MSRI Nov 23, 2021

Hydrodynamic Scale of the Toda Lattice

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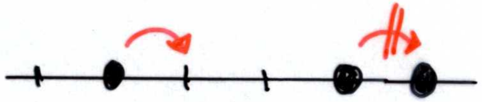
TUM ünchen

hydrodynamic scale (ballistic)

TASEP



- local conservation laws ‖ particle number $\sum_j \eta_j$, density η_j
- all ✓ $\quad \mathcal{L}\eta_j = \underbrace{\eta_{j-1}(1-\eta_j)}_{\text{current density}} - \eta_j(1-\eta_{j+1})$
- stationary measures, transl. invariant ‖ Bernoulli $\mathbb{P}(\eta_j = 1) = \rho$

  1 parameter
- average field, current ‖ $\mathbb{E}(\eta_j) = \rho$

  ‖ $\mathbb{E}(\eta_{j-1}(1-\eta_j)) = \rho(1-\rho)$

⇝ propagation of local stationarity

$$\| \quad \partial_t \rho + \partial_x(\rho(1-\rho)) = 0 \quad \|$$

Euler

- convergence 1985-2005 $\|$ entropy solution $\|$ shocks

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same program for TODA $\Big\{$ average currents $\Big\}$

⇒ convergence to be accomplished ⇐

∞ parameters

$$\partial_t \rho_n + \partial_x \underline{g_n(\rho_0, \rho_1, \dots)} = 0 \qquad n = 0,1,2 \qquad \rho_n = \rho_n(x,t)$$

- Toda lattice   integrable

common structure:

XXZ, δ-Bose gas, sinh-Gordon

quantum Toda, ...

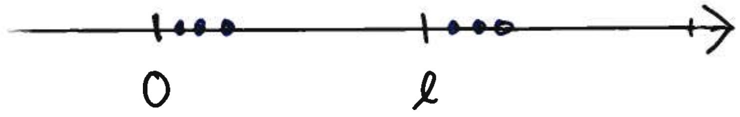
$$q_j, p_j \in \mathbb{R}^2$$

$$H = \sum_{j=1}^{N} \left\{ \tfrac{1}{2} p_j^2 + e^{-(q_{j+1}-q_j)} \right\}$$

$\rightarrow$ periodic b.c. $q_{j+N} = q_j + \ell$ $\ell \in \mathbb{R}$

$$p_{j+N} = p_j$$

// confining //



- Flaschka variables

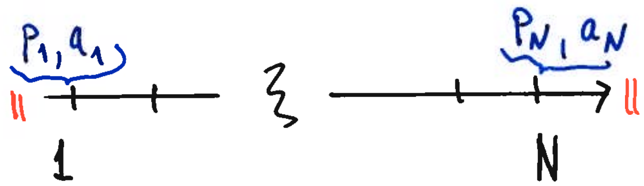
$$a_j = e^{-r_j/2}, \qquad r_j = q_{j+1} - q_j = -2 \log a_j$$

$$\frac{d}{dt} a_j = \tfrac{1}{2} a_j (p_j - p_{j+1}), \quad \frac{d}{dt} p_j = a_{j-1}^2 - a_j^2,$$

$$j = 1, \dots, N, \quad a_{-1} = a_N, \quad p_{N+1} = p_1$$

$$T_N = (\mathbb{R} \times \mathbb{R}_+)^N$$

- lattice field theory



$$\frac{d}{dt} \sum_{j=1}^{N} r_j(t) = 0$$ conserved, local

more conservation laws Lax matrix Flaschka 1974

$$L_N = \begin{bmatrix} p_1 & a_1 & 0 & & a_N \\ a_1 & \ddots & \ddots & & 0 \\ 0 & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & a_{N-1} \\ a_N & & & a_{N-1} & p_N \end{bmatrix} \qquad B_N = \frac{1}{2} \begin{bmatrix} 0 & -a_1 & 0 & & a_N \\ a_1 & 0 & \ddots & & \\ & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & \ddots & -a_{N-1} \\ -a_N & & 0 & a_{N-1} & 0 \end{bmatrix}$$

$$\frac{d}{dt} L_N = [B_N, L_N]$$

eigenvalues

$$L_N \psi_j = \lambda_j \psi_j$$

$\Rightarrow$ eigenvalues are conserved NOT local charges with local density

$$Q^{[n],N} = \mathrm{tr}(L_N)^n = \sum_{j=1}^{N} ((L_N)^n)_{j,j} = \sum_{j=1}^{N} Q_j^{[n],N}$$

