Invariant measures for multilane exclusion process

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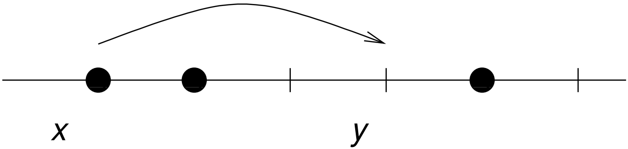
Outline

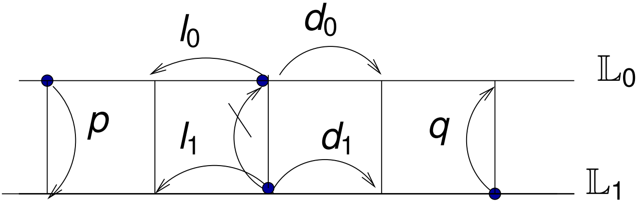
Basic example : The SEP



- Configuration η of particles : for $z \in S$ (countable), $\eta(z) = 0$ or 1. From each site x, choice of y with $p(x, y)$, (translation invariant if $p(x, y) = p(y - x)$, n.n. if $p(y - x) = 0$ for $|y - x| \neq 1$).
  ASEP ("asymmetric simple exclusion process") if $\sum_x xp(x) > 0$.
  TASEP ("Totally asymmetric simple exclusion process") if $p(1) = 1$ for $S = \mathbb{Z}$.
- According to (independent) exponential clocks, jump from x to y if possible (*exclusion rule*).

Our model : multi-lane exclusion

Sites $V = \mathbb{Z} \times W$ with $W = \{0, \cdots, n-1\}$. For $i \in W$, the i'th lane of V is $\mathbb{L}_i := \{x \in V : x = (x(0), x(1)),\ x(0) \in \mathbb{Z},\ x(1) = i\}$ State space $\mathcal{X} = \{0,1\}^V$, with n.n. jumps on V.



We assume $(d_0 + l_0)(d_1 + l_1) > 0$, so particles can always move on both lanes. But they cannot go from $\mathbb{L}_0$ to $\mathbb{L}_1$ if $p = 0$, nor from $\mathbb{L}_1$ to $\mathbb{L}_0$ if $q = 0$. If $p = q = 0$, 2 independent SEP's on each lane. Thus if $p + q \neq 0$, interaction between the two lanes. For $i \in W$, $\gamma_i := d_i - l_i$ is the mean drift on lane i. Because of symmetries we assume w.l.o.g. that

$$\gamma_0 \geq 0, \quad \gamma_0 + \gamma_1 \geq 0, \quad p \geq q, \quad p > 0 \tag{1}$$

Why this model ? Which questions ?

- Two interpretations
  - Traffic-flow modeling : V is a highway, with lanes $\mathbb{L}_i$ on which cars have different speeds and different directions. The steps between the lanes are the direction a car can follow to change lane.
  - Particle species : $i \in W$ is a particle species, the dynamics within each species is a SEP on $\mathbb{Z}$, and a lane change becomes a spin flip for a particle to change species. By the exclusion rule, a particle cannot change its species if there is already a particle of the other species sitting at the same site. This is the only interaction between the two species.
- An intermediate model between $\mathbb{Z}$ and $\mathbb{Z}^2$: already new phenomena

Questions :

- Equilibrium : Invariant measures ? in this talk http ://arxiv.org/abs/2105.12974
- Out of equilibrium : hydrodynamics ? In preparation

Invariant measures for SEP : what is known ?

[Lig] *Interacting particle systems.* Springer, 2005.

[FLS] Ferrari, P. A., Lebowitz, J. L., Speer, E. (2001). Blocking measures for asymmetric exclusion processes via coupling Bernoulli, 7 no. 6, 935–950.
[BLM] Bramson, M., Liggett, T. M. and Mountford, T. (2002). Characterization of stationary measures for one-dimensional exclusion processes. Ann. Probab. 30, 1539–1575.
[BM] Bramson, M. and Mountford, T. (2002). Stationary blocking measures for one-dimensional nonzero mean exclusion processes. Ann. Probab. 30, 1082–1130.

[BL] Bramson, M. and Liggett, T. M. (2005). Exclusion processes in higher dimensions : stationary measures and convergence. Ann. Probab. 33, 2255–2313.

Invariant measures for SEP : what is known ?

[Theorem VIII.2.1, Lig]

ν_α : product measure on S with marginals

$$\nu_\alpha\{\eta : \eta(x) = 1\} = \alpha(x) \tag{2}$$

(a) If $\forall y \in S$, $\sum_x p(x,y) = 1$, then $\nu_\alpha \in \mathcal{I}$ for every constant $\alpha \in [0,1]$ (Bernoulli product measures).
(b) If $\pi(.)$ satisfies

$$\pi(x)p(x,y) = \pi(y)p(y,x), \quad \forall x, y \in S \tag{3}$$

or equivalently

$$\alpha(x)(1-\alpha(y))p(x,y) = \alpha(y)(1-\alpha(x))p(y,x) \tag{4}$$

$$\text{Then} \quad \nu_\alpha \in \mathcal{I} \quad \text{where} \quad \alpha(x) = \frac{\pi(x)}{1+\pi(x)} \tag{5}$$

[Theorem VIII.3.9, Lig]

$$\text{If } S = \mathbb{Z}^d,\ p(x,y) = p(y-x), \quad (\mathcal{I} \cap \mathcal{S})_e = \{\nu_\alpha,\ \alpha \in [0,1]\} \tag{6}$$

Invariant measures for SEP on $\mathbb{Z}$: what is known ?

$S = \mathbb{Z}$, $p(x, y) = p(y - x)$, and irreducibility, i.e. $\forall x, y \in \mathbb{Z}$, $x \to_p y$.
A proba. measure μ on $\{0, 1\}^{\mathbb{Z}}$ is a *blocking measure* if it concentrates on configurations η s.t.

$$\sum_{x<0} \eta(x) + \sum_{x>0} [1 - \eta(x)] < +\infty$$

and it is a *profile measure* if

$$\lim_{x\to-\infty} \mu\{\eta : \eta(x) = 1\} = 0 \quad \text{and} \quad \lim_{x\to+\infty} \mu\{\eta : \eta(x) = 1\} = 1$$

Every blocking measure is a profile measure, but not conversely.

