Quantitative Tracy-Widom law for Wigner matrices

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Definition generalized Wigner matrix

Real symmetric ($\beta = 1$) Wigner matrix $H_N = \frac{X_N}{\sqrt{N}}$, $X_N = (x_{ij})_{i,j=1}^N$:

1. $\{x_{ij}, x_{ii} | i < j\}$ are independent centered random variables and $x_{ij} = x_{ji}$.
2. $$\mathbb{E}[|x_{ij}|^2] = s_{ij}^{(2)}\,,$$
   with $C^{-1} \leq s_{ij}^{(2)} \leq C$ and $\sum_i s_{ij}^{(2)} = N + 1$.
3. $$\max_{i<j} \left\{ \mathbb{E}[|x_{ij}|^k], \mathbb{E}[|x_{ii}|^k] \right\} < C_k, \qquad k \geq 3$$

Complex Hermitian ($\beta = 2$): also $\mathbb{E}[(x_{ij})^2] = 0$.
Special cases: Gaussian orthogonal ensemble (GOE, $\beta = 1$):

$$x_{ij} \sim \mathcal{N}_{\mathbb{R}}(0,1), \qquad x_{ii} \sim \mathcal{N}_{\mathbb{R}}(0,2).$$

Gaussian unitary ensemble (GUE, $\beta = 2$):

$$x_{ij} \sim \mathcal{N}_{\mathbb{C}}(0,1), \qquad x_{ii} \sim \mathcal{N}_{\mathbb{R}}(0,1).$$

Explicit formula for joint eigenvalue distribution.

Wigner's Semicircle law

Eigenvalues of H_N:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N.$$

Empirical spectral distribution (ESD):

$$\mathrm{d}\mu_N(x) := \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}(x)\mathrm{d}x \Longrightarrow \mathrm{d}\mu_{sc}(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]}\mathrm{d}x.$$

Semicircle law is universal.

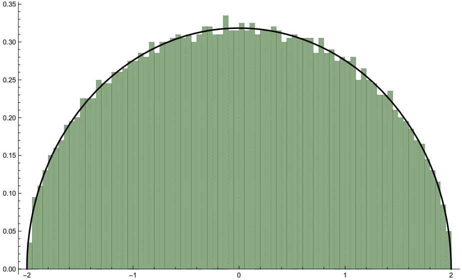

Figure: Histogram of the eigenvalues of a $N = 2000$ GOE

Asymptotics of largest eigenvalue [Bai-Yin '88]:

$$\text{fourth moment exists:} \qquad \lambda_N \to 2, \qquad \text{almost surely.}$$

Typical eigenvalue spacing $N^{-2/3}$ near edge.

$$\text{normalized fluctuations:} \qquad N^{2/3}(\lambda_N - 2).$$

Tracy-Widom Law ['94, '95]:

$$\lim_{N\to\infty} \mathbb{P}^{\mathrm{GUE}}\Big(N^{2/3}(\lambda_N - 2) < r\Big)$$
$$= \mathrm{TW}_2(r) := \exp\Big(-\int_r^\infty (x-r)^2 \big(q(x)\big)^2 \mathrm{d}x\Big).$$

where $q(x)$ is the solution of a Painlevé II equation. GOE: $\mathrm{TW}_1(r)$.

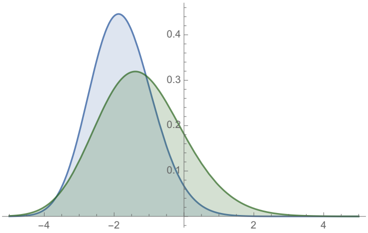

Figure: Tracy-Widom laws

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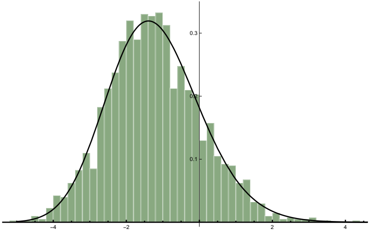

Figure: Histogram of 2000 samples $N = 1000$ GOE

Special case: Gaussian ensemble

Joint eigenvalue distribution:

$$p_{N,\beta}(\lambda_1,\cdots,\lambda_N)=\frac{1}{Z_N}\prod_{i<j}|\lambda_i-\lambda_j|^\beta \mathrm{e}^{-\frac{\beta}{4}N\sum_{i=1}^N\lambda_i^2}.$$

GUE ($\beta=2$): determinantal point process with k-th point correlation

$$p_{N,2}^{(k)}(\lambda_1,\cdots,\lambda_k)=\det[K_N(\lambda_i,\lambda_j)]_{i,j=1}^k,$$

where K_N is expressed in terms of Hermite orthogonal polynomials.

$$K_N(x,y)=\sqrt{N}\sum_{k=0}^{N-1}\phi_k(\sqrt{N}x)\phi_k(\sqrt{N}y)e^{-\frac{N(x^2+y^2)}{4}}.$$

Normalization at the edge:

$$K_N^{\mathrm{edge}}(x,y):=\frac{1}{N^{2/3}}K_N\left(2+\frac{x}{N^{2/3}},2+\frac{y}{N^{2/3}}\right)\Longrightarrow K_{\mathrm{Airy}}(x,y),$$

where

$$K_{\mathrm{Airy}}(x,y)=\frac{\mathrm{Ai}(x)\mathrm{Ai}'(y)-\mathrm{Ai}'(x)\mathrm{Ai}(y)}{x-y}.$$

Tracy-Widom law for Wigner matrix

Question: Is Tracy-Widom law universal?

Edge universality: $$\lim_{N\to\infty} \mathbb{P}\Big(N^{2/3}(\lambda_N - 2) < r\Big) = \mathrm{TW}_\beta(r).$$

Some references:
[Soshnikov '99] (distributions are symmetric and sub-Gaussian decaying);
[Péché-Soshnikov '07, '08] (partially remove symmetric condition);
[Tao-Vu '10, '11] (three moments matching);
[Erdős-Yau-Yin '12](removed the vanishing third moment condition);
[Lee-Yin '14](necessary and sufficient condition, heavy tail).
....
Related models: generalized Wigner matrices, sample covariance matrices, random band matrices, adjacency matrices of random graphs,...