- $N \to \infty$

$L_N \to L \,, \quad Q_j^{[n],N} \to Q_j^{[n]} = (L^n)_{j,j} \,, \; n = 1, 2, \ldots$ random walk expansion

local

and $\quad Q_j^{[0]} = r_j$

- local conservation law $\quad \dfrac{d}{dt} Q_j^{[n]} = J_j^{[n]} - J_{j+1}^{[n]}$

$$J_j^{[0]} = Q_j^{[1]}$$

$$J_j^{[n]} = (L^n L^{\downarrow})_{j,j}$$

? all ?

Henrici, Kappeler (2010)

stationary measures GGE, generalized Gibbs ensemble

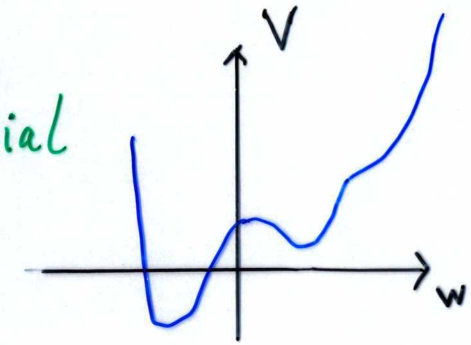
volume constraint: $\delta\left(\sum_{j=1}^{N} r_j - \ell N\right) \cong \prod_{j=1}^{N} e^{-P r_j}$, pressure $P > 0$

GGE:

$$\underbrace{\frac{1}{Z}\prod_{j=1}^{N} dp_j\, da_j\, \frac{2}{a_j}\,(a_j)}_{\text{a priori}}{}^{2P}\; \underbrace{e^{-\operatorname{tr}(V(L_N))}}_{\text{B-weight}}$$

$$V(w) = \sum_{n=1}^{\infty} \mu_n w^n$$

confining potential

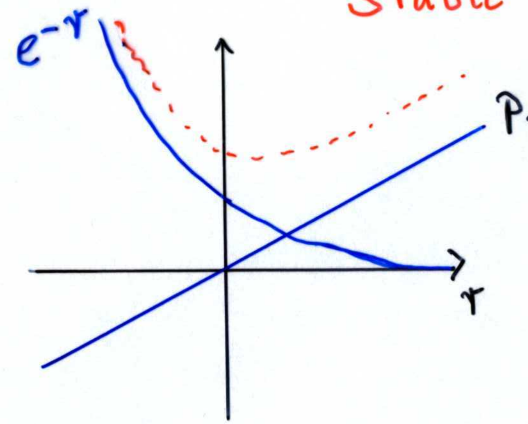


L_N is random Jacobi matrix under GGE

thermal

$V(w) = \frac{1}{2}\beta w^2$, $\{L_{jj} = p_j\}$ i.i.d Gauss

$\{L_{jj+1} = a_j\}$ i.i.d. χ_{2P}



Lax density of states (DOS)

$L_N \psi_j = \lambda_j \psi_j \quad , \; j = 1, \dots, N$

empirical

$$\rho_N(w) = \frac{1}{N} \sum_{j=1}^{N} \delta(w - \lambda_j) \xrightarrow[N \to \infty]{} \nu \, \rho_p(w) \quad \text{a.s.}$$

Guionnet, Menin (2021)

$$\rightarrow \quad \mathbb{E}_{P,V}\left(Q_0^{[n]}\right) = \begin{cases} \int dw \, \rho_p(w) \, w^n & , \; n = 1, 2, \dots \\ \nu & n = 0 \end{cases}$$

average stretch

Dumitriu, Edelman (2002) H.S. (2019)

⇝ free energy functional

$$\mathcal{F}(\rho) = \underbrace{\int \rho V - \int \log|w - w'| \rho(w) \rho(w')}_{\text{energy}} + \underbrace{\int \rho \log \rho}_{\text{entropy}}$$

log gas

map

$P, V \to \nu, \rho_p$

$\leftarrow$?

