Inhomogeneous Interacting Particle Systems

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be wise
– multiparametrize!

I. Alternating Sign Matrices and Izergin-Korepin determinant

6V model: Izergin-Korepin 1980s

ASM: Kuperberg 1996

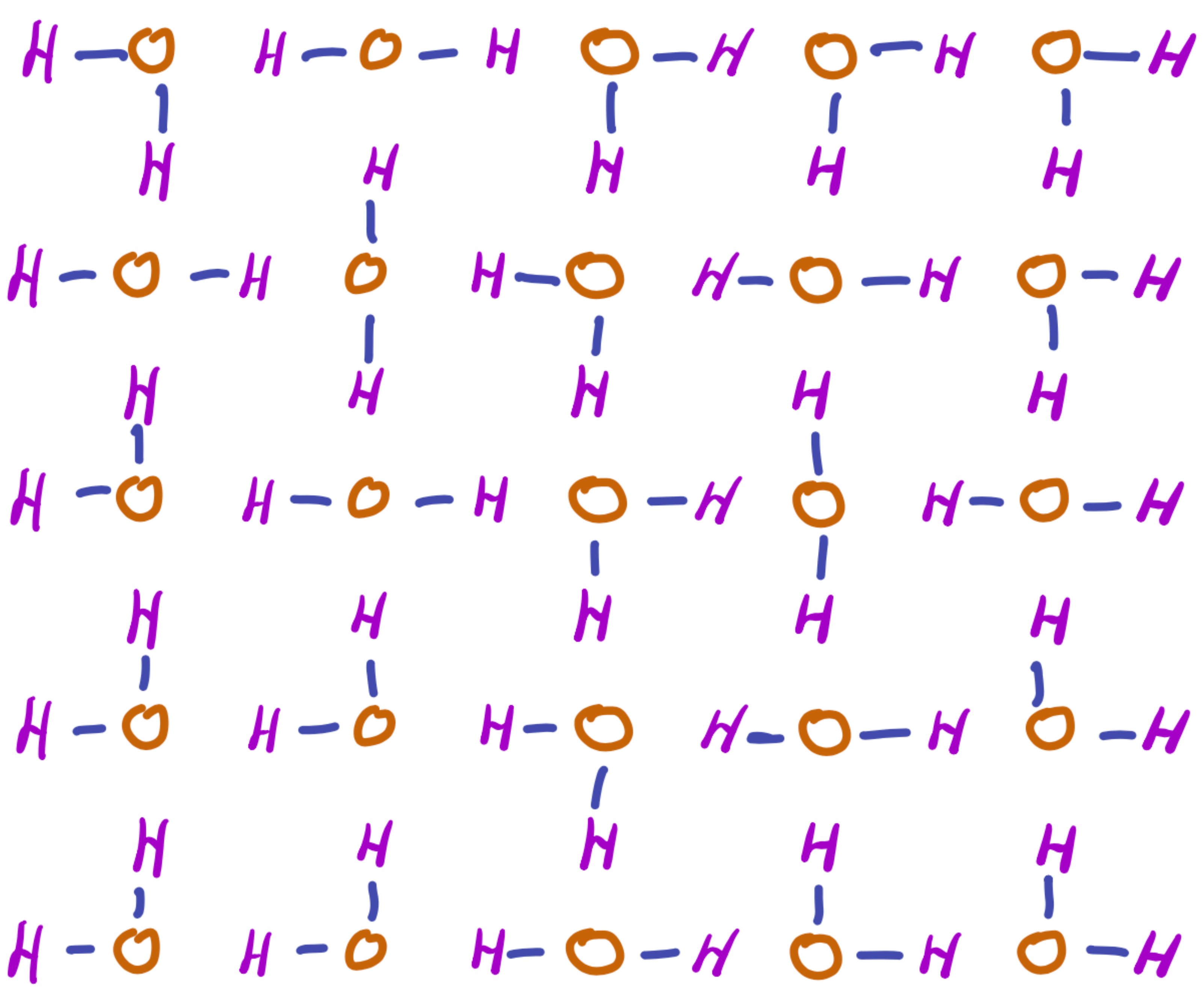
arXiv:math/9712207

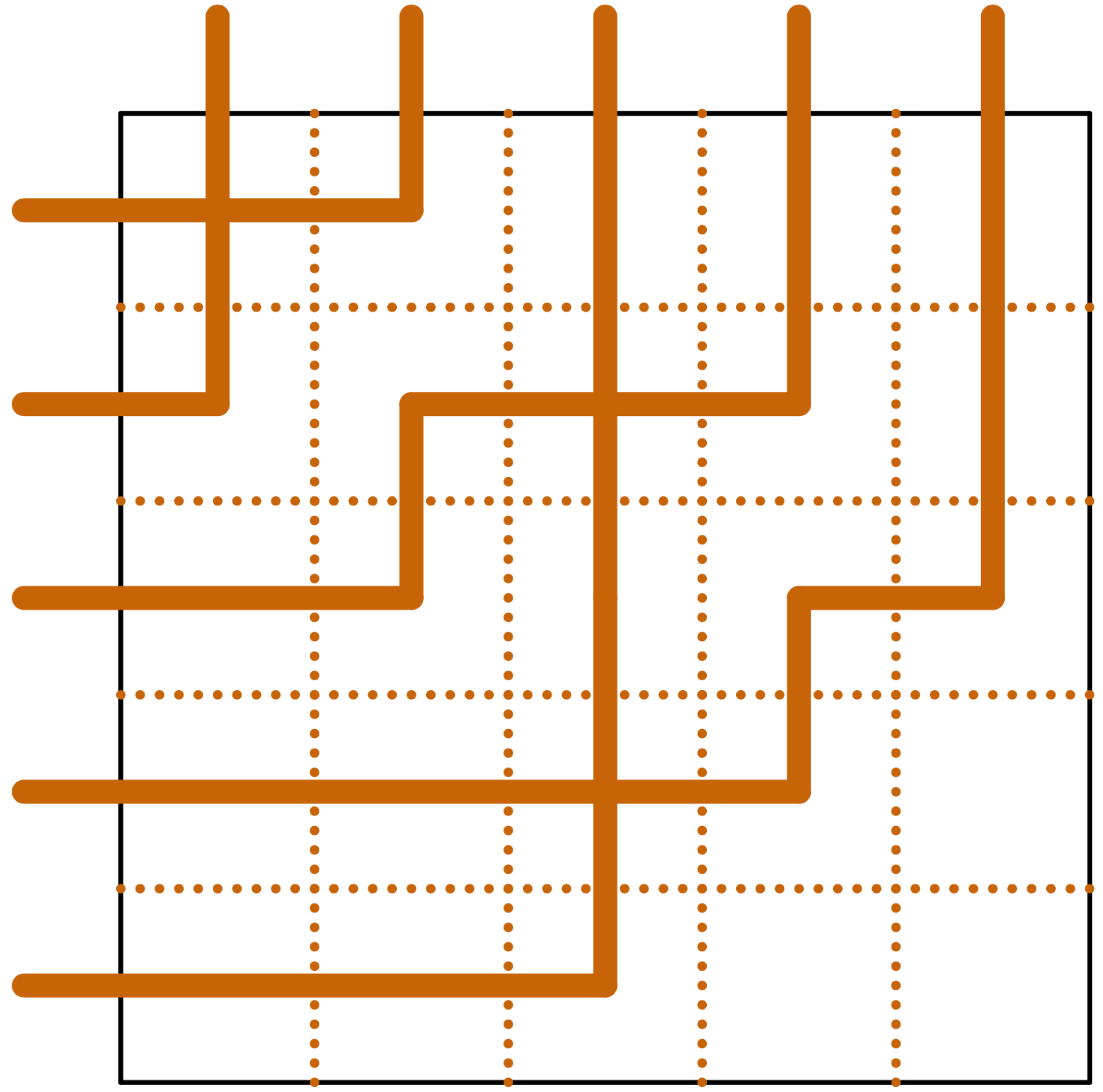
Enumeration of Alternating Sign Matrices (ASM)

ASM $\longleftrightarrow$ Square Ice $\longleftrightarrow$ Six Vertex Model with Domain Wall Boundary Conditions

$$\begin{matrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{matrix}$$

the sum of each row and column is 1 and the nonzero entries in each row and column alternate in sign

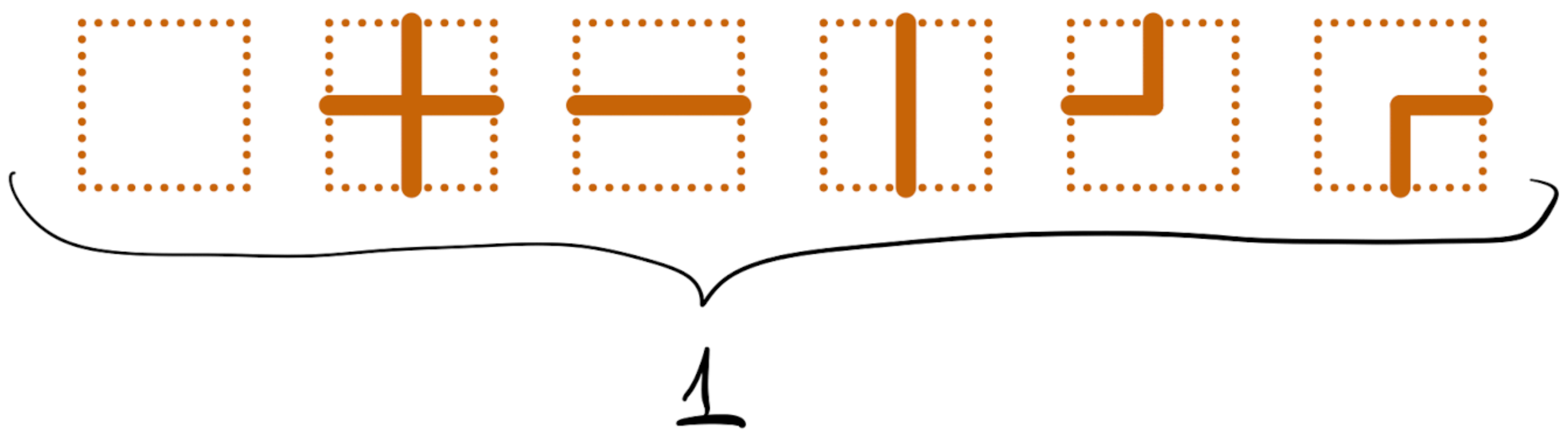




[Mills, Robbins, Rumsey 1983]: $\#ASM(n) = \prod_{k=0}^{n-1} \frac{(3k+1)!}{(n+k)!} = \frac{1!\,4!\,7!\cdots(3n-2)!}{n!\,(n+1)!\cdots(2n-1)!}$. **(1, 2, 7, 42, 429, 7436)**

[Zeilberger 1992]; [Kuperberg 1995] - two different proofs, the second involves multiparametrization