Invariant measures for SEP on $\mathbb{Z}$: what is known ?

Assuming w.l.o.g. $\sum_x xp(x) \geq 0$, we have

1. [BLM] Either (i) $\mathcal{I}_e = \{\nu_\rho,\ \rho \in [0,1]\}$, or (ii) $\mathcal{I}_e = \{\nu_\rho,\ \rho \in [0,1]\} \cup \{\mu_n,\ n \in \mathbb{Z}\}$, where μ_0 is a profile measure, and $\mu_n = \tau_n \mu_0$.
2. [Lig] If $\sum_x xp(x) = 0$, then (i) occurs.
3. [FLS] If $\sum_x xp(x) > 0$; $p(x)$ and $p(-x)$ are decreasing for $x \geq 1$; for $a < 1$, $a^x p(x) \geq p(-x)\, \forall x \geq 1$; then there exists a blocking measure.
4. [BM] If $\sum_x xp(x) > 0$ and $p(.)$ is finite range, then (ii) occurs and μ_0 is a blocking measure.
5. [BLM] If $\sum_x xp(x) > 0$; $p(x)$ and $p(-x)$ are decreasing for $x \geq 1$; $p(x) \geq p(-x)\, \forall x \geq 1$; then (ii) occurs and μ_0 is a blocking measure. The coupling of [FLS] is used.
6. [BLM] If $\sum_{x<0} x^2 p(x) = +\infty$, there exists no stationary blocking measure.

An important open problem : determine whether nonblocking stationary profile measures ever exist.

Invariant measures for SEP on $\mathbb{Z}$: $p(1)+p(-1)=1$

- *Translation invariant measures.* Homogeneous product Bernoulli proba. measures $\{\mu_\rho, \rho \in [0,1]\}$, ρ is the average particle density per site ([Theorem VIII.2.1 (a), Lig]).
- *Blocking measures for ASEP, $p(1)=d, p(-1)=l$, $d \neq l$.* Invariant (non translation invariant) proba. measures ([Theorem VIII.2.1 (b), Lig]) :

$$\text{When } l > 0, \text{ for } c > 0, \quad \rho_i^c := \frac{c\left(\frac{d}{l}\right)^i}{1+c\left(\frac{d}{l}\right)^i} \tag{7}$$

When $l = 0 < d$ (TASEP), for $n \in \mathbb{Z}$ and $c \geq 0$,

$$\rho_i^{n,c} := \mathbf{1}_{\{i>n\}} + \frac{c}{1+c}\mathbf{1}_{\{i=n\}}, \quad i \in \mathbb{Z} \tag{8}$$

$\rho_.$ is a solution of (4) iff of the form (7) when $l > 0$, or (8) when $l = 0 < d$.

Invariant measures for SEP on $\mathbb{Z}$: $p(1) + p(-1) = 1$

For such $\rho_{.}$, $\mu^{\rho_\cdot}$ defined by (5) is reversible.

- If $I > 0$, $\mu^{\rho_\cdot}$ with $\rho_{.} = \rho_{.}^{c}$ given by (7) is not extremal invariant.

$\mu^{\rho_\cdot}$ is supported on the set

$$\left\{\eta \in \{0,1\}^{\mathbb{Z}} : \sum_{x>0}[1-\eta(x)] + \sum_{x\leq 0}\eta(x) < +\infty\right\} \qquad (9)$$

$H(\eta) := \sum_{x\leq 0}\eta(x) - \sum_{x>0}[1-\eta(x)]$ is conserved by SEP if initially finite ; SEP restricted to a level set of H is irreducible ;
$H(\tau_n\eta) = H(\eta) + n, \quad \forall n \in \mathbb{Z}$.
Then, for $c > 0$, $n \in \mathbb{Z}$,

$$\widehat{\mu}_n := \mu^{\rho_{.}^{c}}(.\,|H(\eta) = n) \qquad (10)$$

does not depend on $c > 0$, is extremal invariant, and $\widehat{\mu}_n = \tau_n\widehat{\mu}_0$.

- For $I = 0$, $\mu^{\rho_\cdot}$ with $\rho_{.} = \rho_{.}^{n,c}$ given by (8) is extremal invariant iff $c = 0$; again denoted by $\widehat{\mu}_n$.

$$\eta_n^*(x) := \mathbf{1}_{\{x>n\}}, \quad \widehat{\mu}_n := \delta_{\eta_n^*} \qquad (11)$$

Invariant measures for SEP in higher dimensions : what is known ?

Not much is known when $S = \mathbb{Z}^d$.

[BL] gives necessary and sufficient conditions to have $\nu_\alpha \in \mathcal{I}$, which gives examples of stationary product measures that are neither homogeneous nor reversible.
Also : conditions for various types of measures to be invariant, but no characterization.
The last section of the paper is devoted to open problems ; among them one on *the cyclic ladder*.

$\mathcal{I} \cap \mathcal{S}$ for two-lane SEP

two-parameter “Bernoulli product proba. measure” ν^{ρ_0,ρ_1} for $(\rho_0,\rho_1) \in [0,1]^2$, on $\mathcal{X}$ such that

$$\nu^{\rho_0,\rho_1}(\eta(x)=1) = \begin{cases} \rho_0 & x \in \mathbb{L}_0 \\ \rho_1 & x \in \mathbb{L}_1 \end{cases} . \tag{12}$$

- $p = q = 0$: independent SEP’s on the lanes $\Rightarrow \nu^{\rho_0,\rho_1} \in \mathcal{I}$ $\forall(\rho_0,\rho_1) \in [0,1]^2$.
- $p + q \neq 0$: is there a relation between ρ_0 and ρ_1 under which $\nu^{\rho_0,\rho_1} \in \mathcal{I}$? Let

$$\mathcal{F} := \left\{(\rho_0,\rho_1) \in [0,1]^2 : \; p\rho_0(1-\rho_1) - q\rho_1(1-\rho_0) = 0\right\}$$

This is the reversibility equation (4) *in the vertical direction*.