Quantitative Tracy-Widom law

GUE ($\beta = 2$): Determinantal point process. with k-th point correlation

$$p_{N,2}^{(k)}(\lambda_1, \cdots, \lambda_k) = \det[K_N(\lambda_i, \lambda_j)]_{i,j=1}^k,$$

where K_N is expressed in terms of Hermite functions.

Strong uniform convergence estimates [Deift-Gioev '12]

$$\left|K_N^{\mathrm{edge}}(x,y) - K^{\mathrm{Airy}}(x,y)\right| \leq CN^{-2/3}\mathrm{e}^{-cx}\mathrm{e}^{-cy}\,,$$

for $x, y \geq r_0$, some fixed $r_0 \in \mathbb{R}$.

Berry-Esseen estimate [Johnstone-Ma '12]:

$$\sup_{r > r_0}\left|\mathbb{P}^{\mathrm{GUE}}\Big(N^{2/3}(\lambda_N - 2) < r\Big) - \mathrm{TW}_2(r)\right| \leq CN^{-2/3}.$$

GOE($\beta = 1$): slightly different centering and scaling parameters.

$$(N-1)^{1/6}\sqrt{N}\Big(\lambda_N - \sqrt{4 - \frac{2}{N}}\Big)\,.$$

Generalized Wigner matrix [Bourgade '18]:

$$\sup_{r>r_0}\left|\mathbb{P}\Big(N^{2/3}(\lambda_N-2)<r\Big)-\mathrm{TW}_\beta(r)\right|\le N^{-\frac{2}{9}+\omega}.$$

Matrix Ornstein-Uhlenbeck process (interpolation):

$$\mathrm{d}h_{ij}=\frac{1}{\sqrt{N}}\mathrm{d}\beta_{ij}-\frac{1}{2}h_{ij}\mathrm{d}t,\qquad H(0)=\text{Wigner}.$$

Dyson Brownian motion (DBM): $\lambda_1(t)\le\cdots\le\lambda_N(t)$.

$$\mathrm{d}\lambda_i=\frac{1}{\sqrt{N}}\mathrm{d}\beta_i+\Big(-\frac{\lambda_i}{2}+\frac{1}{N}\sum_{i\neq j}\frac{1}{\lambda_i-\lambda_j}\Big)\mathrm{d}t.$$

Part 1: local relaxation estimate (using coupling of Wigner and Gaussian).

$$|\lambda_i(t)-\mu_i(t)|=O(\frac{1}{Nt})\Longrightarrow N^{2/3}|\lambda_N(t)-\mu_N(t)|=O(\frac{1}{N^{1/3}t}).$$

Part 2: remove small t (Green function comparison for short time).

$$N^{2/3}\big|\lambda_N(t)-\lambda_N(0)\big|=O(\frac{t}{N\eta}+\frac{1}{N^2\eta^2}+N^{2/3}\eta+N^{-1}).$$

Optimization: $t=N^{-1/9},\eta=N^{-8/9}$ (see later for η).

Quantitative Tracy-Widom law for Wigner matrix

Theorem (Schnelli-Xu '21)

$$\sup_{r>r_0} \left| \mathbb{P}\Big(N^{2/3}(\lambda_N - 2) < r \Big) - \mathrm{TW}_\beta(r) \right| \leq N^{-\frac{1}{3}+\omega}.$$

1. Not rely on the DBM relaxation (part 1), but Green function comparison for a long time (part 2).
2. Speed of convergence depends on fourth moment of off-diagonal matrix entries and second moment of diagonal entries.
3. Open question: Can rate be improved to $N^{-2/3}$ for all Wigner matrices?

Green function comparison method

Local law for Green function

Definition: **Stieltjes transform** of a probability measure ν on $\mathbb{R}$,

$$m_\nu : \mathbb{C}^+ \to \mathbb{C}^+, \qquad m_\nu(z) = \int_{\mathbb{R}} \frac{\mathrm{d}\nu(x)}{x - z}.$$

Stieltjes transform of the semicircle law satisfies $m_{sc}^2(z) + z m_{sc}(z) + 1 = 0$.

$$\lim_{\eta \searrow 0} \frac{1}{\pi} \mathrm{Im}\, m_{sc}(E + \mathrm{i}\eta) = \rho_{sc}(E)\,.$$

Stieltjes transform of ESD of H_N, $\mu_N(x) = \frac{1}{N}\sum_{j=1}^N \delta_{\lambda_j}$

$$\boxed{m_N(z) = \frac{1}{N}\sum_{j=1}^N \frac{1}{\lambda_j - z} = \frac{1}{N}\mathrm{Tr}\, G(z), \qquad G(z) := \frac{1}{H_N - z},}$$

where G is called the **Green function** or resolvent of H.

Theorem (Erdős-Yau-Yin '12)

For any $z \in S := \{E + \mathrm{i}\eta : |E| \le 5, N^{-1+\epsilon} \le \eta \le 10\}$,

Entrywise local law: $$\max_{i,j} |G_{ij}(z) - \delta_{ij} m_{sc}(z)| \prec \sqrt{\frac{\operatorname{Im} m_{sc}(z)}{N\eta}} + \frac{1}{N\eta}.$$

Averaged local law: $$|m_N(z) - m_{sc}(z)| \prec \frac{1}{N\eta}.$$

Definition For $\mathcal{X}, \mathcal{Y} \ge 0$, write $\mathcal{X} \prec \mathcal{Y}$ or $\mathcal{X} = O_\prec(\mathcal{Y})$ if

$$\mathbb{P}\big(\mathcal{X} \le N^\tau \mathcal{Y}\big) \ge 1 - N^{-D}, \qquad \forall \tau, D > 0.$$

Local law $\Longrightarrow$ rigidity of eigenvalues: $|\lambda_N - 2| \prec N^{-2/3}$.