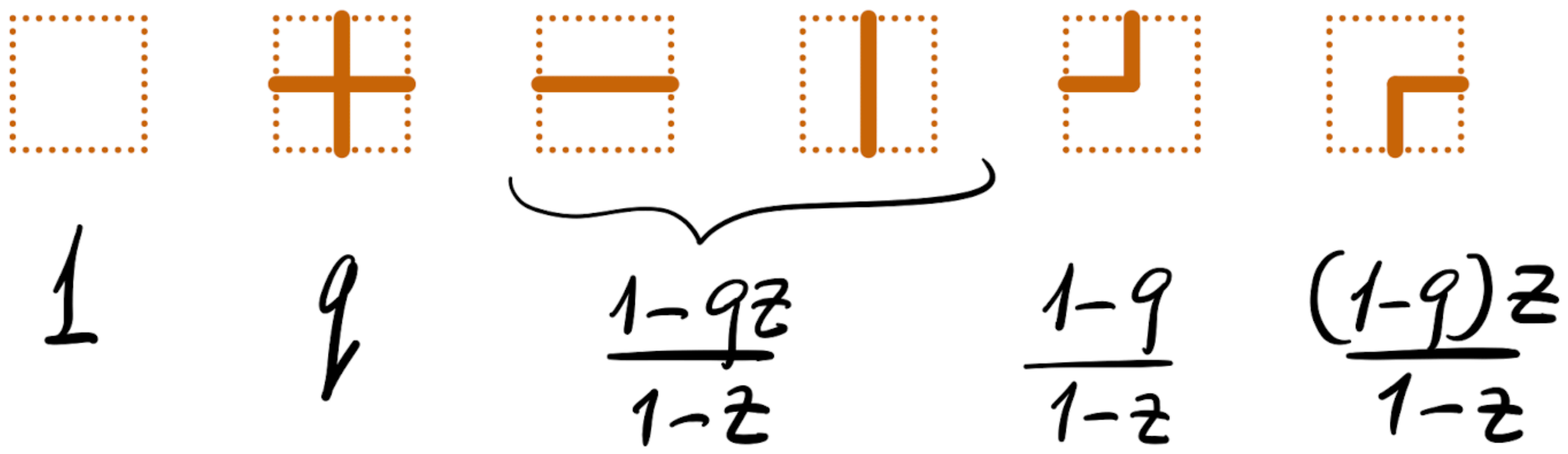
Counting six vertex configurations



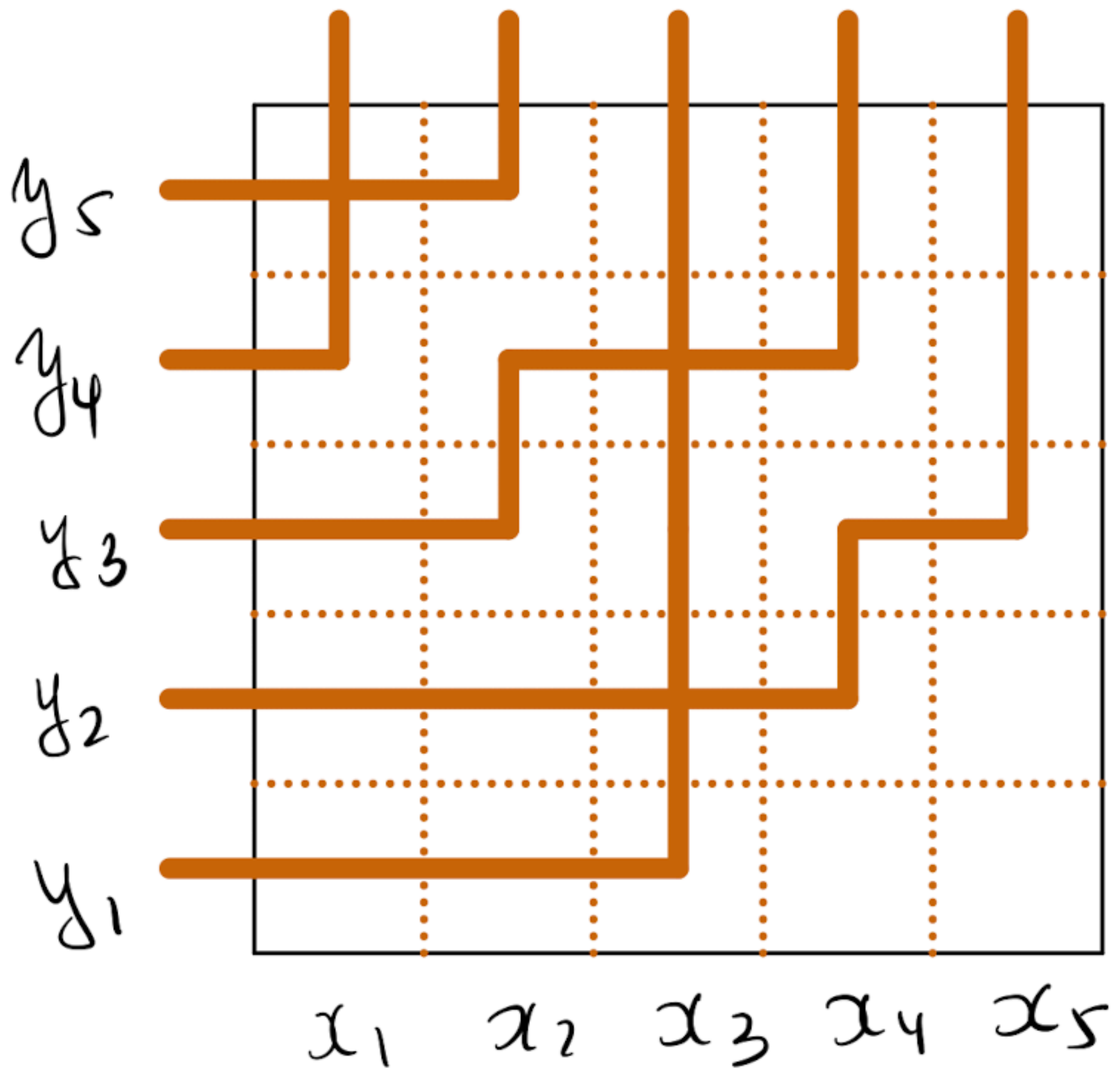
$\#ASM(n)$ is equal to the number of six vertex configurations

with domain wall boundary conditions, counted with uniform weights

Be wise - multiparametrize! **[Kuperberg 1995]**



Here $z = x_i y_j$, the product of a vertical and a horizontal parameter



Theorem [Izergin-Korepin 1980s]

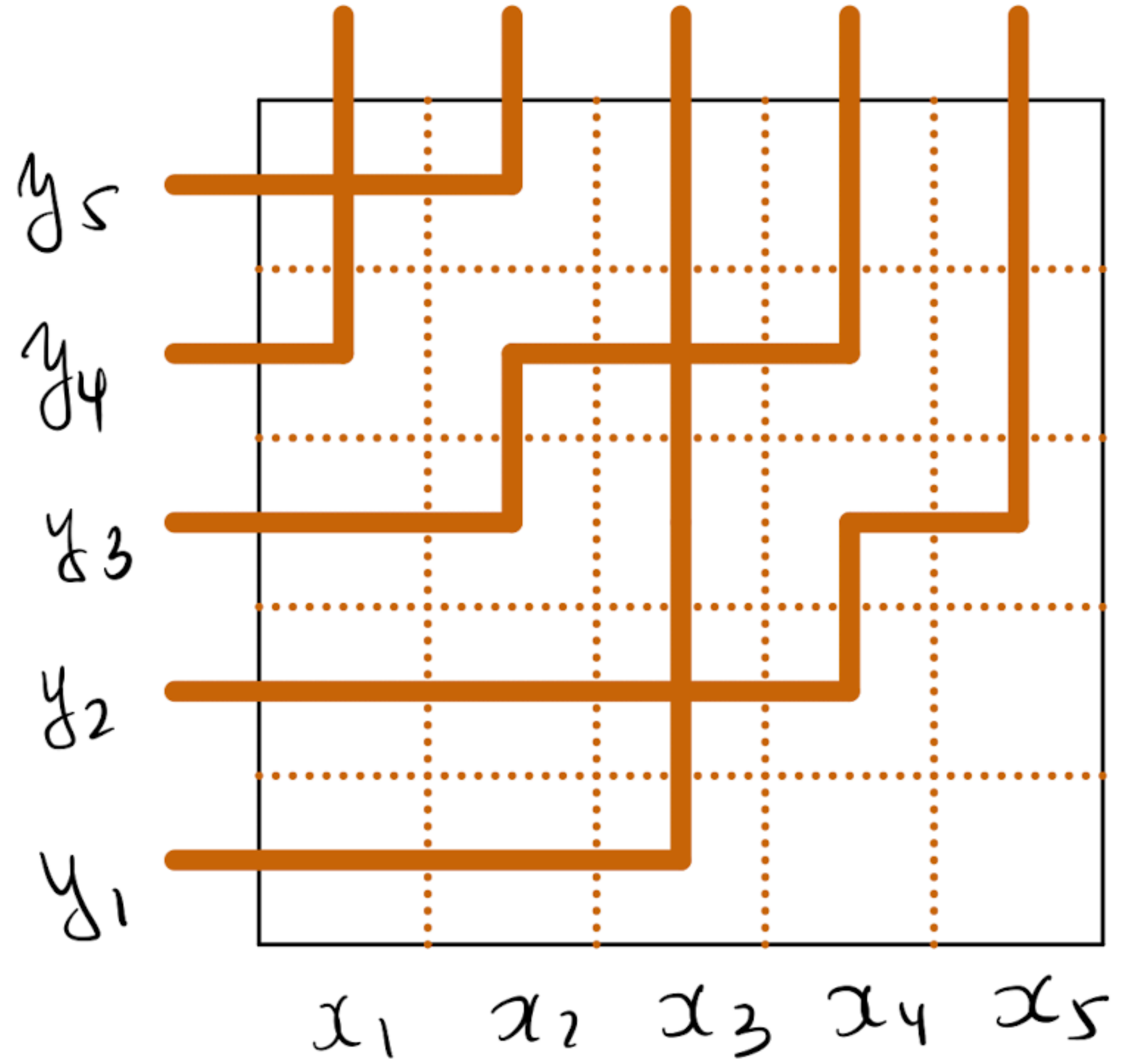
The partition function (weighted sum) is equal to

$$\frac{\prod_{i,j=1}^{n}(1-qx_iy_j)}{\prod_{i<j}(x_i-x_j)(y_i-y_j)} \det\left[\frac{1-q}{(1-x_iy_j)(1-qx_iy_j)}\right]_{i,j=1}^{n}$$

Counting six vertex configurations

[Izergin-Korepin 1980s]

- Partition function is symmetric in x_i, y_j (*Yang-Baxter equation*)
- Recurrence at $x_1 y_1 = 1$ (after clearing the denominator)
- Conditions uniquely determine the function in x_i, y_j
- The determinant satisfies the same properties



$$\frac{\prod_{i,j=1}^{n}(1-qx_iy_j)}{\prod_{i<j}(x_i-x_j)(y_i-y_j)} \det\left[\frac{1-q}{(1-x_iy_j)(1-qx_iy_j)}\right]_{i,j=1}^{n}$$

L'Hospital-like resolution of $\frac{0}{0}$

$$\#ASM(n) = \prod_{k=0}^{n-1}\frac{(3k+1)!}{(n+k)!}.$$

Reduction to ASM

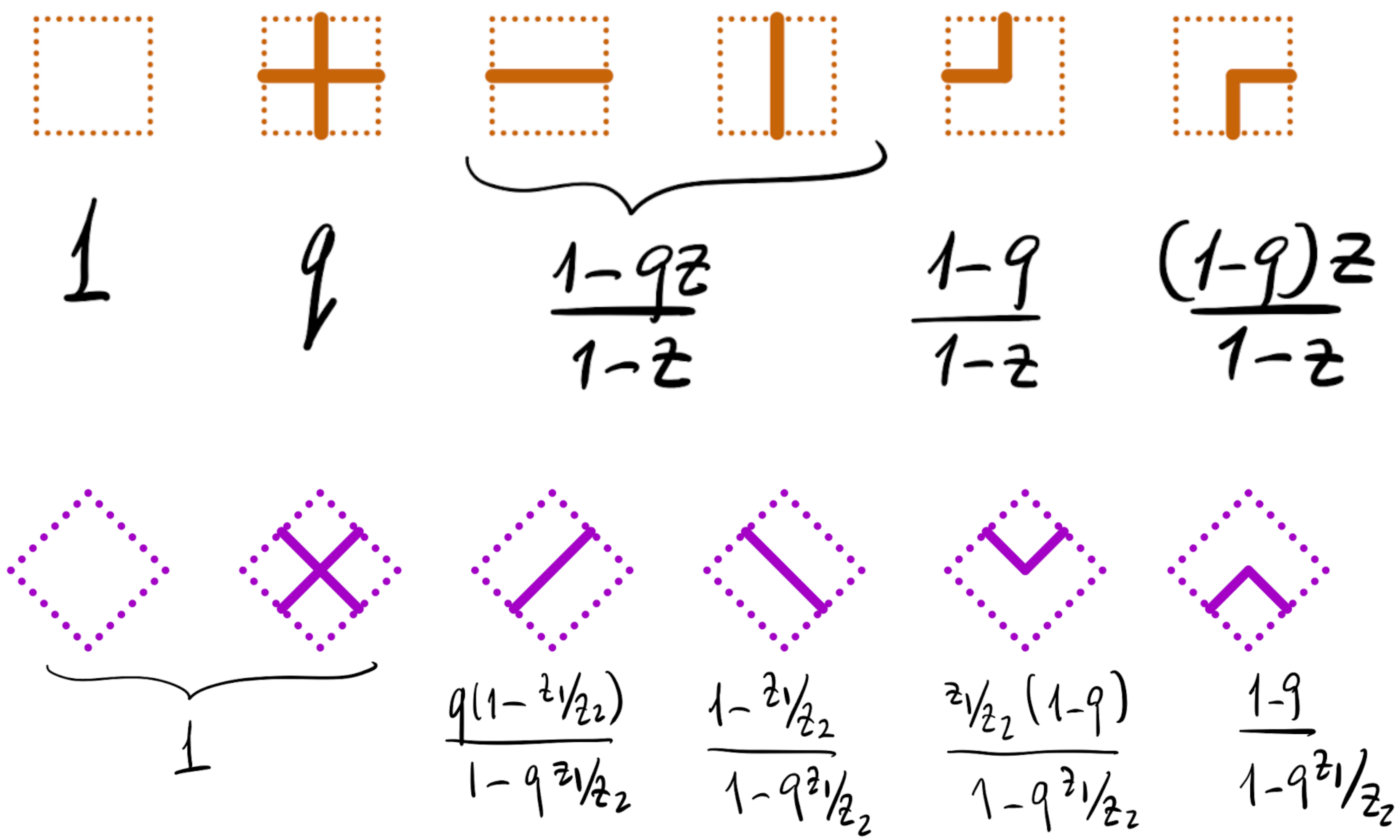
1 q	$\frac{1-qz}{1-z}$	$\frac{1-q}{1-z}$ $\frac{(1-q)z}{1-z}$
$\sqrt{q}$	$\frac{1-qz}{1-z}$	$\frac{1-q}{1-z}\sqrt{z}$
$\sqrt{q}$	$1+q$	$\sqrt{q}$