$\mathcal{F}$ expresses an equilibrium relation for vertical jumps : under ν^{ρ_0,ρ_1}, the mean algebraic “creation rate” on each lane (i.e. resulting from jumps from/to the other lane) has to be 0.

Theorem

$$(\mathcal{I} \cap \mathcal{S})_e = \{\nu^{\rho_0,\rho_1} : (\rho_0, \rho_1) \in \mathcal{F}\} = \{\nu_\rho : \rho \in [0,2]\} \qquad (13)$$

where ρ represents the total mean density over the two lanes :

$$\mathbb{E}_{\nu_\rho}[\eta^0(0) + \eta^1(0)] = \rho \qquad (14)$$

(η^i is the configuration on lane i : for $z \in \mathbb{Z}$, $\eta^i(z) = \eta(z, i)$).

Tools :

- $\mathcal{F}$ can be parametrized by the total density

$$\rho \rightsquigarrow \widetilde{\rho}_0(\rho), \widetilde{\rho}_1(\rho) = 1 - \widetilde{\rho}_0(\rho).$$

For instance, if $p = q \neq 0$,

$$\mathcal{F} = \{(\rho/2, \rho/2) : \rho \in [0,2]\},$$

and if $q = 0 < p$,

$$\mathcal{F} = \{(0, \rho) : \rho \in [0,1]\} \cup \{(\rho - 1, 1) : \rho \in [1,2]\}$$

- Next we define

$$\nu_\rho := \nu^{\widetilde{\rho}_0(\rho),\widetilde{\rho}_1(\rho)} \tag{15}$$

and we have for $i \in \{0, 1\}$,

$$\mathbb{E}_{\nu_\rho}[\eta^i(0)] = \widetilde{\rho}_i(\rho)$$

- To prove that $\nu^{\rho_0,\rho_1} \in \mathcal{I}$:

Separate the horizontal and vertical evolutions. ν^{ρ_0,ρ_1} is stationary non reversible on each lane (by [Theorem VIII.2.1 (a), Lig]) and reversible on each vertical step (by [Theorem VIII.2.1 (b), Lig]) because $(\rho_0, \rho_1) \in \mathcal{F}$.

$$L = \sum_{i \in W} L_h^i + \sum_{z \in \mathbb{Z}} L_v^z \tag{16}$$

where, for $i \in W, z \in \mathbb{Z}$,

$$\begin{aligned} L_h^i f(\eta) &= \sum_{z \in \mathbb{Z}} p\left((z,i),(z+1,i)\right) \eta^i(z)(1-\eta^i(z+1)) \times \\ &\quad \times \left(f\left(\eta^{(z,i),(z+1,i)}\right) - f(\eta)\right) \\ L_v^z f(\eta) &= \sum_{i,j \in W} p\left((z,i),(z,j)\right) \eta^i(z)(1-\eta^j(z)) \left(f\left(\eta^{(z,i),(z,j)}\right) - f(\eta)\right) \end{aligned}$$

L_h^i acts only on η^i, describes the evolution on $\mathbb{L}_i$, i.e. a (single-lane) SEP, for which ν^{ρ_0,ρ_1} is invariant.
L_v^z, acts only on $\{z\} \times W$, describes the motion along $\{z\} \times W$, i.e. the displacements from one lane to another at a fixed spatial location z, for which ν^{ρ_0,ρ_1} is invariant because $(\rho_0, \rho_1) \in \mathcal{F}$.

- To derive extremality, the scheme of proof mainly adapts the standard one (see [Lig]) + additional arguments to deal with discrepancies (their behavior is more tricky for the two-lane SEP) when $q = l_0 = l_1 = 0$.

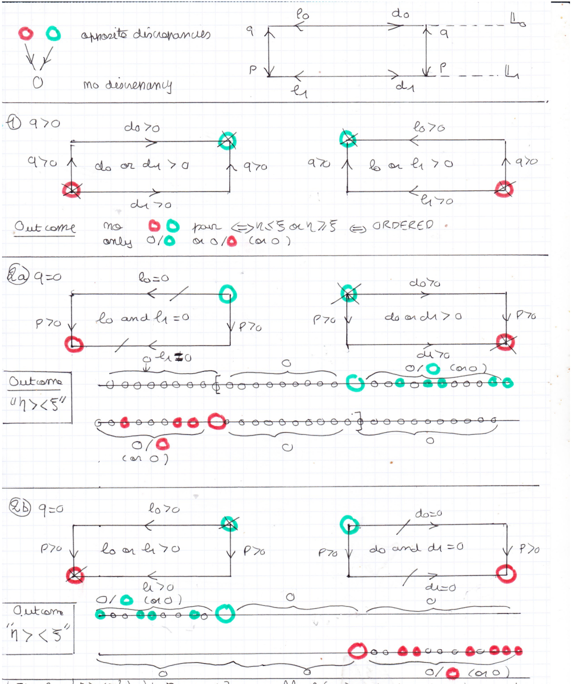
If $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$, at $x \in V$ there is an η *discrepancy* if $\eta(x) > \xi(x)$, a ξ discrepancy if $\eta(x) < \xi(x)$, a coupled particle if $\eta(x) = \xi(x) = 1$, a hole if $\eta(x) = \xi(x) = 0$. An η and a ξ discrepancy are *discrepancies of opposite type*.

x and y are p-connected if $x \rightarrow_p y$ or $y \rightarrow_p x$.
η, ξ in $\mathcal{X}$ are *p-ordered* if there exists no $(x, y) \in V \times V$ s.t. x and y are p-connected and (η, ξ) has discrepancies of opposite types at x and y.

Definition

$p(.,.)$ is *weakly irreducible* if, $\forall (x, y) \in V \times V$ s.t. $x \neq y$, x and y are p-connected.

