}, \mathsf{h}_\lambda(\mathcal{K}_\lambda)) < \delta \,\Big|\, Z_\varepsilon(2,0) \geq e^\lambda \Big) = 1.$$
where $\mathsf{h}_\lambda(\rho;t,x) := \lambda^{-1}\log(\lambda^{\frac{1}{2}}\mathsf{Z}(\rho;\lambda t,\lambda x))$.

The main theorem will be concluded if we can show that
$$\lim_{\lambda\to\infty} \mathrm{dist}_\delta(\mathsf{h}_\lambda(\mathcal{K}_\lambda), \mathsf{h}_*) = 0$$
So, what is the $\lambda \to \infty$ limit of $\mathcal{K}_\lambda$?

$\lambda \to \infty$ limit of $\mathcal{K}_\lambda$

Proposition (Gaudreau Lamarre-L.-Tsai 21)

Recall that

$$\mathcal{K}_\lambda = \arg\inf\left\{ \tfrac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} : \mathsf{Z}(\rho; 2\lambda, 0) \geq \frac{1}{\sqrt{\lambda}} e^{\lambda} \right\}.$$

We have

$$\lim_{\lambda\to\infty} \frac{1}{2\lambda} \sup\{\|\rho - sech^2\|_{L^2([0,2\lambda]\times\mathbb{R})} : \rho \in \mathcal{K}_\lambda\} = 0,$$

Heuristic for the proposition

- **Assume** ρ is **time-independent** and write $\rho(t,\cdot) = \varphi(\cdot)$. Then

$$\mathcal{K}_\lambda = \arg\inf \left\{ \|\varphi\|^2_{L^2(\mathbb{R})} : \mathsf{Z}(\varphi; 2\lambda, 0) \geq \frac{1}{\sqrt{\lambda}} e^{\lambda} \right\}.$$

- View the PDE as $\frac{d}{dt}\mathsf{Z}(t) = A^\varphi \mathsf{Z}(t)$, where $A^\varphi = \frac{1}{2}\partial_{xx} + \varphi$. $\mathsf{Z}(\varphi; 2\lambda, 0)$ should grow as $\exp(2\lambda F(\varphi))$ as $\lambda \to \infty$, where

$$F(\varphi) = \sup \left\{ \int_{\mathbb{R}} \varphi g^2 - \frac{1}{2} g'^2 : g \in H^1(\mathbb{R}) \text{ and } \|g\|_{L^2(\mathbb{R})} = 1 \right\}.$$

- Since $\mathsf{Z}(\varphi; 2\lambda, 0) \sim \exp(2\lambda F(\varphi))$, $\mathcal{K}_\lambda$ **approximates**

$$\arg\inf \{ \|\varphi\|^2_{L^2(\mathbb{R})} : F(\varphi) \geq \frac{1}{2} \}.$$

- This set is given by $\{\mathrm{sech}^2(\cdot - v)\}_{v \in \mathbb{R}}$ (see next page).

A L^4 GNS inequality

Lemma (L^4 Gagliardo-Nirenberg-Sobolev inequality)

For $g \in L^2(\mathbb{R})$ and $g' \in L^2(\mathbb{R})$, we have

$$\|g\|_{L^4(\mathbb{R})} \le 3^{-\frac{1}{8}} \|g'\|_{L^2(\mathbb{R})}^{\frac{1}{4}} \|g\|_{L^2(\mathbb{R})}^{\frac{3}{4}}$$

By solving a differential equation, it is known e.g. [Dolbeault-Esteban-Laptev-Loss 14] that the equality holds iff

$$g(x) = a \cdot \mathrm{sech}(b(x-v))$$

for some fixed a, b, v.

Lemma (Gaudreau Lamarre-L.-Tsai 21)

$$F(\varphi) = \sup\left\{ \int_{\mathbb{R}} \varphi g^2 - \frac{1}{2} g'^2 : g \in H^1(\mathbb{R}) \text{ and } \|g\|_{L^2(\mathbb{R})} = 1 \right\}$$
$$\le \frac{1}{2}\left(\frac{3}{4}\right)^{\frac{2}{3}} \|\varphi\|_{L^2(\mathbb{R})}^{\frac{4}{3}}$$

The inequality becomes an equality if and only if $\varphi(x) = \alpha^2 \mathrm{sech}^2(\alpha(x-v))$.