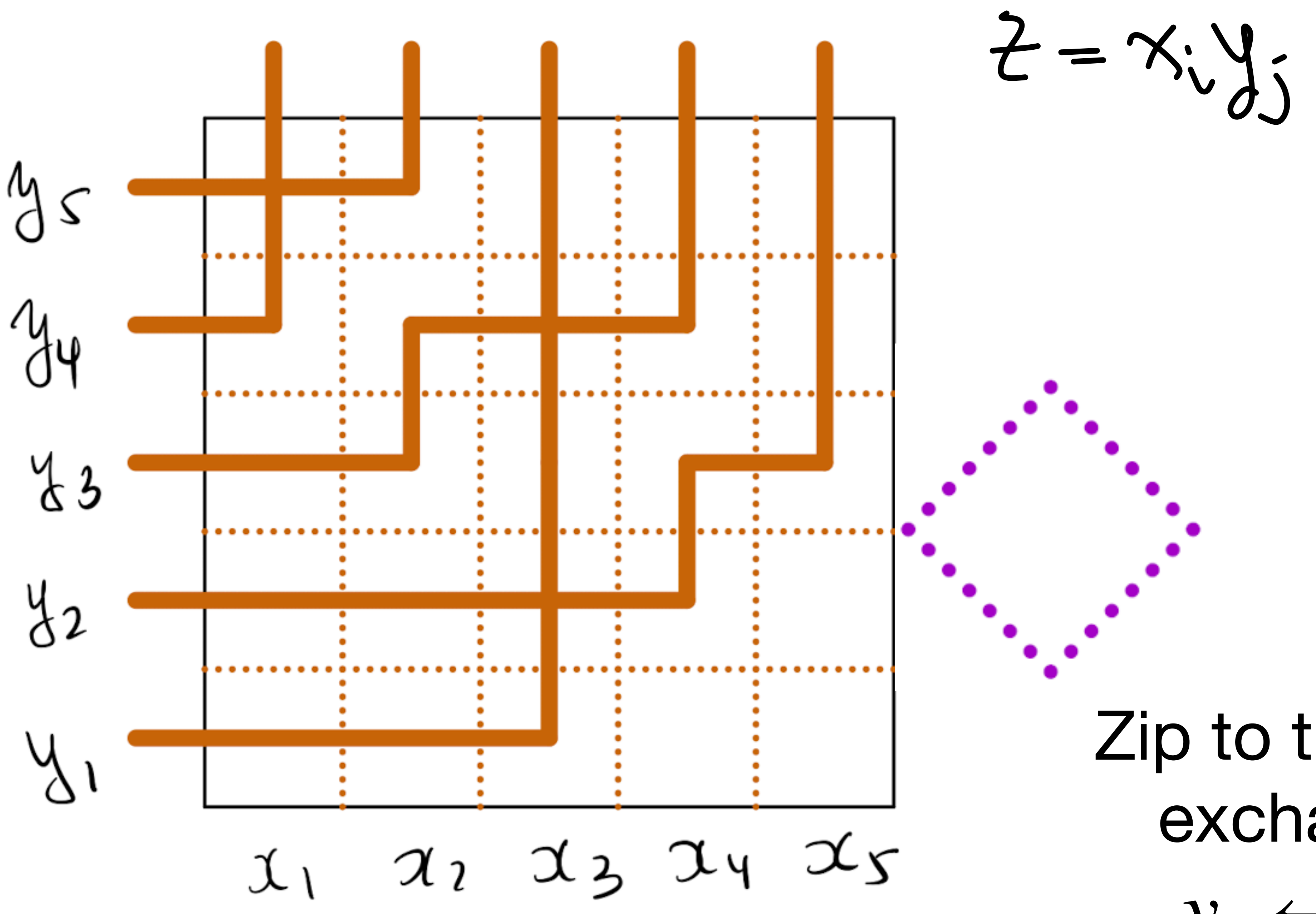
$z=q$

Set $1+q=\sqrt{q}$

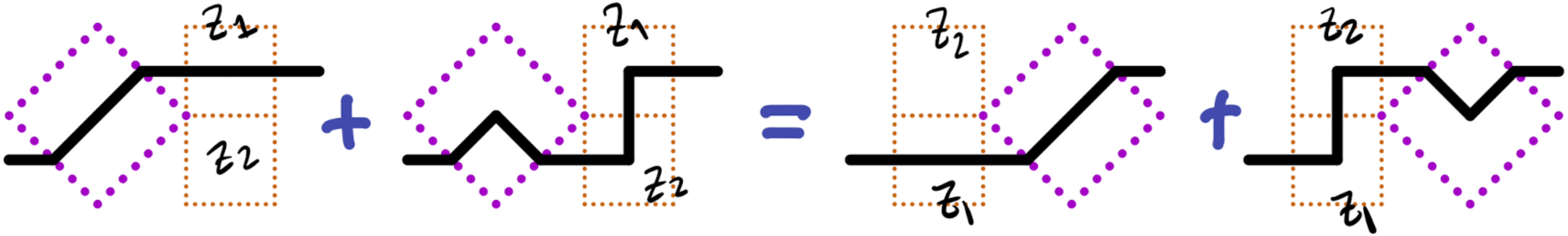
(so q is a cubic root of 1) to get uniform weights

Concrete Yang-Baxter equation





Zip to the left,
exchange
$y_2 \leftrightarrow y_3$



II. TASEP and its multiparameter friends

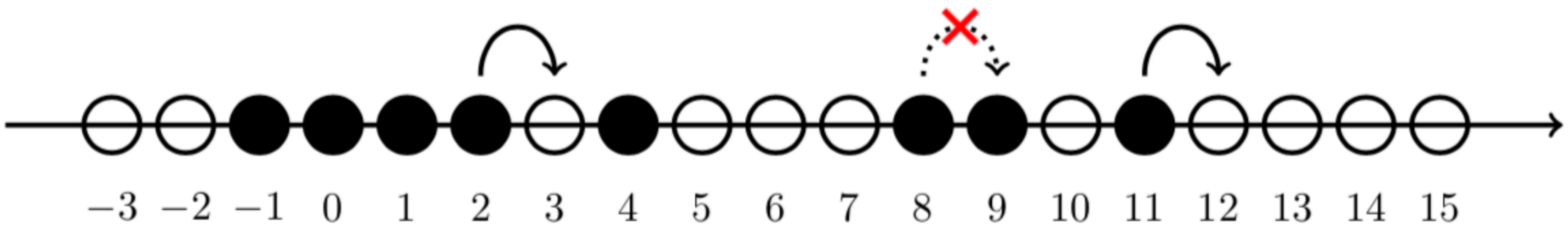
j.w. Alexei Borodin
Alisa Knizel
Axel Saenz
2017-19

arXiv:1703.03857

arXiv:1808.09855

arXiv:1910.08994

TASEP (Totally Asymmetric Simple Exclusion Process)



- TASEP and ASEP (with particles moving in two directions) were introduced in 1969-1970, independently in biology **[C. MacDonald, J. Gibbs, A. Pipkin '69]** and probability **[Spitzer '70]**

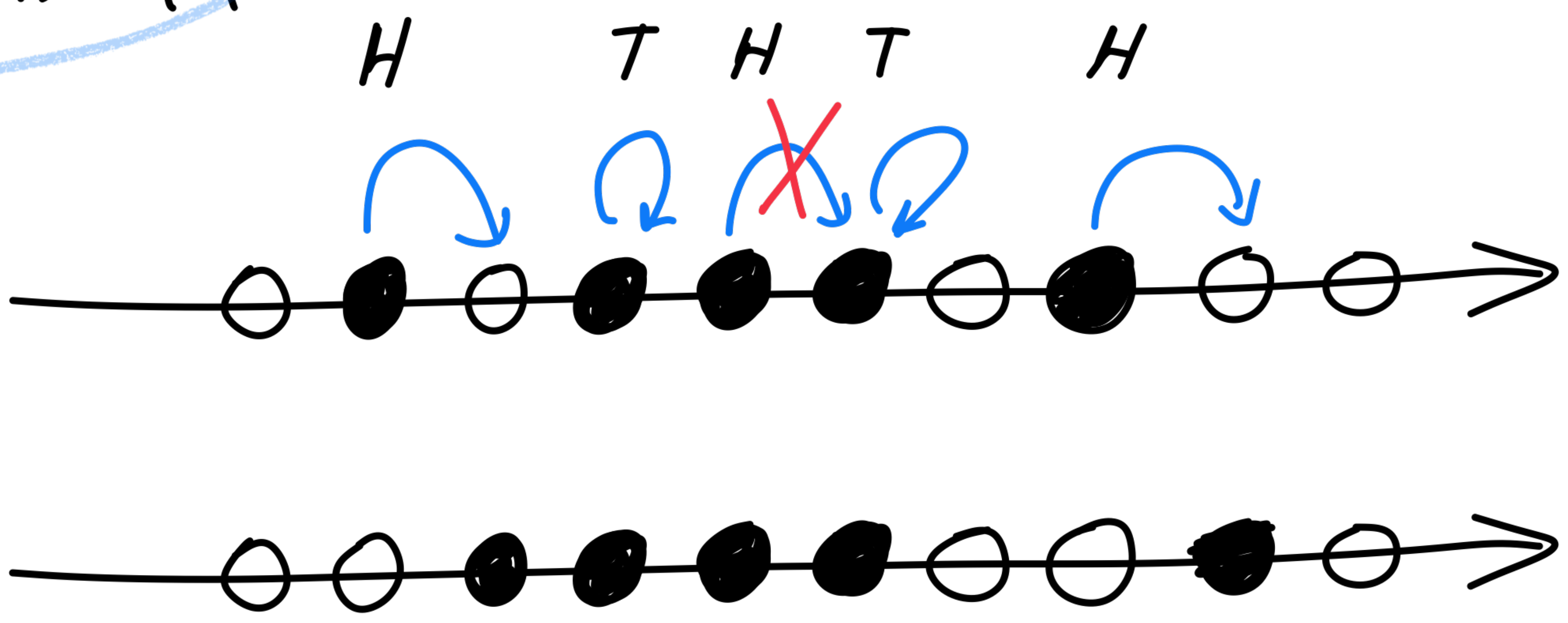
- In 50 years, we understood a lot about TASEP and related systems (in the Kardar-Parisi-Zhang universality class), including limit shapes and fluctuations with general initial data

- New asymptotic results are added every year (KPZ fixed point, Airy sheet, directed landscape, coloured / multispecies processes…)

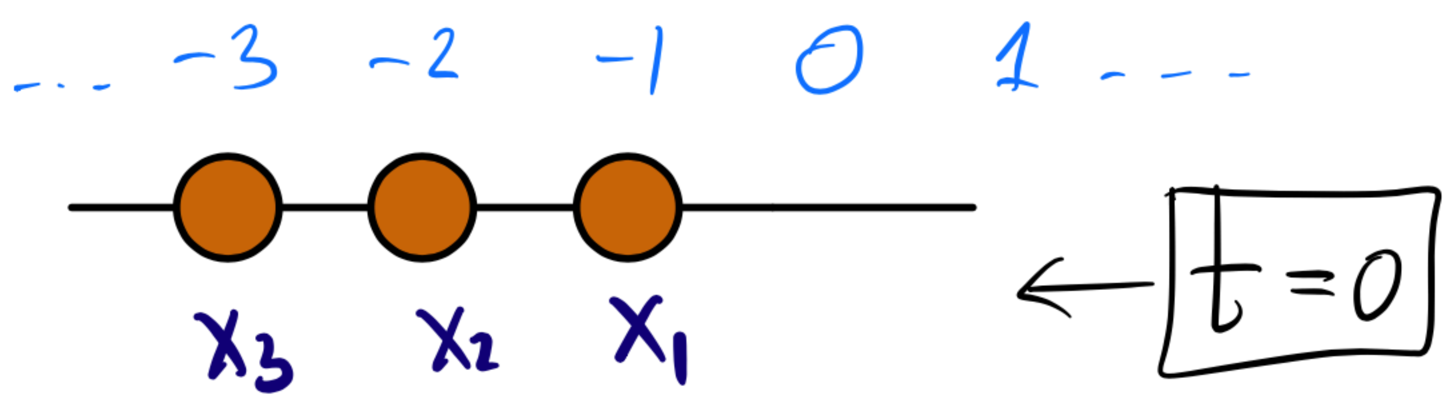
$\mathbb{P}(H) = \frac{\beta}{1+\beta}$

- Following the general principle, let us try to introduce multiple parameters in TASEP. To be more **discrete**, take **Bernoulli TASEP** with sequential update.

- Can add parameters to **particles** or to **space** (*slow bond type deformation*). There is hope of Yang-Baxter integrability, if there is symmetry.