Link largest eigenvalue to Green function

Distribution of largest eigenvalue:

$$(*) := \mathbb{P}\big(N^{2/3}(\lambda_N - 2) < r\big) = \mathbb{P}\big(\lambda_N < 2 + N^{-2/3} r\big), \quad r \geq r_0.$$

Set $E := 2 + N^{-2/3} r,\ E_L := 2 + N^{-2/3+\epsilon}$.

$$(*) = \mathbb{P}\Big(\mathcal{N}(E, \infty) = 0\Big) = \mathbb{P}\Big(\mathcal{N}(E, E_L) = 0\Big) + O(N^{-D}).$$

Note that

$$\mathcal{N}(E, E_L) = \mathrm{Tr} \mathbb{1}_{[E, E_L]}(H) \approx \mathrm{Tr} \mathbb{1}_{[E, E_L]} \star \theta_\eta(H)$$

$$\boxed{\theta_\eta(x) := \frac{\eta}{\pi(x^2 + \eta^2)},\ \eta \ll N^{-\frac{2}{3}}} \qquad = \frac{1}{\pi} \int_E^{E_L} \mathrm{Im}\, \mathrm{Tr}\, G(y + \mathrm{i}\eta) \mathrm{d}y.$$

Choose a cut-off function/smooth indicator function

$$F(x) := \begin{cases} 1, & |x| \leq 1/9, \\ 0, & |x| \geq 2/9, \\ \text{smooth decaying}, & 1/9 < |x| < 2/9. \end{cases}$$

Using that $\mathcal{N}(E, E_L)$ is integer valued, one shows that

$$\mathbb{E}\Big[F\big(\int_{E_-}^{E_L} \mathrm{Im}\,\mathrm{Tr}\,G(y+\mathrm{i}\eta)\mathrm{d}y\big)\Big] \lesssim (*) \lesssim \mathbb{E}\Big[F\big(\int_{E_+}^{E_L} \mathrm{Im}\,\mathrm{Tr}\,G(y+\mathrm{i}\eta)\mathrm{d}y\big)\Big],$$

with $E_- = E - N^{\epsilon'}\eta,\ E_+ = E + N^{\epsilon'}\eta$.
Roughly speaking,

$$\boxed{\mathbb{P}\big(N^{\frac{2}{3}}(\lambda_N - 2) < r\big) \approx \mathbb{E}\Big[F\Big(N\int_{E_1}^{E_2} \mathrm{Im}\, m_N(y+\mathrm{i}\eta)\mathrm{d}y\Big)\Big] + O_{\prec}(N^{\frac{2}{3}}\eta),}$$

where $E_1, E_2 \in [2 - C_1 N^{-\frac{2}{3}}, 2 + C_2 N^{-\frac{2}{3}+\epsilon}]$, and $\eta \ll N^{-\frac{2}{3}}$.

Choosing $\eta = N^{-2/3-\epsilon}$, $\{|G_{ij} - m_{sc}\delta_{ij}|, |m_N - m_{sc}|\} \prec \frac{1}{N\eta} = N^{-1/3+\epsilon}$.

Theorem (Erdős-Yau-Yin '12)

$$\Big|\Big(\mathbb{E} - \mathbb{E}^{\mathrm{G}\beta\mathrm{E}}\Big)\Big[F\Big(N\int_{E_1}^{E_2} \mathrm{Im}\, m_N(y+\mathrm{i}\eta)\mathrm{d}y\Big)\Big]\Big| \le N^{-1/6+c\epsilon}, \quad \eta = N^{-2/3-\epsilon}$$

Green function comparison theorem (GFCT)

For quantitative TW law, we choose

$$\eta = N^{-1+\epsilon} \Longrightarrow \{|G_{ij} - m_{sc}\delta_{ij}|, |m_N - m_{sc}|\} \prec \frac{1}{N\eta} = N^{-\epsilon}.$$

Theorem (Schnelli-Xu '21)

$$\left|\left(\mathbb{E} - \mathbb{E}^{\mathrm{G}\beta\mathrm{E}}\right)\left[F\left(N\int_{E_1}^{E_2} \mathrm{Im}\, m_N(y + \mathrm{i}\eta)\mathrm{d}y\right)\right]\right| \prec N^{-1/3}, \quad \eta = N^{-1+\epsilon}.$$

Leads to quantitative Tracy-Widom law with rate nearly $N^{-1/3}$.

Toy model: Estimate $\mathbb{E}[\operatorname{Im} m_N(z)]$

Proposition

Define $S_{\text{edge}} := \{E + \mathrm{i}\eta : E \in [2 - C_1 N^{-\frac{2}{3}}, 2 + C_2 N^{-\frac{2}{3}+\epsilon}], \eta = N^{-1+\epsilon}\}$,

$$\left|\left(\mathbb{E} - \mathbb{E}^{\mathrm{G}\beta\mathrm{E}}\right)\left[\operatorname{Im} m_N(z)\right]\right| \prec N^{-1/3-\epsilon}, \qquad z \in S_{\text{edge}}.$$

Estimate following from **convergence estimates of correlation kernels and boundedness of Airy kernel:**

$$\boxed{\mathbb{E}^{\mathrm{G}\beta\mathrm{E}}[\operatorname{Im} m_N(z)] = O(N^{-\frac{1}{3}}) \sim \operatorname{Im} m_{sc}(z), \qquad z \in S_{\text{edge}}.}$$

GFCT yields a non-trivial estimate:

$$\mathbb{E}[\operatorname{Im} m_N(z)] = O(N^{-\frac{1}{3}}) \sim \operatorname{Im} m_{sc}.$$