When $l_0 = l_1 = q = 0$, $p(.,.)$ is not weakly irreducible.


opposite discrepancies
0
no discrepancy
ℓ0
d0
L0
q
q
p
p
L1
ℓ1
d1
① q>0
d0>0
q>0
d0 or d1>0
q>0
d1>0
ℓ0>0
q>0
ℓ0 or ℓ1>0
q>0
ℓ1>0
Outcome
no pair ⟺ η≤ξ or η≥ξ ⟺ ORDERED
only 0/ or 0/ (or 0)
②a q=0
ℓ0=0
p>0
ℓ0 and ℓ1=0
p>0
0 ℓ1≠0
d0>0
p>0
d0 or d1>0
p>0
d1>0
Outcome
"η><ξ"
0
0/ (or 0)
0/
(or 0)
0
0
②b q=0
ℓ0>0
p>0
ℓ0 or ℓ1>0
p>0
ℓ1>0
d0=0
p>0
d0 and d1=0
p>0
d1=0
Outcome
"η><ξ"
0/ (or 0)
0
0
0
0
0/ (or 0)

to deal with $l_0 = l_1 = q = 0$, for which $p(.,.)$ is not weakly irreducible :

Definition

For $(\eta, \xi) \in \mathcal{X} \times \mathcal{X}$, we write $\eta >< \xi$ if and only if there exist $x, y \in \mathbb{Z}$ such that $x < y$ and the following hold : (a) there are discrepancies of opposite type at $(x, 1)$ and $(y, 0)$; (b) $\eta^0 \leq \xi^0$ and $\eta^1 \geq \xi^1$ if the discrepancy at $(x, 1)$ is an η discrepancy ; or $\eta^0 \geq \xi^0$ and $\eta^1 \leq \xi^1$ if the discrepancy at $(x, 1)$ is a ξ discrepancy ; (c) There is no discrepancy at $(z, 1)$ if $z > x$, nor any discrepancy at $(z, 0)$ if $z < y$.
We define

$$E_{><} := \{(\eta, \xi) \in \mathcal{X} \times \mathcal{X} : \eta >< \xi\} \tag{17}$$

Thanks to translation invariance,

Lemma

Let $\widetilde{\nu} \in (\widetilde{\mathcal{I}} \cap \widetilde{\mathcal{S}})$. *If* $l_0 = l_1 = q = 0$, *then* $\widetilde{\nu}(E_{><}) = 0$.

Structure of invariant measures for two-lane SEP

Let $\mathcal{D} := \{(\rho,\rho) : \rho \in [0,2]\}$

$(\rho^-,\rho^+) \in [0,2]^2 \setminus \mathcal{D}$ is a *shock.*
A proba. measure μ on $\mathcal{X}$ is a (ρ^-,ρ^+)-*shock measure* if

$$\lim_{n\to-\infty} \tau_n\mu = \nu_{\rho^-}, \quad \lim_{n\to+\infty} \tau_n\mu = \nu_{\rho^+}$$

in the sense of weak convergence. The *amplitude* of the shock (or of the shock measure) is $|\rho^+ - \rho^-|$.

A *partial blocking measure* is a proba. measure whose restriction to one lane is a blocking measure (carrying a $(0,1)$-shock for us), and to the other lane is either full or empty (it is a shock measure).

Theorem

$$\mathcal{I}_e = \{\nu_\rho : 0 \leq \rho \leq 2\} \cup \mathcal{I}_1 \cup \mathcal{I}_2 \tag{18}$$

For $k \in \{1, 2\}$, $\mathcal{I}_k$ *is a (possibly empty) set of shock measures of amplitude* k*, i.e.* $\tau_z \nu_{\rho^-,\rho^+}$, $z \in \mathbb{Z}$ *for some shock* (ρ^-, ρ^+).

For $k = 2$, $(\rho^-, \rho^+) \in \mathcal{B}_2 := \{(0, 2)\}$.
For $k = 1$*, either*
$(\rho^-, \rho^+) \in \mathcal{R}' \subset \mathcal{B}_1 := \{(0, 1), (1, 0), (1, 2), (2, 1)\}$*, or*
$(\rho^-, \rho^+) \in \mathcal{R} \subset [0, 2]^2 \setminus (\mathcal{D} \cup \mathcal{B}_1 \cup \mathcal{B}_2)$.

$\mathcal{I}_1$ may contain *partial* blocking measures, $\mathcal{I}_2$ is stable by translations. Outside degenerate cases, up to translations along $\mathbb{Z}$, $|\mathcal{I}_1| \leq 1$ and $|\mathcal{I}_2| \leq 2$. For a subset of parameter values, we can determine $\mathcal{I}_1$ and $\mathcal{I}_2$, and thus obtain a complete characterization of $\mathcal{I}_e$.

We now give more details. Recall that

$$\gamma_0 \geq 0, \quad \gamma_0 + \gamma_1 \geq 0, \quad p \geq q, \quad p > 0$$

Generic case : $\gamma_0 + \gamma_1 \neq 0$ and $q > 0$

(i) $|\mathcal{R}| \leq 1$ and $\mathcal{R}' = \emptyset$ hence $|\mathcal{I}_1| \leq 1$.
If $\gamma_0 > 0$ and $\gamma_1 > 0$, elements of $\mathcal{I}_2$ are supported on

$$\mathcal{X}_2 \;:=\; \left\{ \eta \in \mathcal{X} : \sum_{x \in V:\, x(0) > 0} [1 - \eta(x)] + \sum_{x \in V:\, x(0) \leq 0} \eta(x) < +\infty \right\} \tag{19}$$

(ii) Assume either : (a) $\theta = d_0/l_0 = d_1/l_1 > 1$; or (b) $l_0 = l_1 = 0$ and $d_0, d_1 > 0$. Then

$$\mathcal{I}_2 := \{\tau_{-z}\check{\nu}_0 : z \in \mathbb{Z}\} \cup \{\tau_{-z}\widehat{\nu}_0 : z \in \mathbb{Z}\} \tag{20}$$

where

(a) (4) has the $(0,1)$-valued solutions

$$\rho^c_{z,i} := \frac{c\theta^z \left(\frac{p}{q}\right)^i}{1 + c\theta^z \left(\frac{p}{q}\right)^i}, \quad (z,i) \in \mathbb{Z} \times W, c > 0 \qquad (21)$$

$\mu^{\rho^c_\cdot}$ is reversible for the two-lane SEP and supported on $\mathcal{X}_2$.
we fix $c > 0$ and define conditioned measures (independent of $c > 0$).