Adding parameters to TASEP. Location-dependent



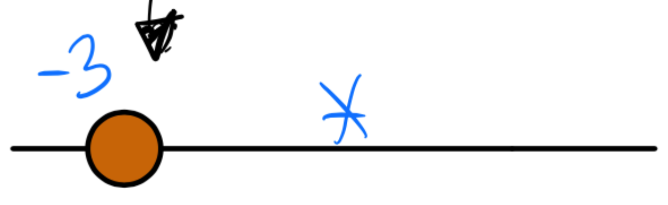
$$b_{-i} = \frac{\beta_i}{1+\beta_i}$$

$$\bar b_i = 1 - b_i$$

i = location of the particle

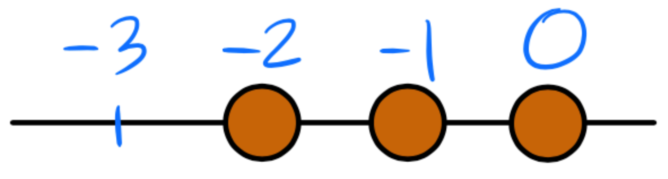
$$\mathbb{P}(x_3(t=2)+3=1) = ?$$

at $t=1$



$\bar b_1$ or $b_1 \bar b_2$ or $b_1 b_2 \bar b_3$

at $t=1$, $b_1 b_2 b_3$



$$\bar b_1 b_2 b_3 + b_1 \bar b_2 b_2 b_3 + b_1 b_2 \bar b_3 b_3 + b_1 b_2 b_3 \left[\bar b_0 + b_0 \bar b_1 + b_0 b_1 \bar b_2\right]$$

$$= 4 b_1 b_2 b_3 - \left[b_1^2 b_2 b_3 + b_1 b_2^2 b_3 + b_1 b_2 b_3^2\right] - \boxed{b_0 b_1^2 b_2^2 b_3}$$

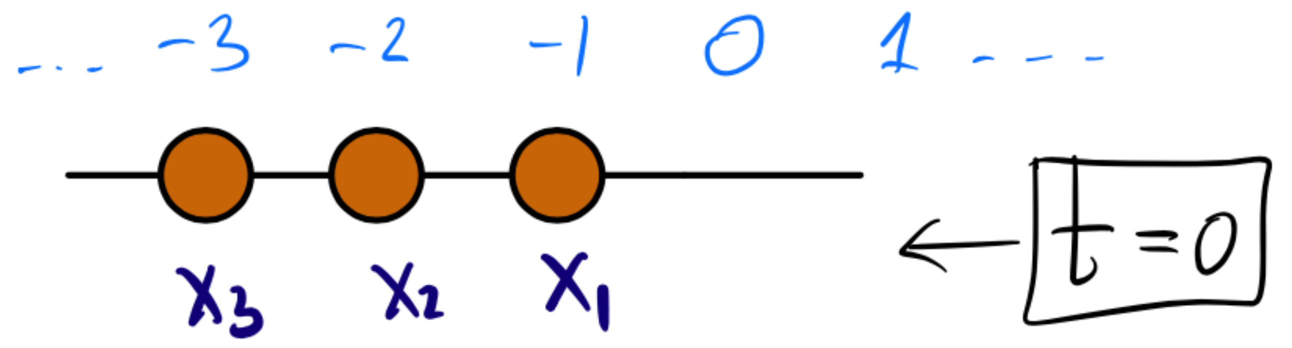
not symmetric in b_i!

At $t=1$, there are two cases, which by $t=2$ lead to nonsymmetric expressions in b_i

To me, **this is why** TASEP in inhomogeneous space (in particular, with slow bond, where only one parameter b_i differs) **is not integrable** and therefore so much more complicated

[Basu-Sidoravicius-Sly 2014], ...
showed that ϵ-slow bond at 0 slows TASEP down for any $\epsilon > 0$

Adding parameters to TASEP. Particle-dependent



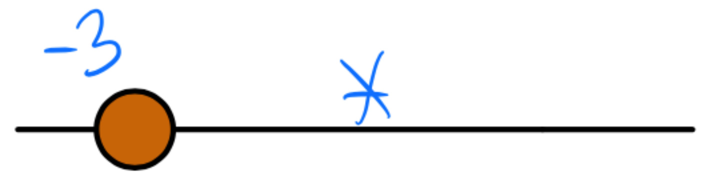
$\boxed{b_i = \frac{\beta_i}{1+\beta_i}}$

$\boxed{\bar b_i = 1 - b_i}$

i = particle's number

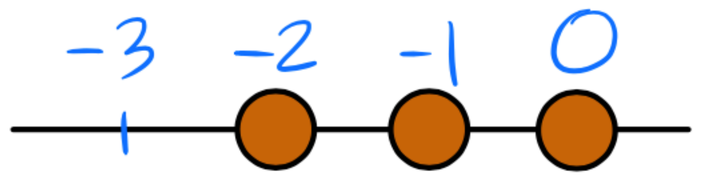
$$\mathbb{P}(x_3(t=2)+3=1) = \,?$$

at $t=1$



$\bar b_1$ or $b_1 \bar b_2$ or $b_1 b_2 \bar b_3$

at $t=1$, $b_1 b_2 b_3$



only change

$$\bar b_1 b_2 b_3 + b_1 \bar b_2 b_2 b_3 + b_1 b_2 \bar b_3 b_3 + b_1 b_2 b_3 [\bar b_1 + b_1 \bar b_2 + b_1 b_2 \bar b_3]$$

$$= 4 b_1 b_2 b_3 - [b_1^2 b_2 b_3 + b_1 b_2^2 b_3 + b_1 b_2 b_3^2]$$

$$- \boxed{b_1^2 b_2^2 b_3^2}$$

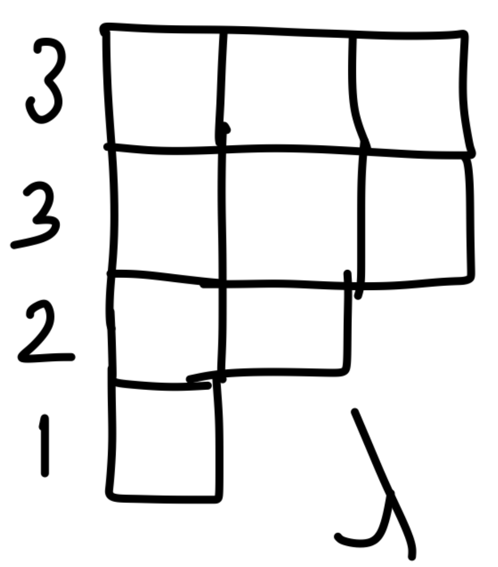
symmetric in b_i!

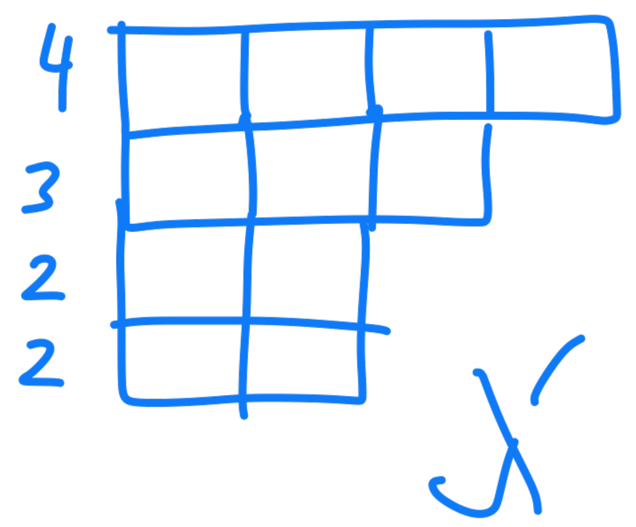
$$s_\lambda(x_1, \ldots, x_N) = \frac{\det[x_i^{\lambda_j+N-j}]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}$$

And indeed, particle-inhomogeneous TASEP is exactly solvable via Schur polynomials and Robinson-Schensted-Knuth **[Vershik-Kerov 1986], [O'Connell 2003]**:

$x_N(t) + N$ is the length of the last part λ_N of the random partition $\lambda_1 \ge \ldots \ge \lambda_N \ge 0$ distributed as

$$\frac{1}{Z} S_\lambda(\beta_1, \ldots, \beta_N)\, S_{\lambda'}(\overbrace{1, \ldots, 1}^{t})$$





Asymptotics and variants

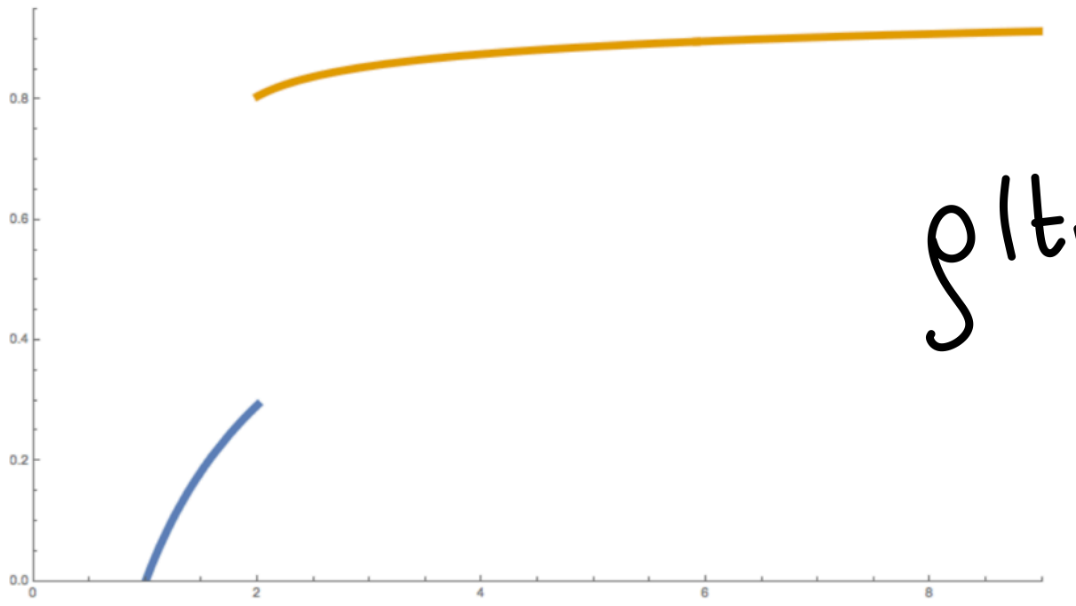
- TASEP with $\beta_i \equiv 1$, **step IC** $x_i(0) = -i,\ i \geq 1$: Tracy-Widom F_2 asymptotics **[Johansson 2000], [Gravner-Tracy-Widom 2002]**

$$\lim_{N\to\infty} \mathbb{P}\left[\frac{x_N(\alpha N) - c_\alpha \cdot N}{\sigma_\alpha \cdot N^{1/3}} \geq -r\right] = F_2(r)$$

- Changing β_i, may induce Baik-Ben Arous-Peche (BBP) phase transition with first slower particles; multiparameters play the role of **spiking** **[Baik 2006]**

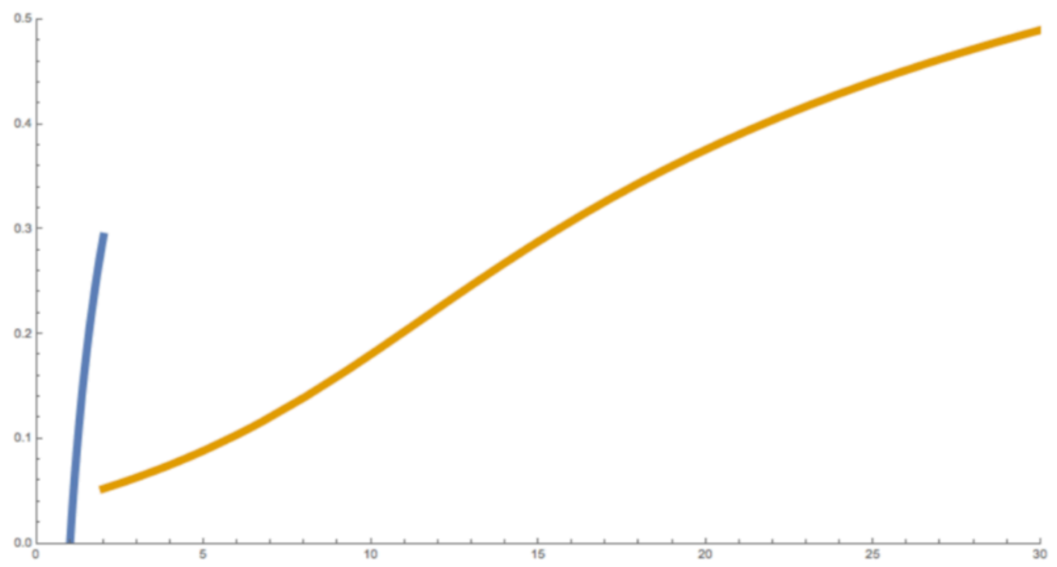
- For arbitrary initial conditions and β_i, is there a KPZ fixed point? (I believe somebody is working on this)

- PushTASEP is solvable with **both** inhomogeneous *particle speeds* **[Borodin-Ferrari 2008]** and in inhomogeneous *space* **[Assiotis 2019], [P. 2019]**