$\rho \geq 0 \; , \; \int \rho = 1$

unique minimizer $\rho^*(P)$ ⇝ $\nu \rho_p = \partial_P \left(P \rho^*(P) \right)$

DOS

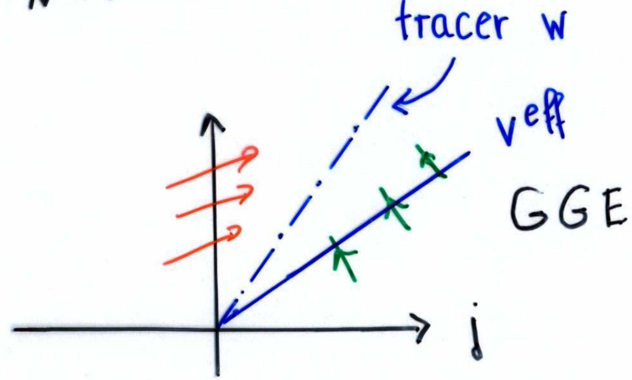
GGE averaged currents

break through Bertini et al (2016)

Doyon et al (2016)

- current DOS

$$\frac{1}{N}\sum_{j=1}^{N} \delta(w-\lambda_j)\left(\sum_{i=1}^{N} a_i \underbrace{\psi_j(i)\,\psi_j(i+1)}_{\text{weight}}\right) \xrightarrow[N\to\infty]{} \rho_J(w)$$



- collision rate ansatz $\quad \rho_J = \rho_p\, v^{eff}$

$$v^{eff}(w) = w + \int_w^{\infty} dw' \log|w-w'|\, \rho_p(w')\left(v^{eff}(w') - v^{eff}(w)\right) - \int_{-\infty}^{w} dw' \log|w-w'|\, \underbrace{\rho_p(w')\left(v^{eff}(w) - v^{eff}(w')\right)}_{\text{rate} \;\geq 0}$$