Cannot be implied by the local law

$$|\operatorname{Im} m_N(z) - \operatorname{Im} m_{sc}(z)| \approx |\operatorname{Im} m_N(z)| \prec \frac{1}{N\eta} = N^{-\epsilon}.$$

Matrix Ornstein-Uhlenbeck process ($\beta = 1$):

$$\mathrm{d}h_{ab}(t) = \sqrt{\frac{1+\delta_{ab}}{N}}\mathrm{d}\beta_{ab} - \frac{1}{2}h_{ab}\mathrm{d}t, \qquad h_{ab}(0) = (H_N)_{ab}\,.$$

In distribution this is equivalent to writing

$$H(t) = \mathrm{e}^{-\frac{t}{2}} H_N + \sqrt{1-\mathrm{e}^{-t}}\mathrm{GOE}\,, \qquad t \in \mathbb{R}_+.$$

Define the time dependent Green function:

$$G(t,z) := (H(t)-z)^{-1}, \qquad m_N(t,z) := \frac{1}{N}\mathrm{Tr}\,G(t,z).$$

Local law for any $t \in \mathbb{R}_+$ and $z \in S_{\mathrm{edge}}$:

$$\boxed{\max_{i \neq j} \left\{ |G_{ij}(t,z)|, |G_{ii}(t,z)+1|, |m_N(t,z)+1| \right\} \prec \Psi := \frac{1}{N\eta} = N^{-\epsilon}.}$$

Here we used that $m_{sc}(2) = -1$.

Using Ito's formula,

$$\mathrm{d}(m_N(t,z)) = \sum_{a,b}(\cdots)\mathrm{d}\beta_{ab} + \frac{1}{2N}\sum_{v,a,b} \mathbf{h_{ab}} G_{va}G_{bv}\mathrm{d}t \Longrightarrow D_1$$

$$+ \frac{2}{N^2}\sum_{v,a,b}\left(G_{va}G_{ab}G_{bv} + G_{vb}G_{bv}G_{aa}\right)\mathrm{d}t \Longrightarrow D_2.$$

Assume for simplicity: $\mathbb{E}h_{ab}^2 = \frac{1+\delta_{ab}}{N}$.

Expectation of the first drift term: $\mathbb{E}[D_1] = \frac{1}{2N}\sum_{v,a,b}\mathbb{E}\Big[h_{ab}G_{va}G_{bv}\Big]$.

k-th cumulant: $s^{(k)}(h) := (-\mathrm{i})^k\left(\frac{\mathrm{d}^k}{\mathrm{d}t^k}\log\mathbb{E}e^{\mathrm{i}th}\right)\Big|_{t=0}$.

$$\mathbb{E}\big[hf(h)\big] = \sum_{k+1=1}^{l}\frac{1}{k!}s^{(k+1)}(h)\mathbb{E}[f^{(k)}(h)] + R_{l+1}.$$

If h is Gaussian, then $s^{(k)}(h) = 0, \forall k \geq 3$ (Stein's lemma).

Cumulant expansions in RMT: [Khorunzhy-Khoruzhenko-Pastur '96], [Boutet de Monvel - Khorunzhy '99], [Lytova-Pastur '09], [He-Knowles '17], ...

Truncate cumulant expansion at fourth order ($s_{ab}^{(k+1)}$ normalized)

$$\mathbb{E}[D_1] = \frac{1}{2N} \sum_{a,b,v} \Big(\sum_{k+1=2}^{4} \frac{s_{ab}^{(k+1)}(t)}{k! N^{\frac{k+1}{2}}} \mathbb{E}\Big[\frac{\partial^k (G_{va} G_{bv})}{\partial h_{ab}^k} \Big] \Big) + O_{\prec}(\frac{1}{\sqrt{N}}).$$

$$\boxed{\text{Rules:} \qquad \frac{\partial G_{ij}}{\partial h_{ba}} = -\frac{G_{ib} G_{aj} + G_{ia} G_{bj}}{1+\delta_{ab}}, \quad G_{ij} = G_{ji}.}$$

Exact cancellation of second order:

$$\frac{1}{2N^2} \sum_{v,a,b} s_{ab}^{(2)}(t) \mathbb{E}\Big[\frac{\partial (G_{va} G_{bv})}{\partial h_{ab}} \Big] = -\mathbb{E}[D_2] \qquad (s_{ab}^{(2)}(t) \equiv 1).$$

Adding up $\mathbb{E}[D_1] + \mathbb{E}[D_2]$,

$$\frac{\mathrm{d}\mathbb{E}[m_N(t,z)]}{\mathrm{d}t} = 0 \cdot K_2 + K_3 + K_4 + O_{\prec}(\frac{1}{\sqrt{N}}),$$

where

$$K_3 := \frac{\sqrt{N}}{N^3} \sum_{v,a,b} s_{ab}^{(3)} \mathbb{E}\Big[\frac{\partial^2 (G_{va} G_{bv})}{\partial h_{ab}^2} \Big], \quad K_4 := \frac{1}{N^3} \sum_{v,a,b} s_{ab}^{(4)} \mathbb{E}\Big[\frac{\partial^3 (G_{va} G_{bv})}{\partial h_{ab}^3} \Big].$$

Examples from third and fourth order:

$$\sqrt{N}\frac{1}{N^3}\sum_{v,a,b} s^{(3)}_{ab}\mathbb{E}\Big[G_{va}G_{bv}G_{aa}G_{bb}+G_{va}G_{av}G_{ab}G_{bb}+\cdots\Big];$$

$$\frac{1}{N^3}\sum_{v,a,b} s^{(4)}_{ab}\mathbb{E}\Big[G_{va}G_{av}G_{aa}(G_{bb})^2+G_{va}G_{ab}G_{bv}G_{aa}G_{bb}+\cdots\Big].$$

Abstract form of averaged product of Green function entries:

$$\mathcal{Q}_d \ni Q_d(t,z) := \frac{1}{N^m}\sum_{v_1=1}^{N}\cdots\sum_{v_m=1}^{N} c_{v_1,\ldots,v_m}\Big(\prod_{i=1}^{n} G_{x_i y_i}(t,z)\Big),$$