$$\begin{aligned} \check{\nu}_n &:= \mu^{\rho^c_\cdot}(\,.\,|H_2(\eta) = 2n\,) = \tau_n \check{\nu}_0, \quad n \in \mathbb{Z} \\ \widehat{\nu}_n &:= \mu^{\rho^c_\cdot}(\,.\,|H_2(\eta) = 2n+1\,) = \tau_n \widehat{\nu}_0, \quad n \in \mathbb{Z} \qquad (22) \end{aligned}$$

where now

$$H_2(\eta) := \sum_{x \in V:\, x(0) \leq 0} \eta(x) - \sum_{x \in V:\, x(0) > 0} [1 - \eta(x)] \qquad (23)$$

(b)

$$\breve{\nu}_0 = \delta_{\breve{\eta}} \quad ; \quad \widehat{\nu}_0 = \frac{q}{p+q}\delta_{\widehat{\eta}^0} + \frac{p}{p+q}\delta_{\widehat{\eta}^1}$$

where for $x \in V$,

$$\begin{aligned} \breve{\eta}\,(x) &= 1_{\{x(0)>0\}} \\ \widehat{\eta}^0\,(x) &= 1_{\{x(0)>0\}} + 1_{\{x=(0,0)\}} \\ \widehat{\eta}^1\,(x) &= 1_{\{x(0)>0\}} + 1_{\{x=(0,1)\}}. \end{aligned}$$

(iii) A complete description of $\mathcal{I}_e$:

reduced parameters $(d, r) \in [0,1] \times [0,1]$ (by (1)) :

$$r := \frac{q}{p}, \quad d := \frac{\gamma_0}{\gamma_0 + \gamma_1} \text{ if } \gamma_0 + \gamma_1 \neq 0$$

and set

$$r_0 := \frac{1 - 2\sqrt{-7 + \sqrt{52}}}{1 + 2\sqrt{-7 + \sqrt{52}}} = 0,042 \cdots \tag{24}$$

$\exists\ \mathcal{Z} \subset [0,1] \times [0,1]$, open, containing $\{1/2\} \times (0, r_0)$, such that $\mathcal{R} = \mathcal{R}' = \emptyset$, $\forall (d,r) \in \mathcal{Z}$. In particular, if $r \in (0, r_0)$, $d_1 = \lambda d_0$ and $l_1 = \lambda l_0$ with λ close enough to 1, then (18) holds with $\mathcal{I}_2$ as in (ii).

Case 2 : $\gamma_0 + \gamma_1 = 0$ and $q > 0$

(i) Assume $\gamma_0 = \gamma_1 = 0$.
Then $\mathcal{R} = \mathcal{R}' = \mathcal{I}_2 = \emptyset$, hence $\mathcal{I}_e = \{\nu_\rho : \ \rho \in [0, 2]\}$.

Remark. When $p = q$, the dynamics is symmetric and the result is well-known. However when $p \neq q$, the two-lane SEP is not a symmetric exclusion process, and our result is new.

(ii) Assume $p = q$.
The model is diffusive and nongradient, and we conjecture that the only invariant measures are Bernoulli.

(iii) Assume $\gamma_0 \neq 0$, $\gamma_1 \neq 0$ and $p \neq q$.
Then $\mathcal{R} = \emptyset$ and $|\mathcal{R}'| \leq 2$.

Case 3 : $q = 0$

A complete description of $\mathcal{I}_e$ when $\gamma_0 \neq \gamma_1$:

(i) (a). If $\gamma_0 > 0$ and $\gamma_1 > 0$, then $\mathcal{R}' = \{(0,1);(1,2)\}$;
$\mathcal{R} = \emptyset$ if $\gamma_0 \neq \gamma_1$, or contained in $\{(3/2, 1/2)\}$ if $\gamma_0 = \gamma_1$.
$\mathcal{I}_1$ consists of partial blocking measures.
$\mathcal{I}_2 = \emptyset$ unless $l_0 = l_1 = 0$.
(b) If $l_0 = l_1 = 0$, $\mathcal{I}_2$ consists of blocking measures.

(ii) If $\gamma_1 < 0 < \gamma_0$, then $\mathcal{R}' = \{(1,0),(1,2)\}$, $\mathcal{R} = \mathcal{I}_2 = \emptyset$.
$\mathcal{I}_1$ consists of partial blocking measures.

(iii) If $\gamma_0 = 0 < \gamma_1$, then $\mathcal{R}' = \{(0,1)\}$, $\mathcal{R} = \mathcal{I}_2 = \emptyset$.
$\mathcal{I}_1$ consists of partial blocking measures.

Remark. In case (i)(a) $\mathcal{I}_2 = \emptyset$ *even* though the drifts are both strictly positive, in sharp contrast with the one-dimensional case.

Details for invariant measures when $q = 0$

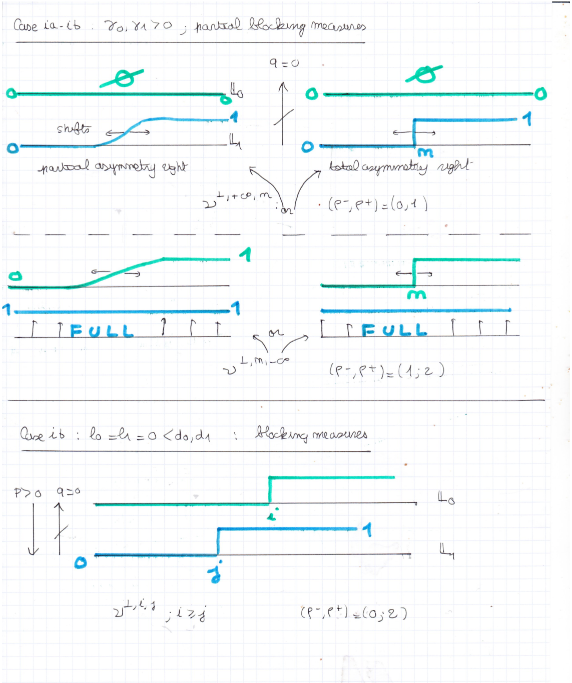
$$\eta^*_n(x) := \mathbf{1}_{\{x>n\}}, \tag{25}$$

By extension, $\eta^*_{-\infty}$ and $\eta^*_{+\infty}$ respectively denote the configurations with all 1's and all 0's.