Some Challenges for the proof

- Characterization of $\mathcal{K}_\lambda$.
    - $\mathcal{K}_\lambda$ is not empty.
    - $\mathcal{K}_\lambda$ only contains non-negative symmetric and decreasing function in space.
    - L^2-norm estimate of $\rho \in \mathcal{K}_\lambda$.

$\mathcal{K}_\lambda$ is not empty

- Recall that

$$\mathcal{K}_\lambda = \arg\inf\Big\{\tfrac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} : \mathsf{Z}(\rho;2\lambda,0) \geq \frac{1}{\sqrt{\lambda}}e^{\lambda}\Big\}.$$

- The problem is that $\{\|\rho\|_{L^2([0,2\lambda]\times\mathbb{R})} \leq r\}$ is not compact in the L^2 topology.

- **Remedy:** we can find larger space $\mathcal{B}$ such that
  - The map $\mathsf{Z} : \rho \to \mathsf{Z}(\rho;2\lambda,0)$ is continuous from $\mathcal{B}$ to $\mathbb{R}$.
  - $\{\|\rho\|_{L^2([0,2\lambda]\times\mathbb{R})} \leq r\}$ forms a **compact** set in $\mathcal{B}$.

L^2-norm estimate

The following lemma intrinsically follows from [Chen 10].

Lemma

For φ continuous and bounded,

$$\lim_{\lambda\to\infty} \lambda^{-1} \log \mathbb{E}_{0\to 0}\left[\exp\left(\int_0^\lambda \varphi(B_b(s))\right)\right] = F(\varphi).$$

As a consequence,

$$\mathsf{Z}(\mathrm{sech}^2; 2\lambda, 0) = p(2\lambda, 0)\mathbb{E}_{0\to 0}\left[\exp\left(\int_0^{2\lambda} \mathrm{sech}^2(B_b(s))ds\right)\right] \sim e^\lambda.$$

Corollary

For $\rho \in \mathcal{K}_\lambda$, $\frac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} \leq \frac{4}{3} + o_\lambda(1)$.

- What does $\mathsf{Z}(\rho; 2\lambda, 0) \geq \frac{1}{\sqrt{\lambda}} e^{\lambda}$ tell about the L^2 norm of ρ.

Proposition

We have

$$\mathsf{Z}(\rho; 2\lambda, 0) \leq C \exp\Big(\int_0^{2\lambda} F(\rho(r, \cdot)) dr\Big).$$

Proof idea.

Assume that $Z(0, x) = f(x) \in C_c^\infty(\mathbb{R})$ and $\rho \in C_c^\infty(\mathbb{R}^2)$. Then

$$\partial_r \mathsf{Z}(r, x) = \frac{1}{2} \partial_{xx} \mathsf{Z}(r, x) + \rho(r, x) \mathsf{Z}(r, x).$$

Multiply both sides by $\mathsf{Z}(r, x)$ and integrate in x,

$$\frac{1}{2} \partial_r \|\mathsf{Z}(r, \cdot)\|_{L^2(\mathbb{R})}^2 \leq F(\rho(r, \cdot)) \|\mathsf{Z}(r, \cdot)\|_{L^2(\mathbb{R})}^2$$

Integrate in r, $\|\mathsf{Z}(t, \cdot)\|_{L^2(\mathbb{R})}^2 \leq \exp\left(2 \int_0^t F(\rho(r, \cdot)) dr\right) \|f\|_{L^2(\mathbb{R})}^2$

...

