Riemann Hilbert Problems: open problems session

Andrei Prokhorov

MSRI

11/16/2021

Painlevé equations in the Hamiltonian form

$$H_{VI} = \frac{p^2}{2} + \sum_{\ell=0}^{3} g_\ell^2 \wp(q + \omega_\ell)$$

$$H_V = \frac{p^2}{2} - \frac{\alpha}{\sinh^2(q/2)} - \frac{\beta}{\cosh^2(q/2)} + \frac{\gamma t}{2} \cosh(q) + \frac{\delta t^2}{8} \cosh(2q)$$

$$H_{IV} = \frac{p^2}{2} - \frac{1}{2}\left(\frac{q}{2}\right)^6 - 2t\left(\frac{q}{2}\right)^4 - 2(t^2 - \alpha)\left(\frac{q}{2}\right)^2 + \beta\left(\frac{q}{2}\right)^{-2}$$

$$H_{III} = \frac{p^2}{2} - \frac{\alpha}{4} \mathrm{e}^{q} + \frac{\beta t}{4} \mathrm{e}^{-q} - \frac{\gamma}{8} \mathrm{e}^{2q} + \frac{\delta t^2}{8} \mathrm{e}^{-2q}$$

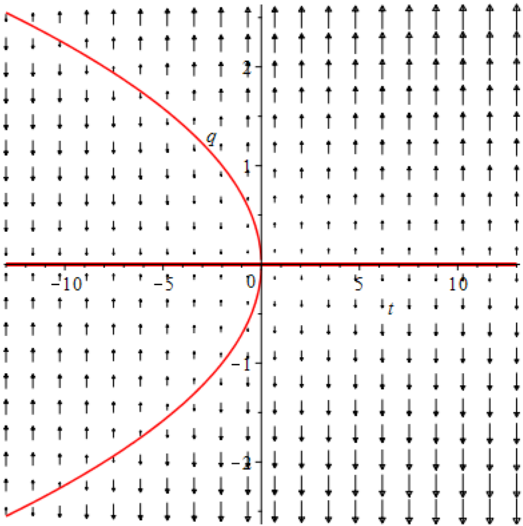
$$H_{II} = \frac{p^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2 - \alpha q$$

$$H_I = \frac{p^2}{2} - 2q^3 - tq$$

Questions:

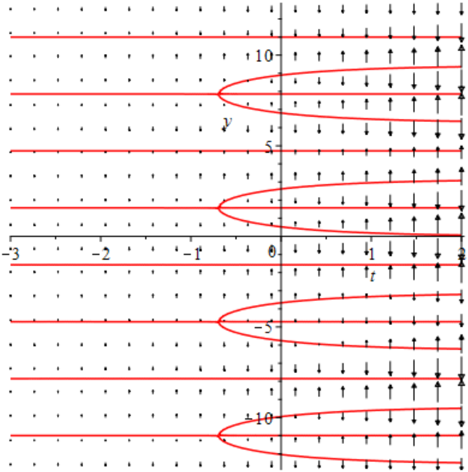
1. Description of asymptotic behaviors motivated by Hamiltonian interpretation
2. Connection problem for tau functions (PIII(D6), PIV, PV)
3. Large parameter asymptotic of rational solutions (PV, PVI)

Painlevé-II force field



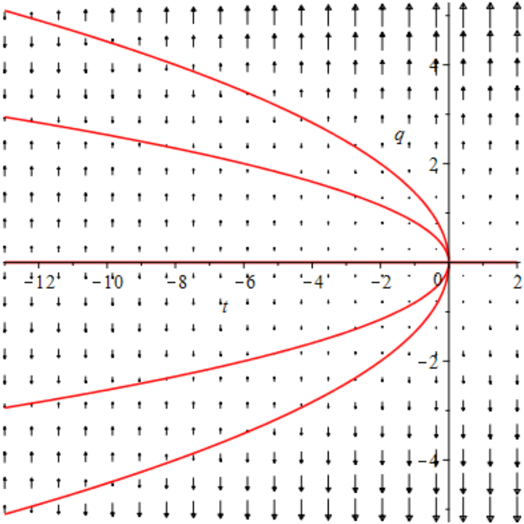
$\alpha = 0$

Painlevé III(D6): force field



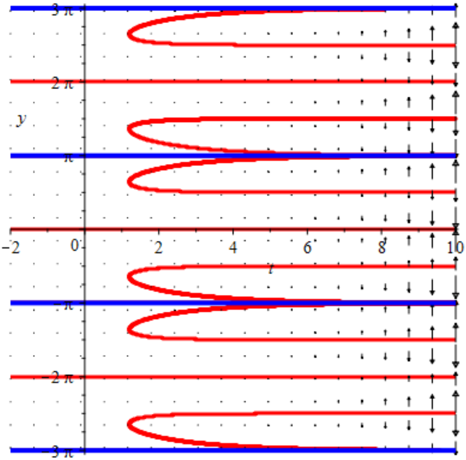
$q = iy, \quad \alpha = -\frac{i}{2}, \quad \beta = -\frac{i}{2}, \quad \gamma = \frac{1}{2}, \quad \delta = -\frac{1}{2},$
$F(y,t) = e^{t}\cos(y) - e^{2t}\sin(2y)$

Painlevé IV: force field



$\alpha = \beta = 0$

Painlevé V: force field



$$q = iy, \quad \alpha = 0, \quad \beta = -1, \quad \gamma = -2, \quad \delta = -2,$$

$$F(y,t) = -\frac{\sin(\frac{y}{2})}{\cos^3(\frac{y}{2})} - e^t \sin(y) - e^{2t}\sin(2y)$$