Degree $d := \#\{i : x_i \neq y_i\}, \quad x_i, y_i \in \{v_1,\cdots,v_m\} =: \mathcal{I},$

Naive bound: $|G_{ij}-m_{sc}\delta_{ij}| \prec \Psi = N^{-\epsilon} \Longrightarrow |Q_d(t,z)| \prec \Psi^d + N^{-1}.$

$$\mathfrak{n}(v_j) := \#\{i:\ x_i = v_j\} + \#\{i:\ y_i = v_j\},\ \forall v_j \in \mathcal{I}.$$

Definition Q_d is **unmatched** (denoted by Q_d^o), if exists $v_j \in \mathcal{I}$ s.t. $\mathfrak{n}(v_j)$ is odd. Otherwise, Q_d is **matched**.
Third order terms from K_3 are unmatched, since $\mathfrak{n}(a) = \mathfrak{n}(b) = 3$.

Lemma (size of unmatched terms)

$$|\mathbb{E}[Q_d^o(t,z)]| \prec N^{-1}, \qquad t \in \mathbb{R}_+, z \in S_{\mathrm{edge}}$$

1. True for spectral parameter z in a broader domain S.
2. True with distinct spectral parameters z_i in the product of Green function entries.
3. GOE: eigenbasis Haar distributed in $O(N)$ independent of eigenvalues:

$$\mathbb{E}[G_{ab}] = \mathbb{E}\Big[\sum_{j=1}^{N} \frac{O_{aj}O_{bj}}{\lambda_j - z}\Big] = \sum_{j=1}^{N} \mathbb{E}\Big[\frac{1}{\lambda_j - z}\Big] \times \mathbb{E}[O_{aj}O_{bj}] = 0. \quad (a \neq b)\,.$$

In general Weingarten calculus.

Expansion mechanism I

Example: $Q_2 := \frac{1}{N^2} \sum_{a,b} \mathbf{G_{aa}} G_{ab} G_{ba}, \quad \mathfrak{n}(a) = 4, \mathfrak{n}(b) = 2.$

Using $zG_{ij} = (HG)_{ij} - \delta_{ij}$ and cumulant expansion formula,

$$2\mathbb{E}[Q_2] \approx z\mathbb{E}[Q_2] = -\frac{1}{N^2} \sum_{a,b} \mathbb{E}[\mathbf{1} G_{ab} G_{ba}] + \frac{1}{N^2} \sum_{a,b,k} \mathbb{E}[\mathbf{h_{ak}} \mathbf{G_{ka}} G_{ab} G_{ba}]$$

$$= -\frac{1}{N^2} \sum_{a,b} \mathbb{E}[G_{ab} G_{ba}] + \frac{1}{N^3} \sum_{a,b,k} \mathbb{E}[\frac{\partial G_{ka} G_{ab} G_{ba}}{\partial h_{ak}}] + \cdots$$

Rules: $\frac{\partial G_{ij}}{\partial h_{ba}} = -\frac{G_{ib} G_{aj} + G_{ia} G_{bj}}{1 + \delta_{ab}}, \quad G_{ij} = G_{ji}.$

Second order (new summation index k which is matched):

$$\frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[\frac{\partial G_{ka}G_{ab}G_{ba}}{\partial h_{ak}}], \qquad \mathfrak{n}(k)=2, \mathfrak{n}(a)=4, \mathfrak{n}(b)=2.$$

2nd. $G_{ka}\frac{\partial G_{ab}}{\partial h_{ak}}G_{ba} = -G_{ka}\big(G_{aa}G_{kb}+G_{ak}G_{ab}\big)G_{ba}$: degree $d\geq 3$;

3rd. $G_{ka}G_{ab}\frac{\partial G_{ba}}{\partial h_{ak}}$: $d\geq 3$.

1st. $\frac{\partial G_{ka}}{\partial h_{ak}}G_{ab}G_{ba} = -G_{ka}G_{ka}G_{ab}G_{ba} - G_{kk}G_{aa}G_{ab}G_{ba}$.

Leading term of degree $d=2$

$$-\frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[G_{kk}G_{aa}G_{ab}G_{ba}] = \frac{1}{N^2}\sum_{a,b}\mathbb{E}[G_{aa}G_{ab}G_{ba}] \Longleftarrow \mathbb{E}[Q_2]$$

$$-\frac{1}{N^2}\sum_{a,b}\mathbb{E}[(m_N+1)G_{aa}G_{ab}G_{ba}] \Longrightarrow \Psi^3.$$

$$\frac{1}{N^2}\sum_{a,b}\mathbb{E}[\mathbf{G_{aa}}G_{ab}G_{ba}] \approx -\frac{1}{N^2}\sum_{a,b}\mathbb{E}[\mathbf{1}G_{ab}G_{ba}]$$
$$-\frac{1}{N^2}\sum_{a,b}\mathbb{E}[G_{aa}G_{ab}G_{ba}(m_N+1)] + \sum_{Q_{d'}\in\mathcal{Q}_{d'},d'\geq 3}\mathbb{E}[Q_{d'}] + \cdots$$

New abstract form:

$$Q_d := \frac{1}{N^{\#\mathcal{I}}}\sum_{\mathcal{I}} c_{\mathcal{I}}\Big(\prod_{i=1}^{n_1} G_{x_i y_i}\Big)(m_N+1)^{n_2}, \quad d := \#\{x_i \neq y_i\} + n_2.$$

In general, choosing $G_{x_1 y_1} = G_{aa}$,

$$\mathbb{E}[Q_d] = \mathbb{E}\Big[Q_d\big(G_{x_1 y_1} \to -1\big)\Big] + \sum_{Q_{d'}\in\mathcal{Q}_{d'},d'\geq d+1}\mathbb{E}[Q_{d'}] + \cdots$$