□

Key step

- If $\rho \in \mathcal{K}_\lambda$,

$$\frac{1}{2\lambda}\|\rho\|^2_{L^2([0,2\lambda]\times\mathbb{R})} \leq \frac{4}{3} + o_\lambda(1).$$

- For $\rho \in \mathcal{K}_\lambda$, we have

$$\begin{aligned}\frac{1}{\sqrt{\lambda}}e^\lambda \leq \mathsf{Z}(\rho; 2\lambda, 0) &\leq C\exp\left(\int_0^{2\lambda} F(\rho(r,\cdot))dr\right)\\ &\leq C\exp\left(\int_0^{2\lambda}\frac{1}{2}\left(\frac{3}{4}\right)^{\frac{2}{3}}\|\rho(r,\cdot)\|^{\frac{4}{3}}_{L^2(\mathbb{R})}dr\right)\\ &\leq C\exp\left(\int_0^{2\lambda}\frac{1}{4}\left(\|\rho(r,\cdot)\|^2_{L^2(\mathbb{R})} + \tfrac{2}{3}\right)dr\right) \leq Ce^{\lambda + o_\lambda(1)}\end{aligned}$$

  - $F(\rho(r,\cdot))$ can not be far from $\frac{1}{2}(\frac{3}{4})^{\frac{2}{3}}\|\rho(r,\cdot)\|^{\frac{4}{3}}_{L^2(\mathbb{R})}$.
  - $\|\rho(r,\cdot)\|^2_{L^2(\mathbb{R})}$ can not be far away from $\frac{4}{3}$.

Lemma

Consider a sequence of symmetric decreasing function $\{\varphi_n\}_{n=1}^{\infty}$ *satisfying* $\|\varphi_n\|^2_{L^2(\mathbb{R})} = \frac{4}{3}$ *and* $F(\varphi_n) \to \frac{1}{2}$*, then* $\varphi_n \to sech^2$ *in* $L^2(\mathbb{R})$*.*

This is enough to show $\mathcal{K}_\lambda \to \mathrm{sech}^2$. To conclude the limit shape, need to show $\mathsf{h}_\lambda(\mathcal{K}_\lambda) \to \mathsf{h}_*$ where $\mathsf{h}_\lambda(\rho; t, x) := \lambda^{-1} \log(\lambda^{\frac{1}{2}} \mathsf{Z}(\rho; \lambda t, \lambda x))$.

We show

- $\mathsf{h}_\lambda(\mathcal{K}_\lambda)$ and $\mathsf{h}_\lambda(\mathrm{sech}^2)$ is close.
- $\mathsf{h}_\lambda(\mathrm{sech}^2) \to \mathsf{h}_*$.

- We have

$$\mathsf{h}_\lambda(\operatorname{sech}^2) = \lambda^{-1} \log \lambda^{\frac{1}{2}} \mathsf{Z}(\operatorname{sech}^2; \lambda t; \lambda x)$$
$$= \lambda^{-1} \log \mathbb{E}_{\lambda x \to 0}\Big[\exp\Big(\int_0^{\lambda t} \operatorname{sech}^2(B_b(s) ds)\Big)\Big] - \frac{x^2}{2t} - \lambda^{-1} \log \sqrt{4\pi}$$

- Let η be the hitting time of zero. We have

$$\mathbb{E}_{\lambda x \to 0}\Big[\exp\Big(\int_0^{\lambda t} \operatorname{sech}^2(B_b(s) ds)\Big)\Big] \approx \mathbb{E}\Big[\exp\big(\tfrac{1}{2}(\lambda t - \eta)\big)\Big].$$

- We have $\mathbb{P}(\eta \approx \lambda s) \approx \exp(-\frac{\lambda x^2 (t-s)}{2st})$. Hence the limit is

$$\sup\Big\{\frac{1}{2}(t-s) - \frac{x^2(t-s)}{2st}\Big\} = \mathsf{h}_*(t, x) + \frac{x^2}{2t}.$$

A final remark

- The physics work [Krajenbrink-Le Doussal 21] solves the finite λ limit shape (conditioning on $h_\varepsilon(2,0) > \lambda$ for fixed λ and send $\varepsilon \to 0$) by solving the $\{P, Q\}$ system

$$\partial_t Q = \frac{1}{2}\partial_x^2 Q + PQ^2,$$
$$-\partial_t P = \frac{1}{2}\partial_x^2 P + P^2 Q.$$

which (formally) can be seen from the variational formula. A major problem would be building (rigorous) relation between the solution to the $\{P, Q\}$ system and the large deviation of the KPZ equation.

Thank you!