Slowdown

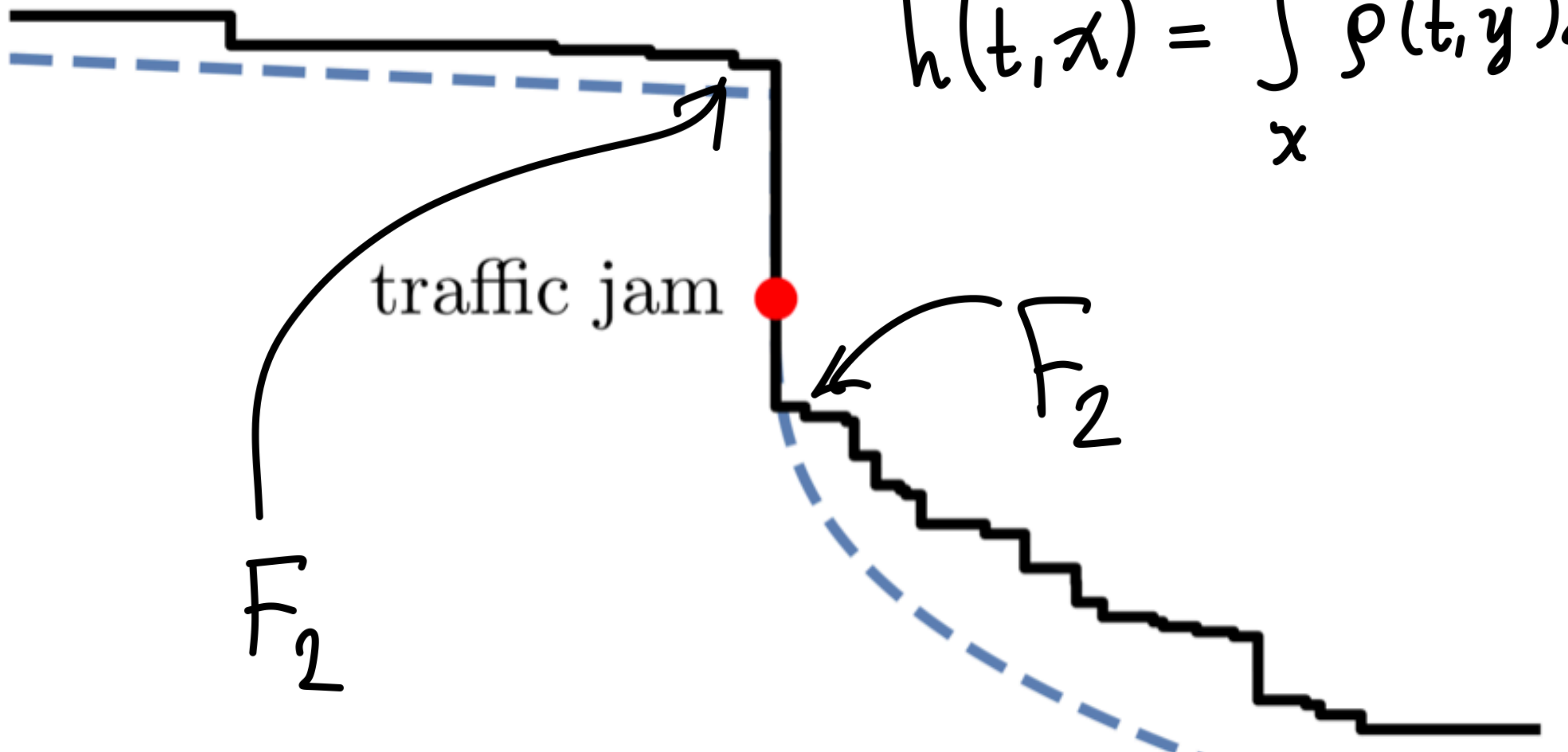
$\rho(t,x)$



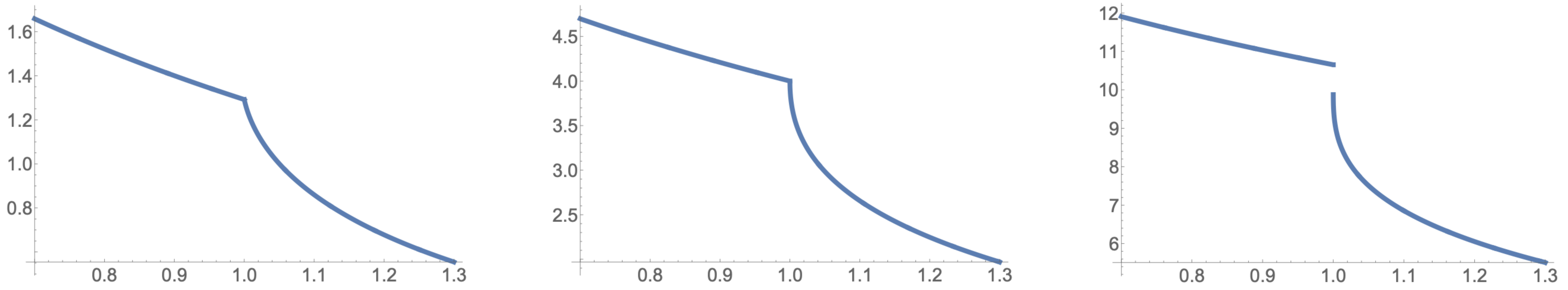
Speedup

- There is also *generalized TASEP* in continuous inhomogeneous space **[Borodin-P. 2017], [Knizel-P.-Saenz 2018]**

$$h(t,x) = \int_x^\infty \rho(t,y)\,dy$$



Phase transition at a traffic jam



Let $\xi(y) = \mathbf{1}_{y\le 1} + \frac{1}{2}\cdot \mathbf{1}_{y>1}$. The traffic jam appears after $t = 12L$. Let $x = 1 + 10\varepsilon(L)$.

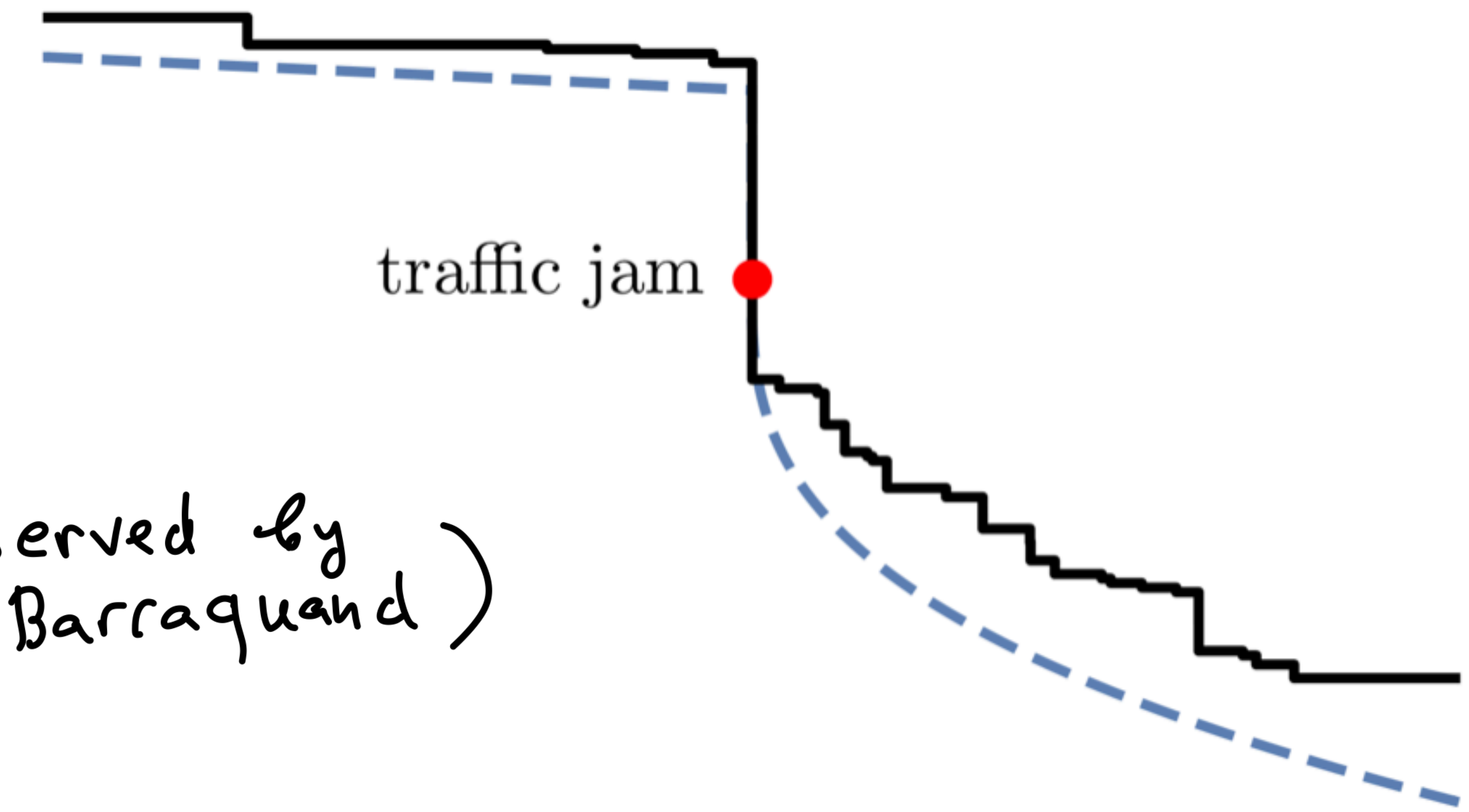
- If $\varepsilon(L) \ll L^{-4/3}$, then $\dfrac{h_{cont}(12L, x) - 4L}{2^{-2/3} c L^{1/3}} \to F_{GUE}$;
- If $\varepsilon(L) \gg L^{-4/3}$, then $\dfrac{h_{cont}(12L, x) - \mathfrak{h}(12, x)L}{c L^{1/3}} \to F_{GUE}$;
- If $\varepsilon(L) = 10^{-4/3}\delta L^{-4/3}$, then $\dfrac{h_{cont}(12L, x) - 4L}{2^{-2/3} c L^{1/3}} \to F^{(\delta)}_{GUE}$ (next slide).

$+$ multitime fluctuations are described by the Airy$_2$ kernel or its δ-deformation

Phase transition at a traffic jam

$$F^{(\delta)}_{GUE}(r) = \det\left(1 - K^{(\delta)}\right)_{(r,+\infty)}$$

$$K^{(\delta)}(r,r') = \frac{1}{(2\pi\mathbf{i})^2}\int_{e^{-2\pi\mathbf{i}/3}\infty}^{e^{2\pi\mathbf{i}/3}\infty} dw \int_{e^{-\pi\mathbf{i}/3}\infty}^{e^{\pi\mathbf{i}/3}\infty} dz\, \frac{1}{z-w} \times \exp\left\{\frac{z^3}{3} - \frac{w^3}{3} - zr + wr' - \frac{\delta}{z} + \frac{\delta}{w}\right\}$$



- If $\delta = 0$, we have $F^{(\delta)}_{GUE} = F_{GUE}$;
- As $\delta \to +\infty$, $F^{(\delta)}_{GUE}(r + 2\delta^{\frac{1}{2}}) \to F_{GUE}(2^{-\frac{2}{3}} r)$. (observed by Barraquand)

III. Markov operators for permuting parameters

j.w. Axel Saenz
Mikhail Tikhonov
2019
Matthew Nicoletti
(2021+)