Expansion mechanism II

Example: $Q_3 := \frac{1}{N^2}\sum_{a,b}\mathbb{E}[\mathbf{G_{ab}}G_{ab}G_{ba}], \quad \mathfrak{n}(a)=\mathfrak{n}(b)=3.$

$$2\mathbb{E}[Q_3] \approx z\mathbb{E}[Q_3] = -\frac{1}{N^2}\sum_{a,b}\mathbb{E}[\delta_{ab}G_{ab}G_{ba}] + \frac{1}{N^2}\sum_{a,b,k}\mathbb{E}[\mathbf{h_{ak}}\mathbf{G_{kb}}G_{ab}G_{ba}]$$

$$=\frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[\frac{\partial G_{kb}G_{ab}G_{ba}}{\partial h_{ak}}] + O_\prec(N^{-1}) + \cdots$$

Leading terms: 1. $\partial G_{kb}/\partial h_{ak}$:

$$-\mathbb{E}[m_N Q_3] = \mathbb{E}[Q_3] - \mathbb{E}[(m_N+1)Q_3].$$

2. $\partial G_{ab}/\partial h_{ak}$ and $\partial G_{ba}/\partial h_{ak}$:

$$-\frac{1}{N^3}\sum_{a,b,k}G_{kb}G_{kb}G_{ba}G_{aa}, \qquad -\frac{1}{N^3}\sum_{a,b,k}G_{kb}G_{ab}G_{bk}G_{aa}$$

Combining with Expansion mechanism I,

$$\mathbb{E}[Q_3] = \frac{1}{N^2}\sum_{a,b}\mathbb{E}[\mathbf{G_{ab}} G_{ab} G_{ba}] = \frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[G_{kb} G_{kb} G_{ba}] + \frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[G_{kb} G_{ab} G_{bk}] - \mathbb{E}[(m_N+1)Q_3] + \cdots .$$

In general, choosing $G_{x_1 y_1} = G_{a y_1}$ with $y_1 \neq a$,

$$\mathbb{E}[Q_d] = \sum_{\substack{2\le i\le n_1 \\ x_i=a, y_i\neq a}} \mathbb{E}\Big[Q_d\big(x_1, x_i \to k\big)\Big] + \sum_{\substack{2\le i\le n_1 \\ x_i\neq a, y_i=a}} \mathbb{E}\Big[Q_d\big(x_1, y_i \to k\big)\Big] + \sum_{Q_{d'}\in\mathcal{Q}_{d'}, d'\ge d+1} \mathbb{E}[Q_{d'}] + \cdots .$$

Observation: (expansion not unique, but okay)

1. # index a in these leading terms (if exist) is $\mathfrak{n}'(a) = \mathfrak{n}(a) - 2$.
2. # leading terms of degree $d = \mathfrak{n}(a) - 1$. (if $\mathfrak{n}(a) = 1$, better bound)

Proof of Lemma

$$\text{Example:} \quad \frac{1}{N^2}\sum_{a,b}\mathbb{E}[\mathbf{G_{ab}}G_{ab}G_{ba}] \prec \Psi^3 + N^{-1} \Longrightarrow \Psi^4 + N^{-1}.$$

Using expansion mechanism II,

$$\frac{1}{N^2}\sum_{a,b}\mathbb{E}[\mathbf{G_{ab}}G_{ab}G_{ba}] = \frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[G_{kb}G_{kb}G_{ba}] \Longrightarrow \mathfrak{n}(a)=1, \text{unmatched}$$

$$+\frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[G_{kb}G_{ab}G_{bk}] + \sum Q^o_{d\geq 4}.$$

Using expansion mechanism II again on the unmatched index a with $\mathfrak{n}(a)=1$

$$\frac{1}{N^3}\sum_{a,b,k}\mathbb{E}[G_{kb}G_{kb}G_{ba}] = \sum Q^o_{d\geq 4} \Longrightarrow \Psi^4 + O(N^{-1}).$$

In general, iteratively use Expansion mechanism II for $\frac{\mathfrak{n}(a)+1}{2}$ times

$$Q^o_{d_0} = \sum Q^o_{d_1\geq d_0+1} = \sum Q^o_{d_2\geq d_0+2}\cdots = \sum Q^o_{d\geq D} = \Psi^D + O(N^{-1}).$$

Fourth order term

Example: $$\frac{1}{N^2}\sum_{a,b}\mathbb{E}[G_{ab}G_{ab}G_{aa}G_{bb}], \qquad \mathfrak{n}(a)=\mathfrak{n}(b)=4.$$

Step 1. Reduction to *trace-like quantities* (neglecting factors m_N+1):

$$\widetilde{Q}_d=\mathbb{E}\Big[\frac{1}{N^{\#\mathcal{I}}}\sum_{\mathcal{I}}c_{\mathcal{I}}\Big(\prod_{i=1}^{n}G_{x_iy_i}\Big)\Big], \quad x_i\neq y_i,\ \mathfrak{n}(v)=2,\ \forall v\in\mathcal{I}.$$

Using Expansion mechanism I:

$$\frac{1}{N^2}\sum_{a,b}\mathbb{E}[G_{ab}G_{ba}\mathbf{G_{aa}}G_{bb}]=-\frac{1}{N^2}\sum_{a,b}\mathbb{E}[G_{ab}G_{ba}\mathbf{G_{bb}}]+\cdots$$
$$=\frac{1}{N^2}\sum_{a,b}\mathbb{E}[G_{ab}G_{ba}]+\cdots.$$

Further expansion will not help.

$$\frac{1}{N^2}\sum_{a,b}\mathbb{E}[G_{ab}G_{ba}]=\frac{1}{N^2}\sum_{k,b}\mathbb{E}[G_{kb}G_{bk}]+\cdots.$$