In cases *(i)–(iii)*, for $n \in \mathbb{Z}$, we denote by $\nu^{\perp,+\infty,n}$ and $\nu^{\perp,n,-\infty}$ the proba. measures on $\mathcal{X}$ defined by : Under $\nu^{\perp,+\infty,n}$, $\eta^0 = \eta^*_{+\infty}$ (i.e. lane 0 is empty) and $\eta^1 \sim \widehat{\mu}_n$, where $\widehat{\mu}_n$ is given by (25) if $l_1 = 0$ (or by ... if partial asymmetry).
Under $\nu^{\perp,n,-\infty}$, $\eta^1 = \eta^*_{-\infty}$ (i.e. lane 1 is full) and $\eta^0 \sim \widehat{\mu}_n$. where $\widehat{\mu}_n$ is given by (25) if $l_0 = 0$ (or by ... if partial asymmetry).

Case (i), (a). We set

$$\mathcal{I}_1 \quad := \quad \left\{\nu^{\perp,+\infty,n} : \; n \in \mathbb{Z}\right\} \cup \left\{\nu^{\perp,n,-\infty} : \; n \in \mathbb{Z}\right\} \tag{26}$$



Case ia-ib : $\gamma_0, \gamma_1 > 0$; partial blocking measures

$\nu^{\perp,+\infty,m}$: or $(\rho^-,\rho^+)=(0,1)$

or $\nu^{\perp,m,-\infty}$ $(\rho^-,\rho^+)=(1;2)$

Case ib : $l_0=l_1=0<d_0,d_1$: blocking measures

$\nu^{\perp,i,j}$; $i \geq j$ $(\rho^-,\rho^+)=(0;2)$

Case (i), (b). Let $\mathbb{B} := \{(i,j) \in \mathbb{Z}^2 : \ i \geq j\}$, and set $\overline{\mathbb{B}} := \mathbb{B} \cup \{(+\infty, n), (n, -\infty) : \ n \in \mathbb{Z}\}$. For $(i,j) \in \overline{\mathbb{B}}$, let $\nu^{\perp,i,j}$ denote the Dirac measure supported on the configuration $\eta^{\perp,i,j}$:

$$\eta^{\perp,i,j}(z,0) = \eta_i^*(z), \quad \eta^{\perp,i,j}(z,1) = \eta_j^*(z) \tag{27}$$

for every $z \in \mathbb{Z}$.

$$\mathcal{I}_2 \ := \ \left\{\nu^{\perp,i,j} : (i,j) \in \mathbb{B}\right\} \tag{28}$$

Case (ii). For $n \in \mathbb{Z}$, we denote by $\nu^{\perp,+\infty,n\leftarrow}$ the proba. measure on $\mathcal{X}$ defined by :

Lane symmetry operator σ defined by $(\sigma\eta)(z,i) = \eta(-z,i)$ for $\eta \in \mathcal{X}$, $(z,i) \in V$.

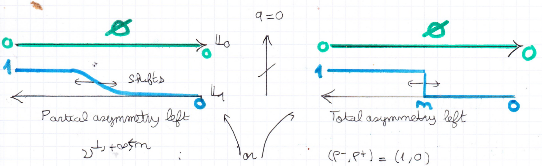
Under $\nu^{\perp,+\infty,n\leftarrow}$, $\eta^0 = \eta^*_{+\infty}$ and $\sigma\eta^1 \sim \widehat{\mu}_n$, where $\widehat{\mu}_n$ is given by (25) if $l_1 = 0$ (or by ... if partial asymmetry).

$$\mathcal{I}_1 := \left\{\nu^{\perp,+\infty,n\leftarrow} : \ n \in \mathbb{Z}\right\} \cup \left\{\nu^{\perp,n,-\infty} : \ n \in \mathbb{Z}\right\} \tag{29}$$

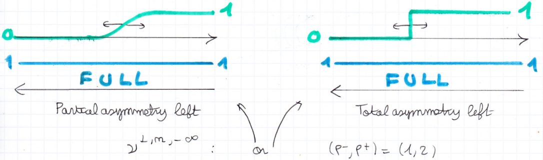
Case (iii).

$$\mathcal{I}_1 := \left\{\nu^{\perp,+\infty,n} : \ n \in \mathbb{Z}\right\} \tag{30}$$

Case ii) : $\gamma_1 < 0 < \gamma_0$; partial blocking measures $\begin{cases} * \ \nu^{\perp,+\infty,\leftarrow m} \\ * \ \nu^{\perp,m,-\infty} \text{ as before} \end{cases}$

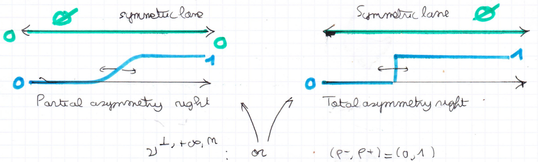


$\nu^{\perp,+\infty,\leftarrow m}$: or $(\rho^-,\rho^+) = (1,0)$



$\nu^{\perp,m,-\infty}$: or $(\rho^-,\rho^+) = (1,2)$

Case iii) $\gamma_0 = 0 < \gamma_1$: partial blocking measures $\nu^{\perp,+\infty,m}$



$\nu^{\perp,+\infty,m}$: or $(\rho^-,\rho^+) = (0,1)$

Questions left open

We do not know if for certain parameter values it is possible to have $\mathcal{I}_1 \neq \emptyset$ with a shock of amplitude 1 that is not a partial blocking measure. In the case $p = q$ it is believed in [BL] that this probably does not occur.

We conjecture that when $pq > 0$, $\gamma_0 > 0$ and $\gamma_1 > 0$, then $\mathcal{I}_2 \neq \emptyset$.

Extension
Our model and approach extend to more general *multi-lane* exclusion processes with an arbitrary (finite) number of lanes.

The cyclic ladder

Assumption

$W = \mathbb{T}_n$ is a torus, and $q(i,j) = Q(j-i)$ for some $Q : \mathbb{T}_n \to [0,+\infty)$ is an irreducible translation-invariant kernel.

For $\rho \in [0,n]$, ν_ρ is the product measure on $\mathcal{X}$ s.t.

$$\forall (z,i) \in \mathbb{Z} \times W, \quad \nu_\rho \{\eta(z,i) = 1\} = \frac{\rho}{n} \tag{31}$$

[BL,page 2309] : a proba. measure on $\mathcal{X}$ is *rotationally invariant* if it is invariant by τ', the translation operator along W.