bare

2-particle scattering shift

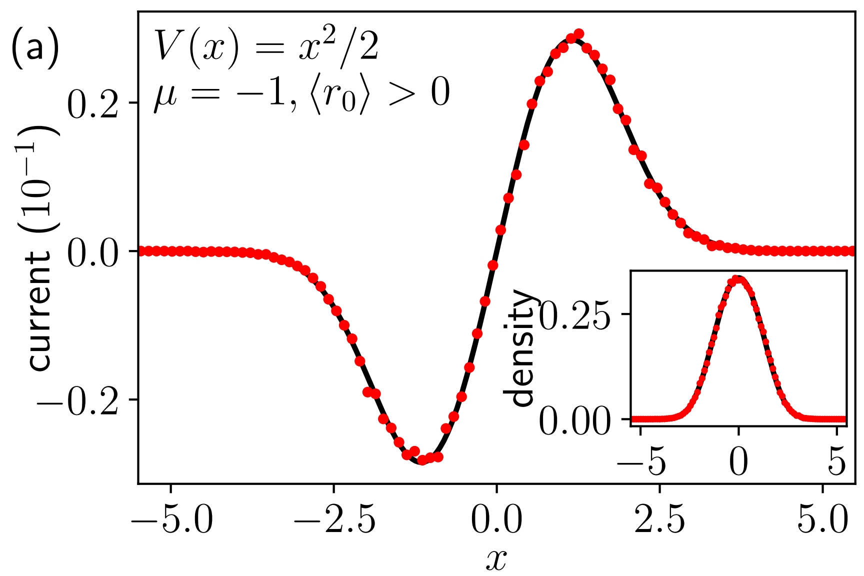
GGE

$\Rightarrow$

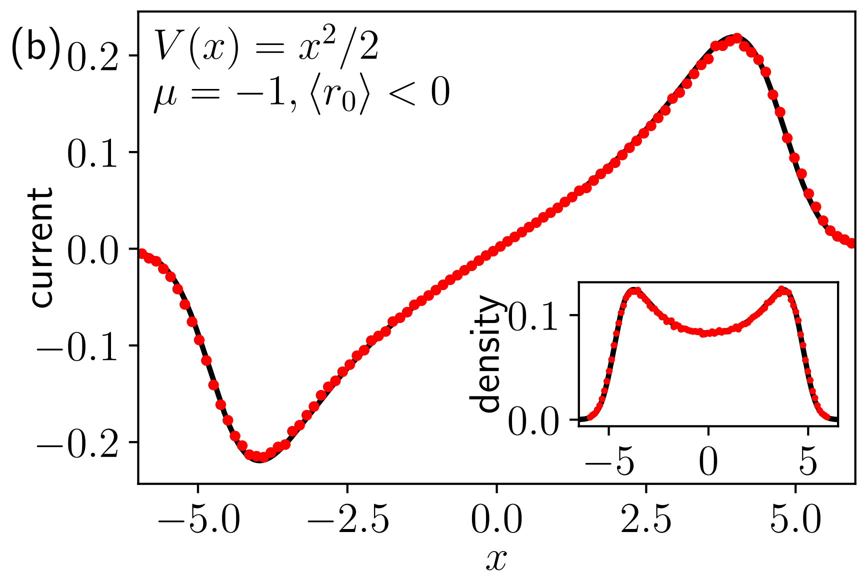
$$\left\| \; v^{eff}(w) = w + \int dw' \log|w-w'|\, \rho_p(w')\left(v^{eff}(w') - v^{eff}(w)\right) \; \right\|$$

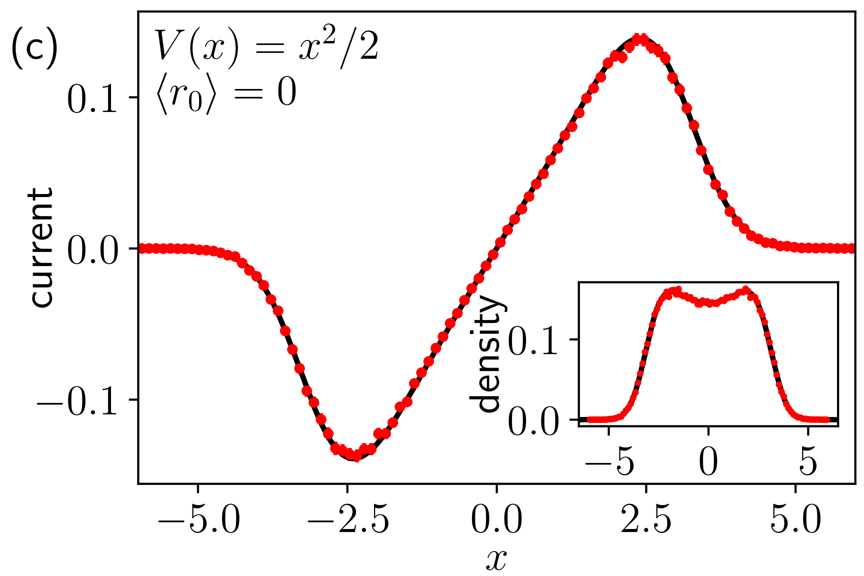
compare KdV

Ken McL

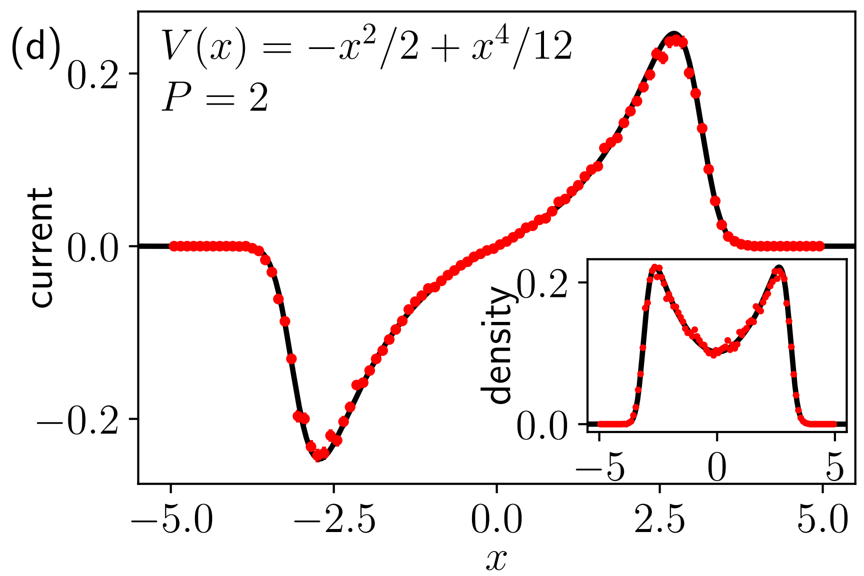


(a)
$V(x) = x^2/2$
$\mu = -1, \langle r_0 \rangle > 0$
current (10^{-1})
0.2
0.0
−0.2
−5.0
−2.5
0.0
2.5
5.0
x
density
0.25
0.00
−5
0
5