arXiv:1907.09155

arXiv:1912.06067

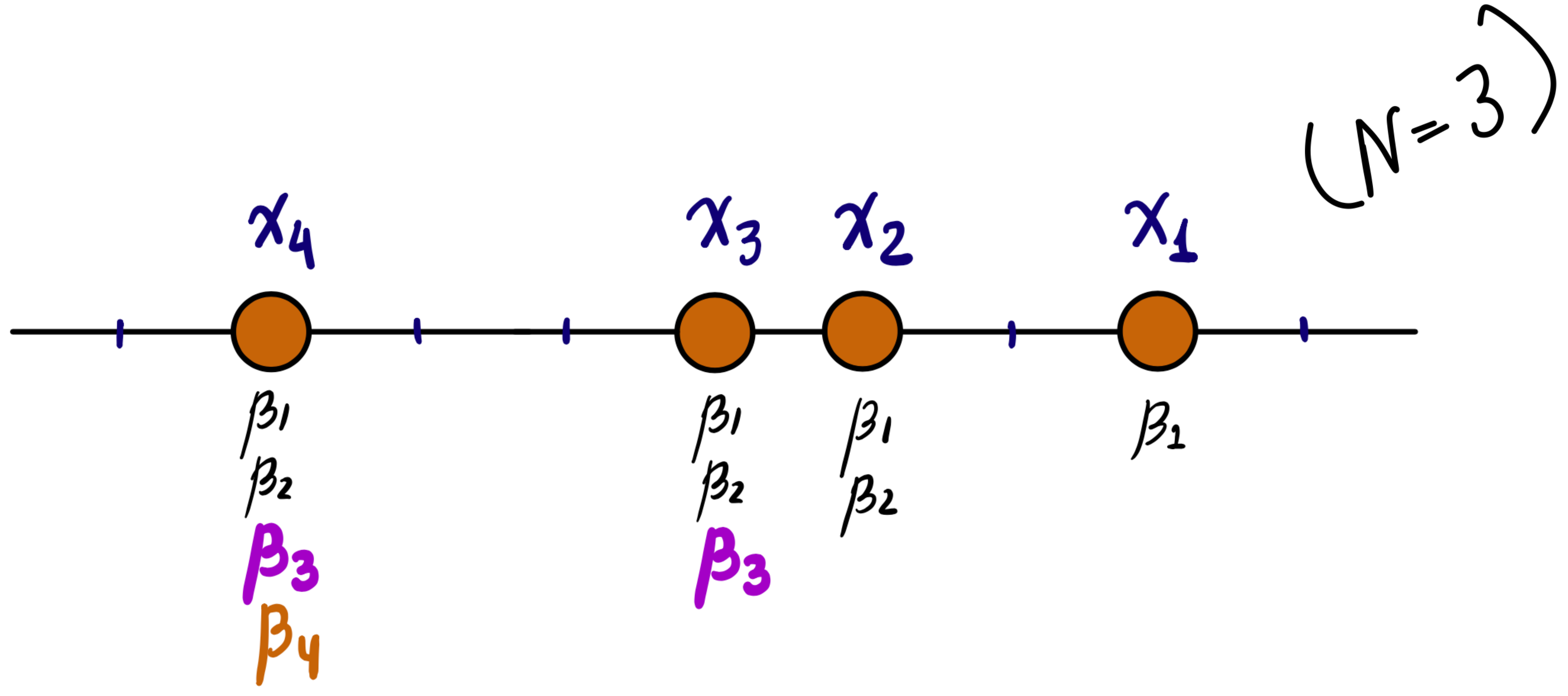
arXiv:1912.08671

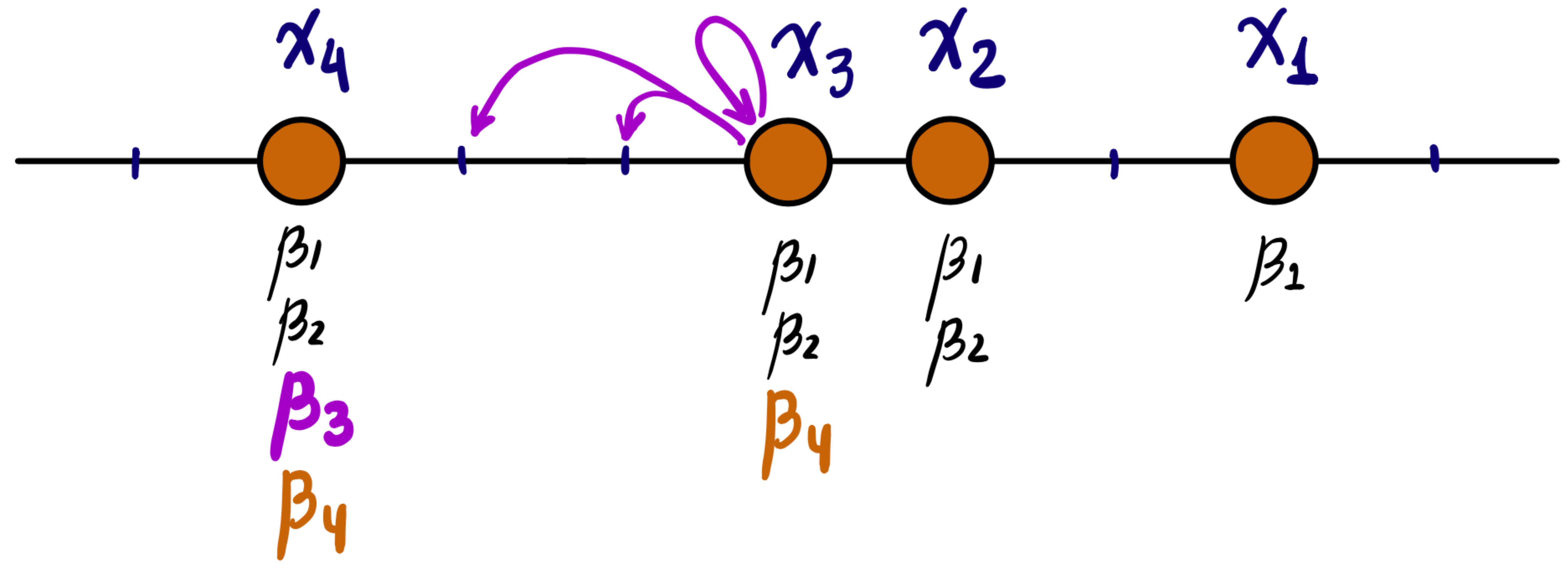
What does symmetry mean probabilistically?

$$s_\lambda(x_1,\dots,x_N) = \frac{\det[x_i^{\lambda_j+N-j}]_{i,j=1}^N}{\prod_{i<j}(x_i-x_j)}$$

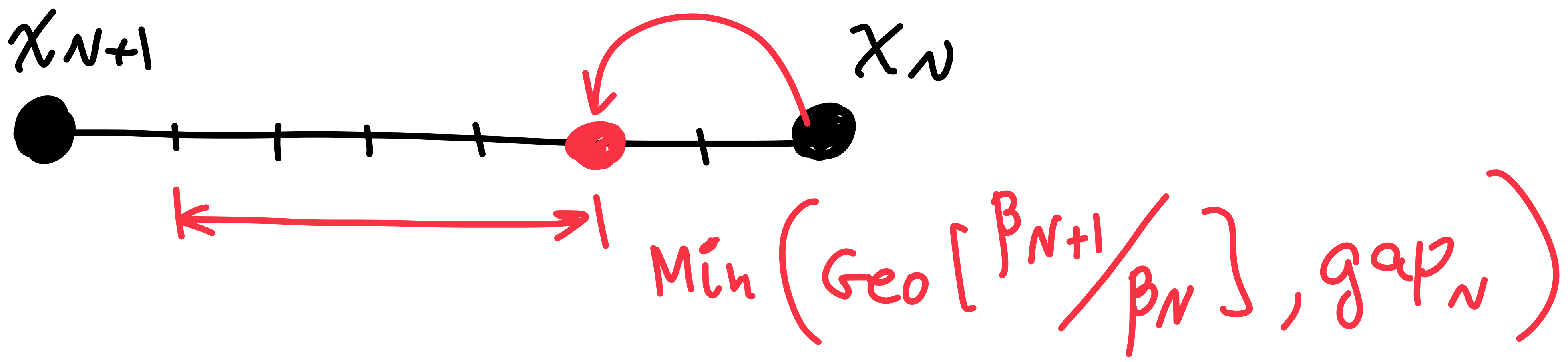
- Distribution of $x_N(t)$ is expressed through Schur symmetric polynomial
- So, x_N depends on $\beta_1,\dots,\beta_N$ symmetrically
- What if we swap β_N, β_{N+1}? Affects only x_N.

$$x_N(t)+N=\lambda_N \sim \frac{1}{Z} S_\lambda(\beta_1,\dots,\beta_N)\, S_{\lambda'}(\overbrace{1,\dots,1}^{t})$$





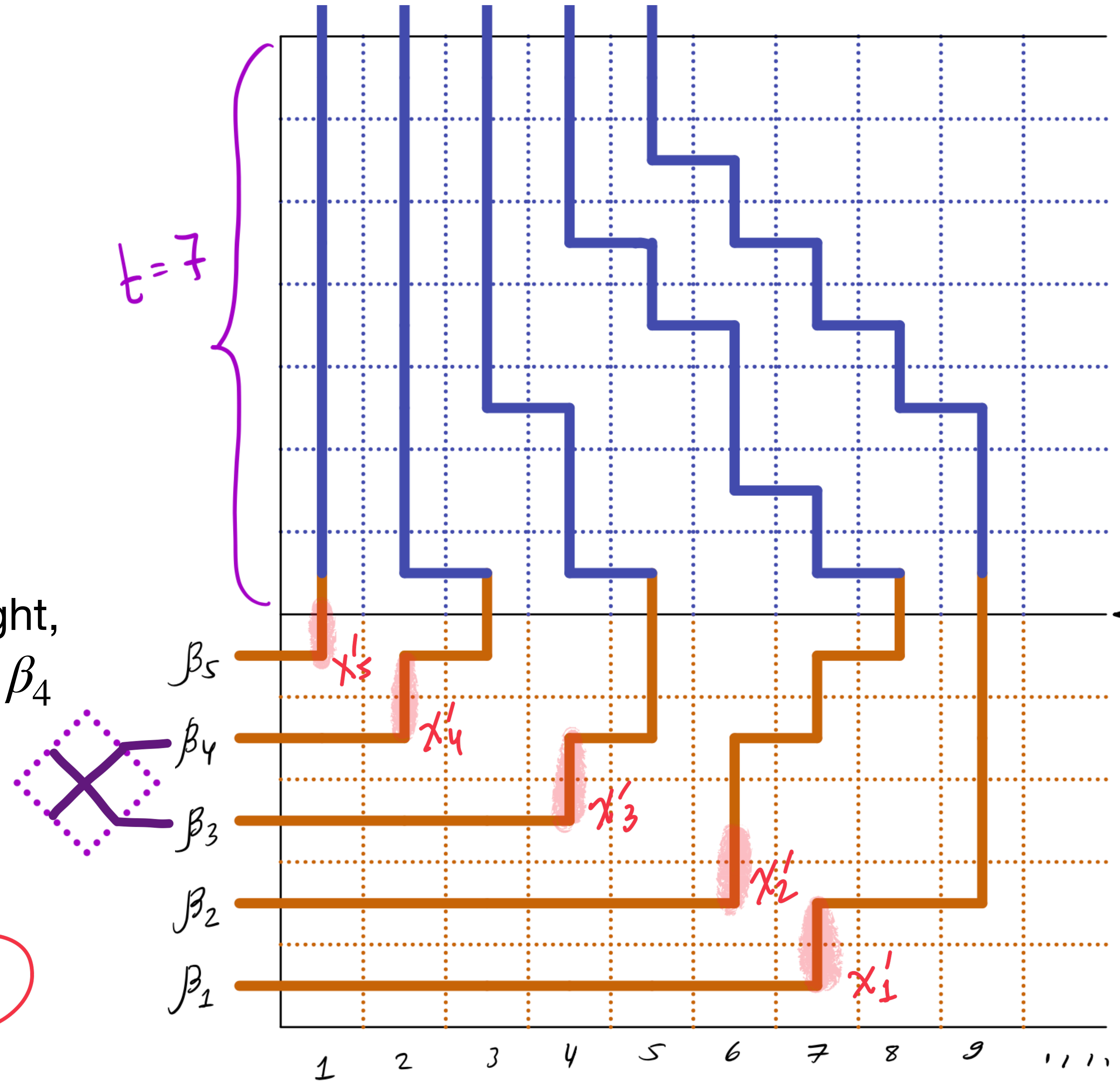
- Sometimes can **couple** the distributions, while only moving x_N
- Possible iff $\beta_{N+1} \le \beta_N$: x_N was faster, and jumps back
- Coupling can be realized through Yang-Baxter equation



Symmetry and vertex models

$$x_N(t)+N=\lambda_N \sim \frac{1}{Z}\, s_\lambda(\beta_1,\dots,\beta_N)\, s_{\lambda'}(\overbrace{1,\dots,1}^{t})$$

$$s_\lambda(x_1,\ldots,x_N)=\frac{\det[x_i^{\lambda_j+N-j}]_{i,j=1}^N}{\prod_{i<j}(x_i-x_j)}$$



above, (paths travel ≤1 at each horizontal)

$\leftarrow \{\lambda_i+N+1-i\}$

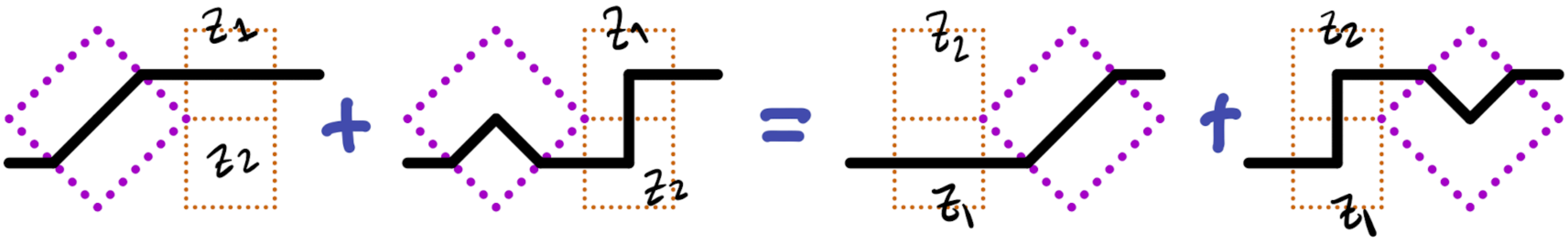
$\lambda=(4,4,2,1,0)$

Zip to the right,
swap $\beta_3 \leftrightarrow \beta_4$

TASEP
$x_i' = x_i + N + 1$

- Semistandard Young tableau
- Nonintersecting paths
- Lozenge tilings
- Free fermion five vertex model

From Yang-Baxter equation to probabilistic operation (Markov map)



- If the weights are positive (this is where $\beta_{N+1} < \beta_N$ comes from), then we can construct a **coupling** between left and right-hand sides
- The coupling gives a rule for randomized update when moving the cross
- Example:

$$\boxed{\begin{array}{c} w(\circ) + w(\triangle) = w(\square) + w(\times) \\ 2 + 2 = 3 + 1 \end{array}}$$