Open question 1. for the ladder process :
when d_i and l_i are independent of i (i.e. the horizontal dynamics is the same on each lane), are *all* invariant measures rotationally invariant ?

Theorem

(0) $(\mathcal{I}\cap\mathcal{S})_e=\{\nu_\rho,\ \rho\in[0,n]\}$*.*
(1) For $k=1,\ldots,n$*, let* $(\rho_k^-,\rho_k^+)=\left(\frac{n-k}{2},\frac{n+k}{2}=n-\rho_k^-\right)$*. Then :*
(a)

$$\mathcal{I}_e=\{\nu_\rho:\ \rho\in[0,n]\}\cup\bigcup_{k=1}^{n}\mathcal{I}_k \tag{32}$$

where $\mathcal{I}_k$ *is a (possibly empty) set of at most* k (ρ_k^-,ρ_k^+)*-shock measures of amplitude* k*.*
(b) If $\forall i\in W,\ \gamma_i>0$*,* $\mathcal{I}_n$ *is supported on* $\mathcal{X}_n$ *(cf.* (19)*).*
(c) If $\forall i\in W,\ d_i/l_i$ *does not depend on* i*,* $\mathcal{I}_n$ *consists of* n *explicit blocking measures* ν_i*.*
(up to horizontal translations)
(2) If $\forall i\in W,\ \gamma_i:=d_i-l_i=0$*, then* $\mathcal{I}_e=\{\nu_\rho:\ \rho\in[0,n]\}$*.*
(3) If d_i *and* l_i *do not depend on* i*, any invariant measure is rotationally invariant.*

Theorem : Detailed scheme of proof

- **Step 1 : comparing an invariant measure with its translate.** Let $\mu \in \mathcal{I}_e$. We prove that $\mu \leq \tau\mu$ or $\tau\mu \leq \mu$ (stochastic order). This is equivalent (Strassen theorem) to a coupling $\overline{\mu}(d\eta, d\xi)$ of $\mu(d\eta)$ and $\tau\mu(d\xi)$ under which $\eta \leq \xi$ or $\xi \leq \eta$ a.s. This step is an adaptation to our model of [BLM] when $q > 0$.

  Main ingredients : attractiveness, weak irreducibility, finite propagation property, characterization of $(\mathcal{I} \cap \mathcal{S})_e$, and space-time ergodicity for the measures in this set.
  Non-weakly irreducible case. When $q = 0$, again additional arguments, different from the translation invariant case, are necessary to fill the gap between $\{\eta \leq \xi\} \cup \{\xi \leq \eta\}$; they involve introducing an intermediate relation : $\eta \bowtie \xi$ iff $\eta >< \xi$, and both the number of $z \in \mathbb{Z}^+$ on lane 1 not occupied by a coupled particle and the number of $z \in \mathbb{Z}^-$ on lane 0 not occupied by a hole are finite.

Scheme of proof

- **Step 2 : mean shock.**
  If $\tau\mu = \mu$, back to $(\mathcal{I} \cap \mathcal{S})_e$. If e.g. $\mu < \tau\mu$, the total number of discrepancies $D(\eta, \xi)$ under $\overline{\mu}$ is constant (extremality). Its expectation is a telescoping sum equal to the difference of mean densities at $\pm\infty$ ("mean" shock).

  Single lane ASEP : simplifying feature.
  Since max density is 1, no choice but $0/1$ mean density at $\pm\infty$, *hence* asymptotic to $\mathcal{B}(0/1)$ at $\pm\infty$. It cannot be 1 at $-\infty$ and 0 at $+\infty$ (HDL for ASEP : not stationary for Burgers but develops rarefaction).

  Thus for single-lane ASEP :

  - $\mathcal{I}_e$ contains only profile measures.
  - The following steps 3–4 not are needed for ASEP.

Scheme of proof

- **Step 3 : mean shock implies shock.**
  From step 2 and Cesaro averaging, $\exists$ limits $\mu_\pm \in (\mathcal{I} \cap \mathcal{S})$ at $\pm\infty$.

  Problem : show that $\mu_\pm \in (\mathcal{I} \cap \mathcal{S})_e$. *Then* $\mu_\pm = \mu_{\rho^\pm}$, i.e. it is a (ρ^-, ρ^+)-*shock measure*.

  By step 2, $|\rho^+ - \rho^-| \in \{1, 2\}$.

  - $|\rho^+ - \rho^-| = 2$: then $\{\rho^-, \rho^+\} = \{0, 2\}$. *Profile measures*, analogous to ASEP. Sometimes explicit blocking measures.
  - $|\rho^+ - \rho^-| = 1$: shock measure. *Problem* : what are possible (ρ^-, ρ^+) ?

Scheme of proof

- **Step 4 : restricting possible shocks.**
  Possible (ρ^-,ρ^+)-shocks ? Analysis of the *flux function* :

$$G(\rho) := \gamma_0\rho_0(1-\rho_0)+\gamma_1\rho_1(1-\rho_1)$$

for a unique (ρ_0,ρ_1) such that

$$(\rho_0,\rho_1)\in\mathcal{F},\quad \rho_0+\rho_1=\rho$$

Remark. Vertical jumps, i.e. with rates (p, q), do not contribute to G.

Necessary conditions.

  - *Flux continuity condition (C) :* $G(\rho^+)=G(\rho^-)$.
  - *Entropy condition (E) :* $\rho^\pm$ optimizer on $[\rho^+\wedge\rho^-,\rho^+\vee\rho^-]$ (e.g. $G(\rho^+)=G(\rho^-)=\min_{\rho\in[\rho^-,\rho^+]}G$ if $\rho^-<\rho^+$).