Calogero-Painlevé systems in the Hamiltonian form

$$\tilde{H}_{VI} = \sum_{j=1}^{n} \left(\frac{p_j^2}{2} + \sum_{\ell=0}^{3} g_\ell^2 \wp(q_j + \omega_\ell) \right) + g_4^2 \sum_{j \neq k} \Big(\wp(q_j - q_k) + \wp(q_j + q_k) \Big).$$

$$\tilde{H}_{V} = \sum_{j=1}^{n} \left(\frac{p_j^2}{2} - \frac{\alpha}{\sinh^2(q_j/2)} - \frac{\beta}{\cosh^2(q_j/2)} + \frac{\gamma t}{2} \cosh(q_j) + \frac{\delta t^2}{8} \cosh(2q_j) \right) +$$

$$+ g_4^2 \sum_{j \neq k} \left(\frac{1}{\sinh^2((q_j - q_k)/2)} + \frac{1}{\sinh^2((q_j + q_k)/2)} \right).$$

$$\tilde{H}_{IV} = \sum_{j=1}^{n} \left(\frac{p_j^2}{2} - \frac{1}{2} \left(\frac{q_j}{2}\right)^6 - 2t \left(\frac{q_j}{2}\right)^4 - 2(t^2 - \alpha) \left(\frac{q_j}{2}\right)^2 + \beta \left(\frac{q_j}{2}\right)^{-2} \right)$$

$$+ g_4^2 \sum_{j \neq k} \left(\frac{1}{(q_j - q_k)^2} + \frac{1}{(q_j + q_k)^2} \right).$$

$$\tilde{H}_{III} = \sum_{j=1}^{n} \left(\frac{p_j^2}{2} - \frac{\alpha}{4} \mathrm{e}^{q_j} + \frac{\beta t}{4} \mathrm{e}^{-q_j} - \frac{\gamma}{8} \mathrm{e}^{2q_j} + \frac{\delta t^2}{8} \mathrm{e}^{-2q_j} \right) + g_4^2 \sum_{j \neq k} \frac{1}{\sinh^2\left((q_j - q_k)/2\right)}.$$

$$\tilde{H}_{II} = \sum_{j=1}^{n} \left(\frac{p_j^2}{2} - \frac{1}{2} \left(q_j^2 + \frac{t}{2} \right)^2 - \alpha q_j \right) + g_4^2 \sum_{j \neq k} \frac{1}{(q_j - q_k)^2}.$$

$$\tilde{H}_{I} = \sum_{j=1}^{n} \left(\frac{p_j^2}{2} - 2q_j^3 - t q_j \right) + g_4^2 \sum_{j \neq k} \frac{1}{(q_j - q_k)^2}.$$

Calogero-Painlevé systems

References:
Takasaki (2001), Bertola, Cafasso, Roubtsov (2018), Rumanov (2013), Grava, Its, Kapaev, Mezzadri (2018), Li (2018)

Questions:

1. Asymptotic of generic solutions of Calogero-Painlevé systems
2. Large parameter asymptotic of rational solutions
3. Connection problem for tau function

Example

Consider the second Calogero-Painlevé system with

$$\alpha = \frac{2-n}{2n}, \quad g_4 = \frac{i}{n}.$$

Introduce τ function

$$\tau(t) = \exp\left(\int_{-\infty}^{t} \left(\tilde{H}_{II} + \frac{\sum_{j=1}^{n} q_j}{2}\right) dt'\right).$$

It is related to the β Tracy-Widom distribution

$$\tau(t) = F_{2n}(-t2^{-\frac{1}{3}})$$

Example

$n = 1$

$$q(t) = -2^{-\frac{1}{3}} \frac{u'(-2^{-\frac{1}{3}} t)}{u(-2^{-\frac{1}{3}} t)}$$

where $u(t)$ is Hastings Mcleod solution.

$$q(t) = -\frac{1}{2t} + O\left(\frac{1}{t^4}\right), \quad t \to +\infty,$$

$$q(t) = -\sqrt{\frac{-t}{2}} + O\left(\frac{1}{|t|}\right), \quad t \to -\infty,$$

$$\tau(t) = \exp\left(\int_{-\infty}^{t} \left(\frac{(q')^2}{2} - \frac{1}{2}\left(q^2 + \frac{t}{2}\right)^2\right) dt'\right).$$

Example

Conjecture(Borot, Nadal, (2012))

$$\tau(t) \simeq 1 - \frac{(n-1)!\exp\left(-\frac{2n\sqrt{2}(-t)^{\frac{3}{2}}}{3}\right)}{(8n)^n \pi \sqrt{2}(-t)^{\frac{3n}{2}}}, \quad t \to -\infty$$

Conjecture(Borot, Eynard, Majumdar, Nadal (2010))

$$\tau(t) \simeq c_n \exp\left(-\frac{nt^3}{24} + \frac{(n-1)t^{\frac{3}{2}}}{3} + \frac{(n + \frac{1}{n} - 3)}{8}\ln t\right), \quad t \to \infty$$

$$c_n = \frac{\sqrt{\frac{1}{n}} e^{n\zeta'(-1)} (2\pi)^{\frac{n-1}{4}} 2^{\frac{37}{16} - \frac{53}{48}\left(n + \frac{1}{n}\right)}}{\prod_{m=1}^{n-1} G(1 + \frac{m}{n})}$$