	$\square$	$\times$
$\circ$	$1/2$	$1/2$
$\triangle$	1	0

	$\square$	$\times$
$\circ$	$3/4$	$1/4$
$\triangle$	$3/4$	$1/4$

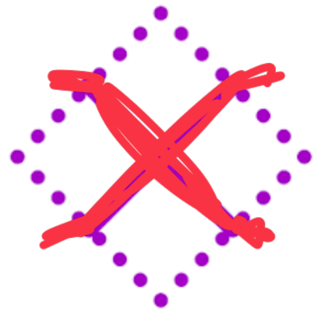
$$w(\triangle)\, p(\triangle \to \square) = w(\square)\, \tilde{p}(\square \to \triangle)$$

etc

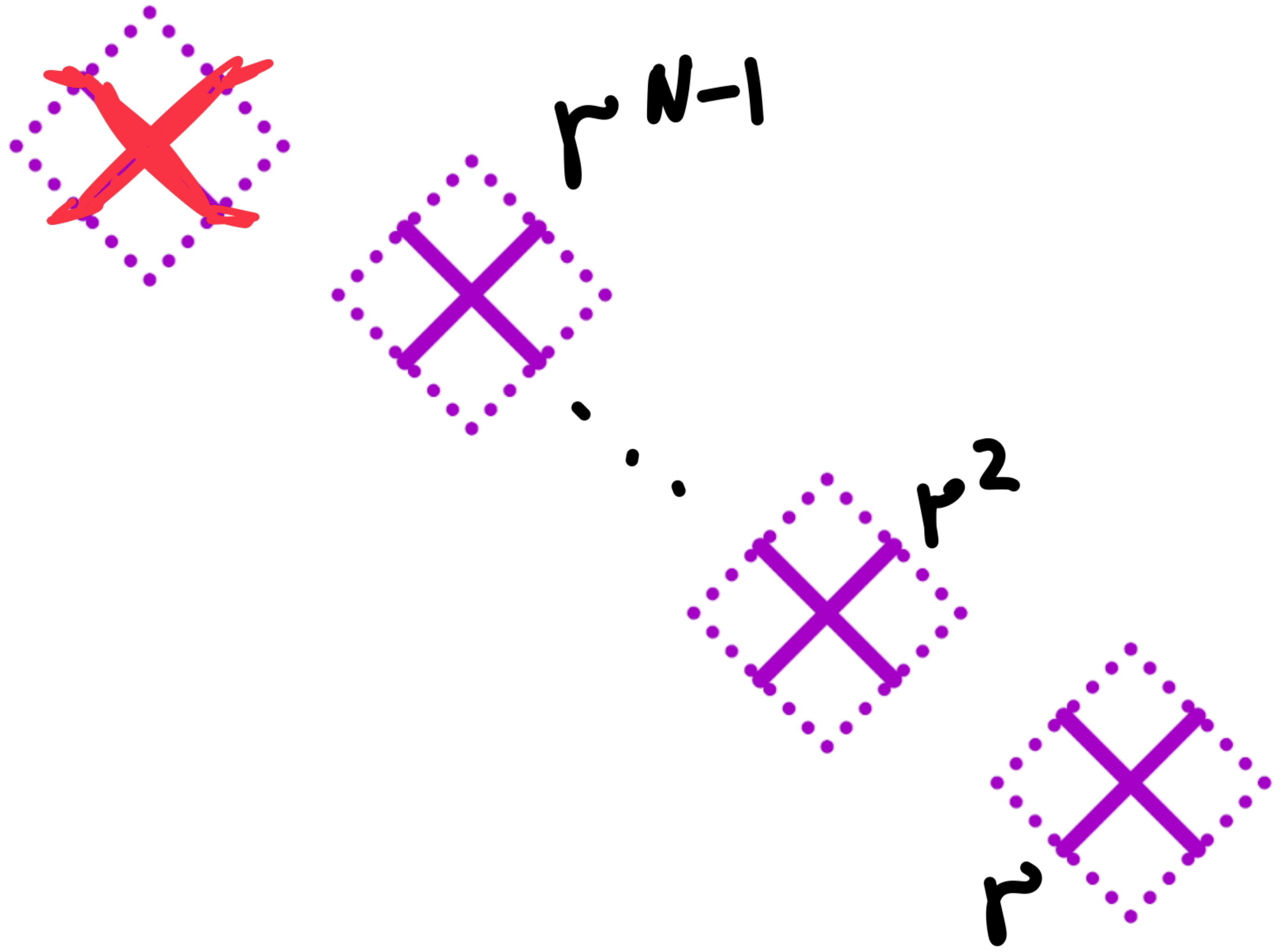
Application to the vertex model

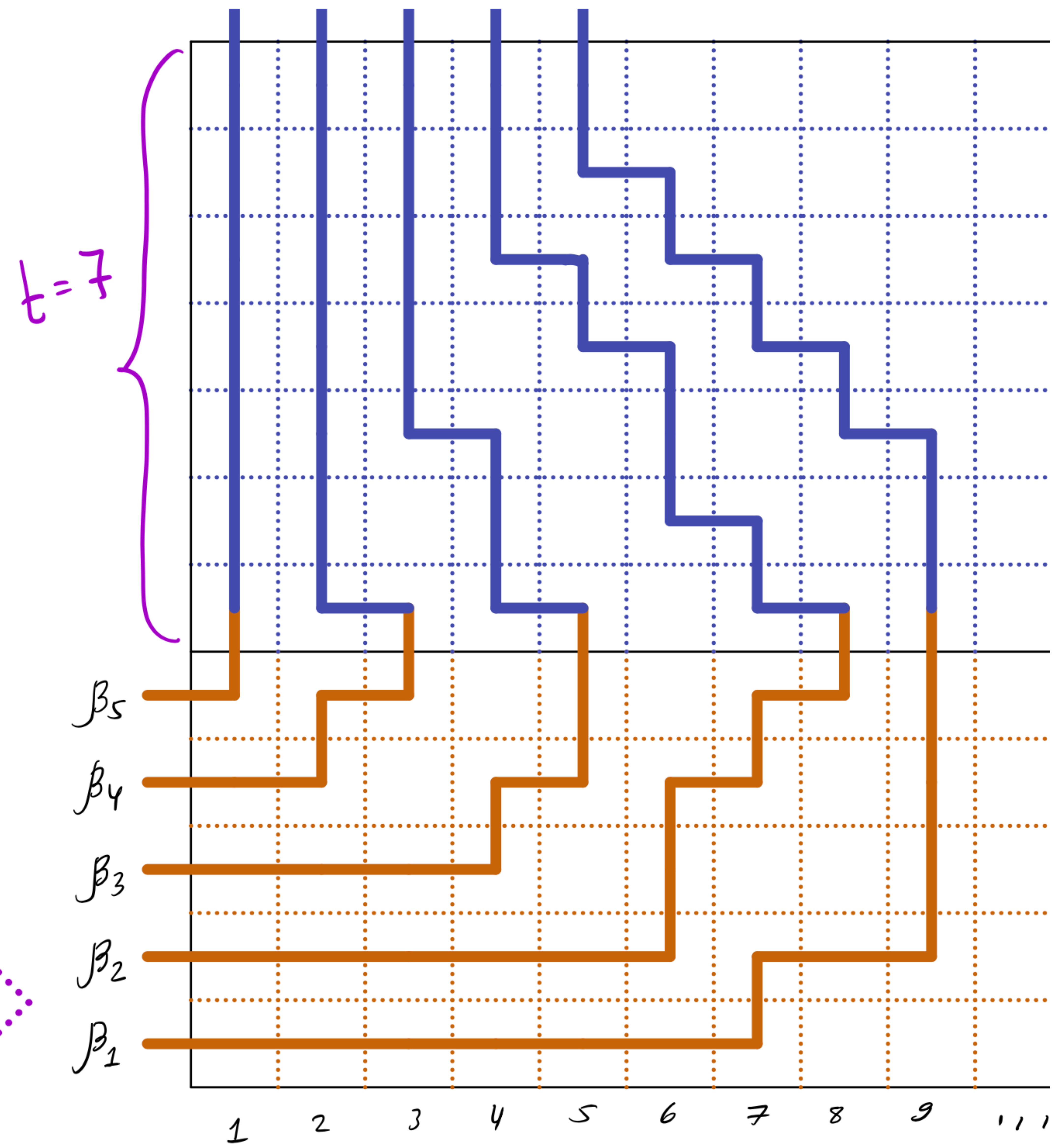
$$\beta_i = \beta r^{i-1},\ 0 < r < 1$$

- Let $\beta_i = \beta r^{i-1}$

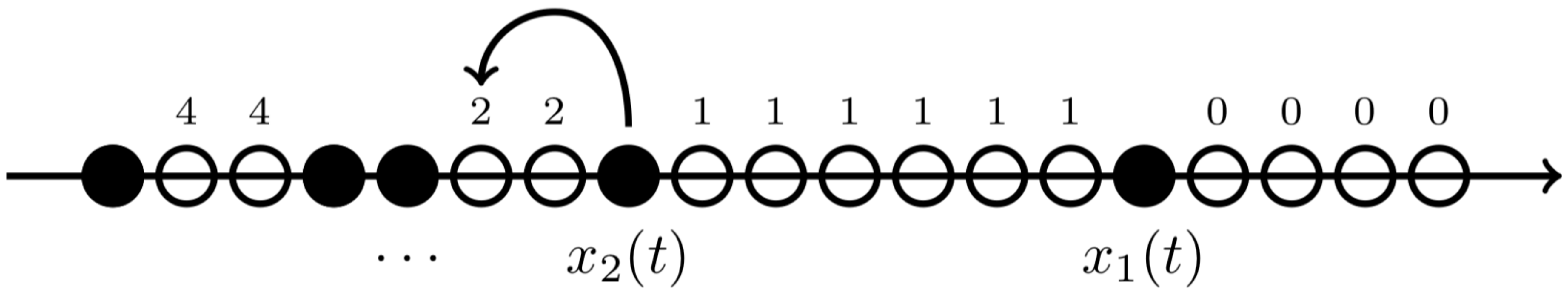


- Move β_1 all the way up
- Continue moving through the "time" part
- Take $r \to 1$, and take a Poisson-like limit
- We get a continuous time Markov operation which continuously makes β smaller
- Then, take another, standard Poisson limit to continuous time TASEP





Mapping TASEP back in time [P.-Saenz 2019]



- **Backward TASEP.** Markov chain on left-packed configurations $x_1 > x_2 > x_3 > \dots$
- Each hole has an independent exponential clock with rate equal to the number m of particles to its right, $\mathbb{P}(\text{wait} > s) = e^{-m\cdot s}$, $s > 0$.
- When the clock at a hole rings, the leftmost of the particles that are to the right of the hole instantaneously jumps into this hole
- Because total rate of jump is proportional to the size of the gap, this is a discrete space inhomogeneous version of the Hammersley process [Hammersley '72], [Aldous-Diaconis '95]

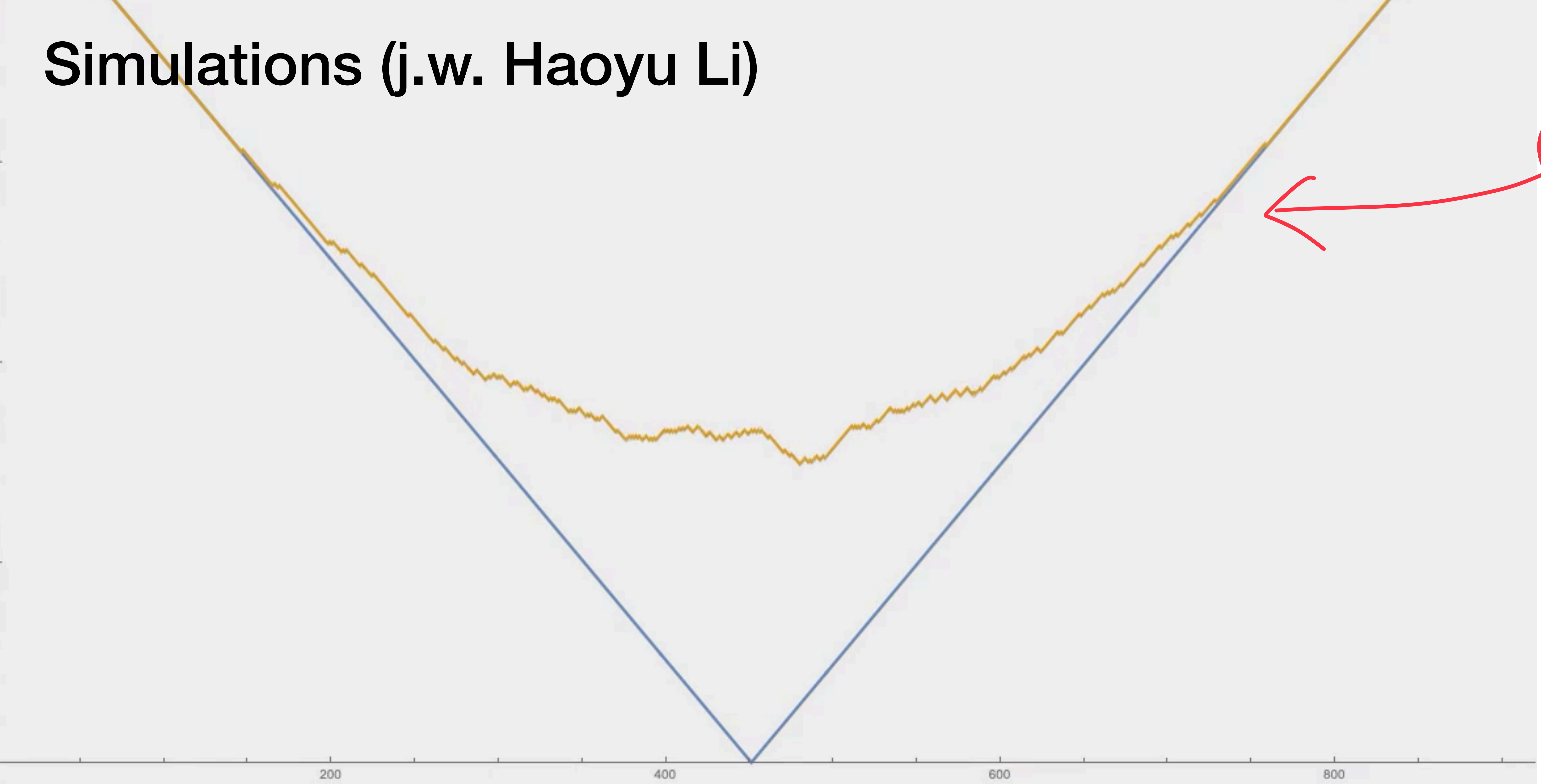
Theorem.