Scheme of proof

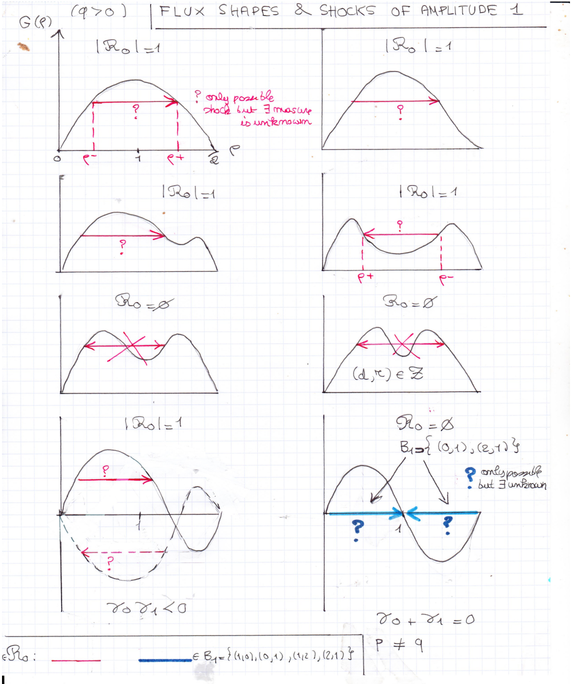
- **Step 4 : restricting possible shocks.** Define

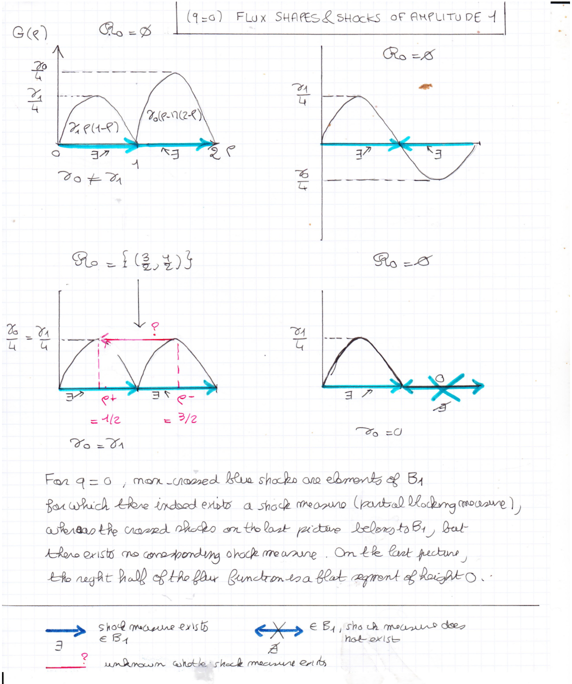
$$\begin{array}{rcl} \mathcal{D} & = & \{(\rho^-,\rho^+)\in[0,2]^2:\ \rho^-=\rho^+\}\quad ([0,2]^2\setminus\mathcal{D}\text{ : shocks}) \\ \mathcal{B}_1 & = & \{(0,1),(1,2),(2,1)\}\quad \text{(partial blockage)} \\ \mathcal{R}_0 & = & \{(\rho^-,\rho^+)\in[0,2]^2\setminus(\mathcal{D}\cup\mathcal{B}_1):\ |\rho^+-\rho^-|=1,\ (C),\ (E)\} \end{array}$$

Proposition

When $\gamma_0+\gamma_1\neq 0$*, let* $d:=\dfrac{\gamma_0}{\gamma_0+\gamma_1}$ *and* $r:=\dfrac{q}{p}$.

- *For all parameter values,* $|\mathcal{R}_0|\leq 1$.
- *For an explicit* $r_0\simeq 0,042$*, there is a neighbourhood* $\mathcal{Z}$ *of* $\{d=1/2\}\times\{r\in[0,r_0)\cup(1/r_0,+\infty]\}$ *such that* $\mathcal{R}_0=\emptyset$ *for all* $(d,r)\in\mathcal{Z}$.


G(ρ)
(q>0)
FLUX SHAPES & SHOCKS OF AMPLITUDE 1
|R₀|=1
? only possible shock but ∃ measure is unknown
?
0
ρ⁻
1
ρ+
2
ρ
|R₀|=1
?
|R₀|=1
?
|R₀|=1
?
ρ+
ρ⁻
R₀=∅
R₀=∅
(d,r) ∈ Z
|R₀|=1
?
1
?
γ₀γ₁<0
R₀=∅
B₁⊃{(0,1),(2,1)}
? only possible but ∃ unknown
?
1
?
γ₀+γ₁=0
p ≠ q
R₀ :
∈ B₁={(1,0),(0,1),(1,2),(2,1)}



(q=0) FLUX SHAPES & SHOCKS OF AMPLITUDE 1

For $q = 0$, non-crossed blue shocks are elements of B_1 for which there indeed exists a shock measure (partial blocking measure), whereas the crossed shocks on the last picture belongs to B_1, but there exists no corresponding shock measure. On the last picture, the right half of the flux function is a flat segment of height 0 .

∃ shock measure exists $\in B_1$

∄ $\in B_1$, shock measure does not exist

? unknown whether shock measure exists

Scheme of proof

- **Step 5 : uniqueness of a shock.**

Proposition

If $|\rho^+ - \rho^-| = k \in \{1, 2\}$*, there are (up to shifts) at most k* (ρ^-, ρ^+)*-shock measures in* $\mathcal{I}_e$*.*

> *Principle of proof.* Show that two shock-measures μ and ν are comparable. Then, extending an argument of [BLM] for ASEP profile measures, squeeze ν between successive translates of μ.

Scheme of proof

- **Step 5 : uniqueness of a shock.**
  Optimal for $k = 2$. Explicit construction of 2 extremal $(0, 2)$-blocking measures for some parameter values.

  Idea. For *blocking* measures, i.e. when

  $$H_2(\eta) := \sum_{x\leq 0}[\eta(x,0)+\eta(x,1)] - \sum_{x>0}[(1-\eta(x,0))+(1-\eta(x,1))]$$

  is finite, then H_2 is a conserved quantity.

  Since $H_2(\tau\eta) = H_2(\eta) - 2$, blocking space $\{H_2 < +\infty\}$ split into odd/even components ; at most one measure on each.

  Remark. The Proposition does *not* require blocking.

Scheme of proof

- **Step 6 : The case $q = 0$.**
  One can compare each lane with an ASEP and use convergence results for ASEP to obtain more information and complete characterization of $\mathcal{I}_e$ in all cases except $\gamma_0 = \gamma_1$.

Thank you for your Attention