Step 2. Estimate for Gaussian ensemble:

$$\Big|\mathbb{E}^{\mathrm{G}\beta\mathrm{E}}\Big[\frac{1}{N^2}\sum_{a,b} G_{ab}G_{ba}\Big]\Big| \le \mathbb{E}^{\mathrm{G}\beta\mathrm{E}}\Big[\frac{1}{N^2}\sum_{a,b}|G_{ab}|^2\Big] \qquad \text{(Ward identity)}$$

$$= \mathbb{E}^{\mathrm{G}\beta\mathrm{E}}\Big[\frac{\mathrm{Im}\, m_N(z)}{N\eta}\Big] = O_\prec(N^{-\frac{1}{3}-\epsilon}).$$

In general, we have

$$|\mathbb{E}^{\mathrm{G}\beta\mathrm{E}}[\widetilde{Q}_d]| \prec \mathbb{E}^{\mathrm{G}\beta\mathrm{E}}\Big[\frac{\mathrm{Im}\, m_N(z)}{(N\eta)^{d-1}}\Big] = O_\prec(N^{-\frac{1}{3}-(d-1)\epsilon}), \qquad d \ge 2.$$

Step 3. Iterative comparison $\Longrightarrow$ hierarchy of $\widetilde{Q}_d$.

$$\frac{\mathrm{d}\mathbb{E}[\widetilde{Q}_d]}{\mathrm{d}t} = \frac{1}{4N^{3/2}}\sum_{a,b} s_{ab}^{(3)}\mathbb{E}\Big[\frac{\partial^3 \widetilde{Q}_d}{\partial h_{ab}^3}\Big] + \frac{1}{12N^3}\sum_{a,b} s_{ab}^{(4)}\mathbb{E}\Big[\frac{\partial^4 \widetilde{Q}_d}{\partial h_{ab}^4}\Big] + \cdots$$

$$= \sum \mathbb{E}[\widetilde{Q}_{d'\ge d+1}] + O_\prec(N^{-1/2}).$$

Hierarchy of $\widetilde{Q}_d$:

d	$\widetilde{Q}_d$	Estimate
2	$\frac{1}{N^2}\mathbb{E}[\operatorname{Tr} G^2]$	$N^{-\frac{1}{3}-\epsilon}$
3	$\frac{1}{N^3}\mathbb{E}[\operatorname{Tr} G^3]$	$N^{-\frac{1}{3}-2\epsilon}$
4	$\frac{1}{N^4}\mathbb{E}[\operatorname{Tr} G^4]$, $\frac{1}{N^4}\mathbb{E}[\operatorname{Tr} G^2 \operatorname{Tr} G^2]$	$N^{-\frac{1}{3}-3\epsilon}$
5	$\frac{1}{N^5}\mathbb{E}[\operatorname{Tr} G^5]$, $\frac{1}{N^5}\mathbb{E}[\operatorname{Tr} G^2 \operatorname{Tr} G^3]$	$N^{-\frac{1}{3}-4\epsilon}$
$\vdots$	$\vdots$	$\vdots$
$D-1$	$\frac{1}{N^{D-1}}\operatorname{Tr} G^{D-1}$, $\frac{1}{N^{D-1}}\operatorname{Tr} G^2 \operatorname{Tr} G^{D-3}$, $\cdots$	$N^{-\frac{1}{3}-(d-1)\epsilon}$
$D \geq \frac{1}{3\epsilon}+1$	$\Psi^D + N^{-1} \leq N^{-\frac{1}{3}-\epsilon}$	$N^{-\frac{1}{3}-D\epsilon}$

Lemma

$$|\mathbb{E}[\widetilde{Q}_d]| \prec N^{-1/3-\epsilon}, \qquad d \geq 2, t \in \mathbb{R}_+, z \in S_{\mathrm{edge}}$$

Proof of GFCT for $\mathbb{E}[\mathrm{Im}\, m_N]$

$$\frac{\mathrm{d}\mathbb{E}[m_N(t,z)]}{\mathrm{d}t} = 0 \cdot K_2 + K_3 + K_4,$$

where

$$K_3 = \sqrt{N}\frac{1}{N^3}\sum_{v,a,b} s_{ab}^{(3)}\mathbb{E}\Big[\frac{\partial^2(G_{va}G_{bv})}{\partial h_{ab}^2}\Big] = \sqrt{N}\sum \mathbb{E}[Q_d^o] = O_\prec(N^{-\frac{1}{2}}),$$

$$K_4 = \frac{1}{N^3}\sum_{v,a,b} s_{ab}^{(4)}\mathbb{E}\Big[\frac{\partial^3(G_{va}G_{bv})}{\partial h_{ab}^3}\Big] = \sum \mathbb{E}[\widetilde{Q}_{d\geq 2}] = O_\prec(N^{-\frac{1}{3}-\epsilon}).$$

Choosing $T = 8\log N$,

$$\begin{aligned}\mathbb{E}[\mathrm{Im}\, m_N(0,z)] =& \mathbb{E}[\mathrm{Im}\, m_N(T,z)] + O_\prec(N^{-\frac{1}{3}-\epsilon}) \\ =& \mathbb{E}^{\mathrm{GOE}}[\mathrm{Im}\, m_N(z)] + O_\prec(N^{-\frac{1}{3}-\epsilon}).\end{aligned}$$

Extension to general F

Set

$$\mathcal{X}(t) := N \int_{E_1}^{E_2} \operatorname{Im} m_N(t, y + \mathrm{i}\eta)\mathrm{d}y.$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{E}[F(\mathcal{X}(t))] = \sum_{a,b,v} \sum_{k+1=3} \frac{1}{k!} \frac{s_{ab}^{(k+1)}(t)}{N^{\frac{k+1}{2}}} \mathbb{E}\Big[\frac{\partial^k F'(\mathcal{X}) \int_{E_1}^{E_2} \operatorname{Im} G_{va} G_{bv} \mathrm{d}y}{\partial h_{ab}^k}\Big]$$

$$= \sum_{a,b} \sum_{k+1=3}^{4} \frac{1}{k!} \frac{s_{ab}^{(k+1)}(t)}{N^{\frac{k+1}{2}}} \mathbb{E}\Big[\frac{\partial^k F'(\mathcal{X}) \Delta \operatorname{Im} G_{ab}}{\partial h_{ab}^k}\Big] + O_\prec\Big(\frac{1}{\sqrt{N}}\Big),$$

where we use $G^2 = \frac{\mathrm{d}}{\mathrm{d}y} G$ and

$$\Delta \operatorname{Im} G := \operatorname{Im} G(t, E_2 + \mathrm{i}\eta) - \operatorname{Im} G(t, E_1 + \mathrm{i}\eta).$$

Third order terms: unmatched and bounded by $O_\prec(N^{-1/2})$.