(b)
$V(x) = x^2/2$
$\mu = -1, \langle r_0 \rangle < 0$
current
0.2
0.1
0.0
−0.1
−0.2
−5.0
−2.5
0.0
2.5
5.0
x
density
0.1
0.0
−5
0
5



(c)
$V(x) = x^2/2$
$\langle r_0 \rangle = 0$
current
0.1
0.0
−0.1
−5.0
−2.5
0.0
2.5
5.0
x
density
0.1
0.0
−5
0
5



(d)
$V(x) = -x^2/2 + x^4/12$
$P = 2$
current
0.2
0.0
−0.2
−5.0
−2.5
0.0
2.5
5.0
x
density
0.2
0.0
−5
0
5

charge - current correlator

n-th charge $Q_j^{[n]} = (L^n)_{jj}$

- infinite system

$$D_{m,n} = \sum_{j \in \mathbb{Z}} \mathbb{E}_{P,V}\left(J_j^{[m]} Q_0^{[n]} \right)^c$$

$\Rightarrow D_{m,n} = D_{n,m}$ $\Leftarrow$ uses only conservation law

$$\partial_P \mathbb{E}_{P,V}\left(J_0^{[n]} \right) = D_{n,0} = D_{0,n} \underset{\Uparrow}{=} \sum_{j \in \mathbb{Z}} \mathbb{E}_{P,V}\left(Q_j^{[1]} Q_0^{[n]} \right)^c$$

$$\underbrace{J_j^{[0]} = Q_j^{[1]}}$$

"conserved current"

valid in general

except spin $\frac{1}{2}$ Fermi-Hubbard

fluctuations of eigenvalues

Dyson Brownian motion

optimal transport

Lambert (2021)

Yoshimura, HS (2020)

$$\partial_P \Big(\mathbb{E}_{P,V}\big(\zeta_0^{[n]}\big) - P \langle \varsigma_1, C^\# \varsigma_n \rangle \Big) = 0$$

$\varsigma_m(w) = w^m$

$C^\#$ covariance operator

$$\begin{array}{lcc} & \parallel & \parallel \\ P \to 0 & 0 & 0 \end{array}$$

$$\mathbb{E}_{P,V}\big(\zeta_0^{[0]}\big) = P \langle \varsigma_1, C^\# \varsigma_n \rangle$$

- dressing transformation

$$T f(w) = \int dw' \log|w-w'| \, f(w')$$

$$f^{dr} = \frac{1}{1 - T\rho_\mu} f$$

multiplication

$$\rho_\mu = \frac{\rho_p}{1 + T\rho_p}$$

$$\rightsquigarrow \quad v^{eff} = \frac{(\varsigma_1)^{dr}}{(\varsigma_0)^{dr}}$$

$\Rightarrow$ v^{eff} is solution of collision rate ansatz $\Leftarrow$

hydrodynamic equations $\quad \nu(x,t)$, $\rho_p(x,t;w)$, $q_1 = \mathbb{E}_{\mathfrak{P},V}(Q_0^{[1]})$

local stretch DOS

$$\partial_t \nu - \partial_x q_1 = 0$$

$$\partial_t(\nu \rho_p) + \partial_x(\rho_p(v^{\text{eff}} - q_1)) = 0$$

- normal form $\quad \partial_t \rho_\mu + \underbrace{\frac{1}{\nu}(v^{\text{eff}} - q_1)}_{\text{diagonal}} \partial_x \rho_\mu = 0$

(continuous spectrum)

numerics iFluid

future

- Toda : → $P, V \Leftrightarrow \nu, \rho_p$ (so far $V(w) \cong w^{2n}, |w| \to \infty$)

  → mixing of GGE (exponential ?)

→ GHD : solution theory (smooth solutions, no shocks ?)

→ dynamics (see MSRI program 2031)

- other integrable many-body systems

  ⇒ Ablowitz - Ladik discrete NLS Grava, Mazzuci (2021), H.S. (2021)

  discrete modified KdV H.S. (2021)

  compact unitary Lax

// Thank You for Listening //