Let μ_t be the distribution of the continuous time TASEP (with step IC) at time t.

Let L_τ be the backward TASEP Markov semigroup.

Then $\mu_t L_\tau = \mu_{t\cdot e^{-\tau}}$, i.e.,
$\sum_{\vec{x}} \mu_t(\vec{x}) L_\tau(\vec{x}, \vec{y}) = \mu_{t\cdot e^{-\tau}}(\vec{y})$.

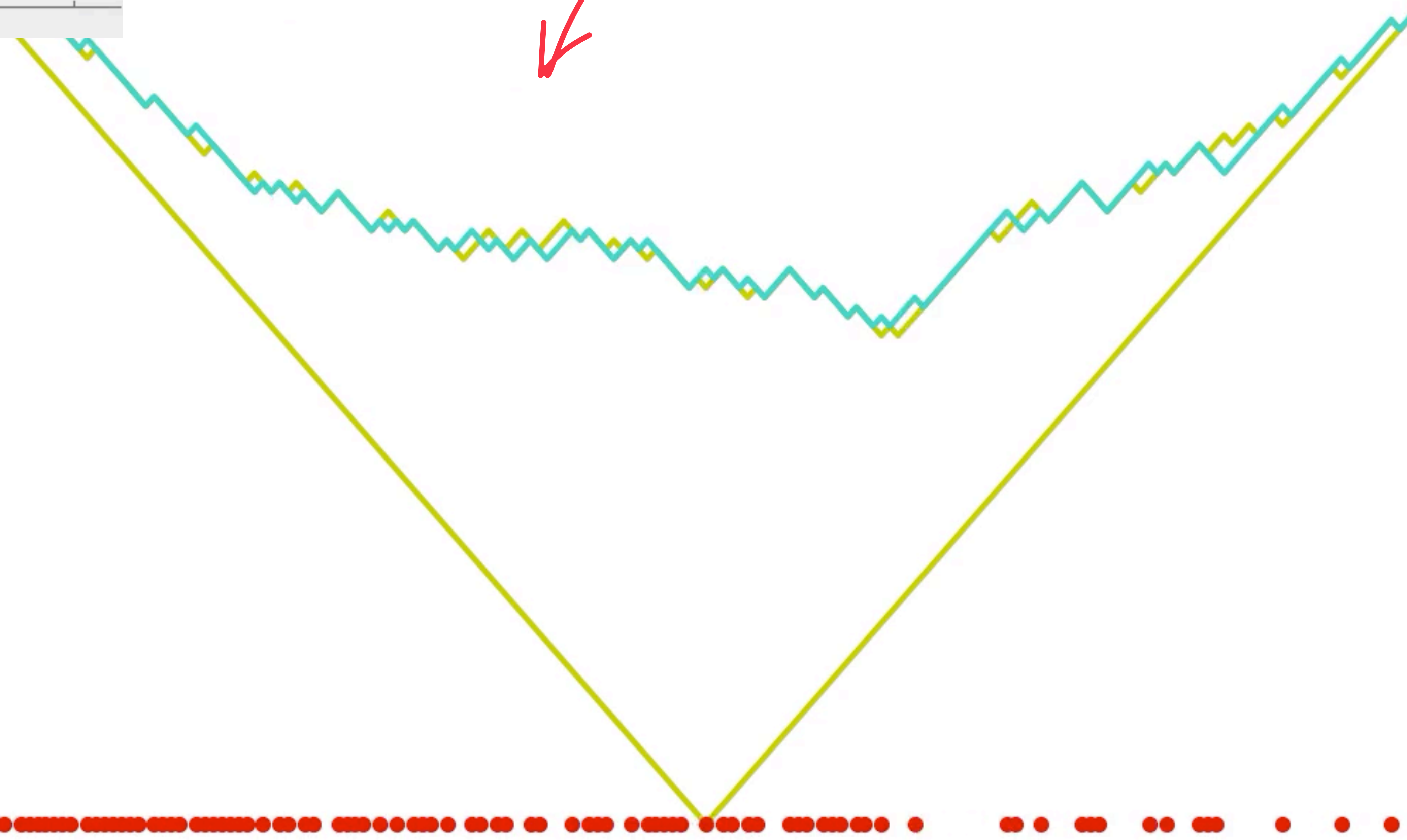
Simulations (j.w. Haoyu Li)



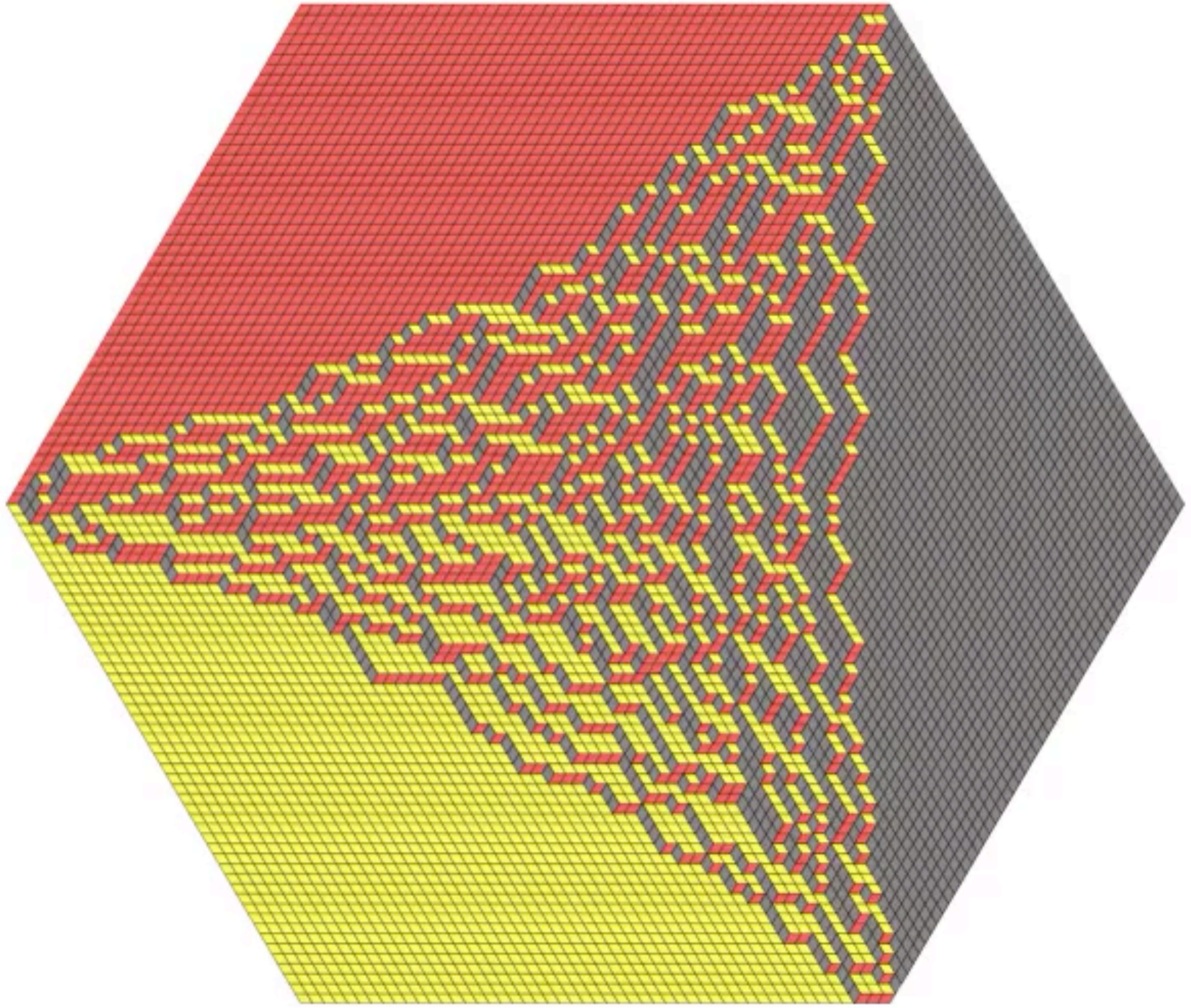
Fwd, then Bwd

Stationary
$= \text{Fwd} + \frac{1}{2} \text{Bwd}$

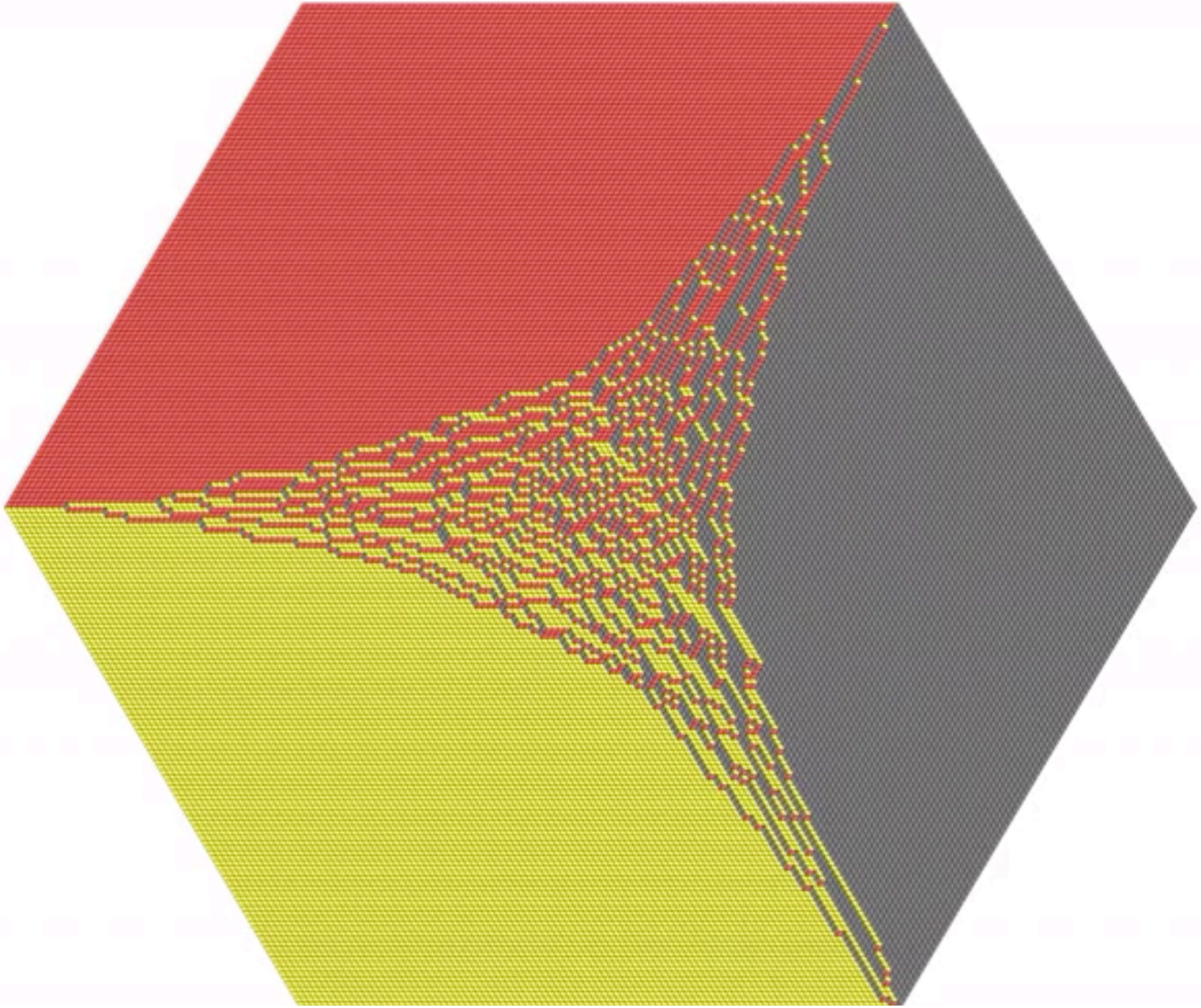
- Under stationary dynamics, how fast do the two KPZ like interfaces decouple?
- Similar result for q-TASEP + there is **duality**



Application to lozenge tilings (j.w. Edith Zhang)



Limit shapes of q^{vol} lozenge tilings:
[Cohn-Kenyon-Propp ’00], [Kenyon-Okounkov ’05]



Summary

- It is worthwhile to insert multiple parameters into integrable models
- Especially if the model stays integrable
- Helps with **enumeration**
- **Spiked** asymptotics and new phase transitions
- Permutations of multiple parameters can be realized as **Markov operators** on the original model
- Leads to interesting symmetries of the model, and new dynamics. Works for TASEP, q-TASEP, polymers, deformed GUE random matrices, …
- In progress - hydrodynamics for six vertex model