Fourth order terms:

$$\begin{aligned} K_4 =& 2s^{(4)}(t) \sum_a \mathbb{E}\Big[F'(\mathcal{X})\Delta \mathrm{Im}\, m_N(z)\Big] \\ &- 2s^{(4)}(t)\mathbb{E}\Big[F''(\mathcal{X})\Big(\Delta \mathrm{Im}\, m_N(z)\Big)^2\Big] \\ &- \frac{2s^{(4)}(t)}{N^2}\mathbb{E}[F'(\mathcal{X})\Delta \mathrm{Im}\,\mathrm{Tr}\, G^2] + \cdots \end{aligned}$$

where $s^{(4)}(t) = \frac{1}{2}\sum_{a\neq b} s_{ab}^{(2)}$.

Recall from the toy model:

$$\mathbb{E}[\mathrm{Im}\, m_N(t,z)] = O(N^{-1/3}) \Longrightarrow |K_4| \prec N^{-1/3}$$

Therefore, we conclude that, $T = 8\log N$,

$$\mathbb{E}[F(\mathcal{X}(0))] = \mathbb{E}[F(\mathcal{X}(T))] + O_{\prec}(N^{-\frac{1}{3}}) = \mathbb{E}^{\mathrm{GOE}}[F(\mathcal{X})] + O_{\prec}(N^{-\frac{1}{3}}).$$

Strategy to prove quantitative Tracy-Widom Law

1. Reduction to Green function comparison, choose $\eta = N^{-1+\epsilon}$.
2. Interpolation between Wigner and Gaussian matrix:

$$H(t) = \mathrm{e}^{-\frac{t}{2}}\mathrm{Wig} + \sqrt{1-\mathrm{e}^{-t}}\mathrm{Gau}$$

3. Compute $\frac{\mathrm{d}\mathbb{E}[F(\mathcal{X})]}{\mathrm{d}t} = \frac{1}{N^{\frac{3}{2}}}\sum_{a,b} s_{ab}^{(3)}\mathbb{E}\left[\frac{\partial^3 F(\mathcal{X})}{\partial h_{ab}^3}\right] + \frac{1}{N^2}\sum_{a,b} s_{ab}^{(4)}\mathbb{E}\left[\frac{\partial^4 F(\mathcal{X})}{\partial h_{ab}^4}\right] + \cdots$
   Exact cancellation of the second order.
4. Unmatched third order: iterative expansions and power counting by the local law.
5. Matched fourth order: iteratively reduce to trace-like functions; estimate of trace-like functions for Gaussian ensemble; iterative comparisons with Gaussian ensemble.

Tools required: local law + expansion mechanism + asymptotic estimates for Gaussian ensemble.

Related work: sample covariance matrix

Sample covariance matrix $\frac{1}{N}X^*X$, where $X = (x_{ij})$ is an $M \times N$ matrix with independent random entries

$$\mathbb{E}[x_{ij}] = 0, \qquad \mathbb{E}\big[|x_{ij}|^2\big] = 1, \qquad \mathbb{E}\big[|x_{ij}|^k\big] < \infty \ (k \geq 3).$$

Complex case: also assume that $\mathbb{E}[(x_{ij})^2] = 0$.
Special Gaussian case: white Wishart ensemble (or Laguerre ensemble).

$$\text{Aspect ratio:} \quad \varrho \equiv \varrho_N := M/N \to \varrho_0 \in (0, \infty)..$$

Marchenko–Pastur Law: ESD converges weakly to

$$\mathrm{d}\mu_{\mathrm{MP}}(x) := \frac{1}{2\pi\varrho_0}\sqrt{\frac{(x - E_0^-)(E_0^+ - x)}{x^2}}\mathbb{1}_{[E_0^-, E_0^+]}(x)\mathrm{d}x + (1 - \varrho_0^{-1})_+\delta_0(\mathrm{d}x),$$

with edges $E_0^{\pm} := (1 \pm \sqrt{\varrho_0})^2$.

Define $\mu_N := (\sqrt{M} + \sqrt{N})^2; \quad \sigma_N := (\sqrt{M} + \sqrt{N})\Big(\frac{1}{\sqrt{M}} + \frac{1}{\sqrt{N}}\Big)^{1/3}$

$$\boxed{\text{Tracy-Widom limit:} \qquad \frac{N\lambda_N - \mu_N}{\sigma_N} \Longrightarrow \mathrm{TW}_\beta.}$$

Quantitative: Gaussian [El Karoui '06, Ma '12]; non-Gaussian [Wang '19, Schnelli-Xu '21]. Linearization of $\frac{1}{N}X^*X$:

$$H(z) = \begin{pmatrix} -zI_N & \frac{1}{\sqrt{N}}X^* \\ \frac{1}{\sqrt{N}}X & -I_M \end{pmatrix},$$

and corresponding Green function:

$$G(z) = (H(z))^{-1} \approx \begin{pmatrix} m_{\mathrm{MP}}(z) & 0 \\ 0 & -\frac{1}{1+m_{\mathrm{MP}}(z)} \end{pmatrix}.$$

Block structure: adapted expansion mechanism.

Thank you for listening!