Transition probabilities and expectation values for multi-species exclusion processes

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ACEMS

ASEP

The asymmetric exclusion process on $\mathbb{Z}$:

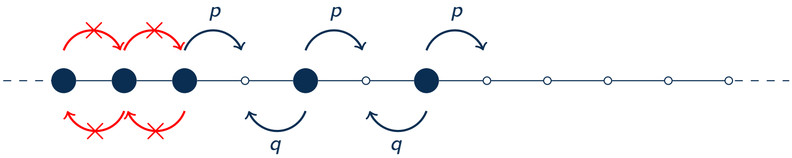


Figure: Configuration of particles and hopping rates in the ASEP on $\mathbb{Z}$

Markov chain:

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\mathbb{P}(\mathcal{C};t) = \sum_{\mathcal{C}'\neq\mathcal{C}} W(\mathcal{C}'\to\mathcal{C})\mathbb{P}(\mathcal{C}';t) - \sum_{\mathcal{C}'\neq\mathcal{C}} W(\mathcal{C}\to\mathcal{C}')\mathbb{P}(\mathcal{C};t),$$

Initial condition:

$$\mathbb{P}(\{\mu\to\nu\};0) = \prod_{i=1}^{n}\delta_{\nu_i,\mu_i}.$$

ASEP transition probability

One particle (Bethe ansatz) eigenfunction:

$$\varphi_z(\nu, t) = \exp\left(-t\frac{z(p-q)^2}{p(1+z)(1+z/\tau)}\right)\left(\frac{1+z}{1+z/\tau}\right)^{\nu-1}, \qquad \tau = \frac{p}{q}$$

Many particles (Tracy-Widom):

$$\mathbb{P}(\{\mu \to \nu\}; t) = \frac{1}{(2\pi \mathrm{i})^n}\oint_{-\tau} \mathrm{d}z_1 \cdots \oint_{-\tau} \mathrm{d}z_n \prod_{i=1}^{n} \frac{p-q}{(1+z_i/\tau)^2} \times \sum_{\pi \in S_n} \prod_{\pi_i < \pi_j} \frac{\tau z_i - z_j}{z_i - \tau z_j} \prod_{i=1}^{n} \varphi_{z_i}(\nu_{\pi_i} - \mu_i, t)$$

ASEP expectation values

$N_x(t)$: the number of particles to have crossed a given site x after time t.

Convenient observable (ASEP self-dual): $Q_x(t) = \tau^{N_x(t)}$ with $\tau = \frac{p}{q}$ and

$$\widetilde{Q}_x(t) = \frac{Q_x(t) - Q_{x-1}(t)}{\tau - 1} = \tau^{N_{x-1}(t)} \mathbf{1}_{x \in \nu_t}$$

Theorem (Borodin-Corwin-Sasamoto (step initial condition),...)

$$\mathbb{E}[\widetilde{Q}_{x_1}(t) \cdots \widetilde{Q}_{x_k}(t)] = \oint \mathrm{d}z_1 \cdots \oint \mathrm{d}z_n \prod_{1 \leq i < j \leq k} \frac{z_i - z_j}{z_i - \tau z_j} \prod_{\alpha=1}^{k} \varphi_{z_i}(x_i, t) \frac{1}{z_i + \tau}$$

Fluctuations of particle flow across the origin follow KPZ statistics given by the Tracy-Widom distribution:

Theorem (Fluctuations of ASEP)

$$\lim_{t \to \infty} P\left(\frac{N_0(t) - vt}{t^{1/3}} > -s \right) = F_{\mathrm{GUE}}(s).$$

Generalisation to multi-species

Outline

- Two-species TASEP
- r-ASEP ($U_\tau(\widehat{sl}_{r+1})$), vertex model approach
- Arndt-Heinzel-Rittenberg model
- Multi-species duality observables

Two-species TASEP

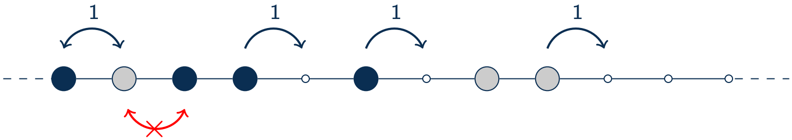


Figure: Configuration of particles and hopping rates in the 2-TASEP on $\mathbb{Z}$

Transition events and rates:

$$(1,0) \mapsto (0,1) \text{ at rate } 1,$$
$$(2,0) \mapsto (0,2) \text{ at rate } 1,$$
$$(2,1) \mapsto (1,2) \text{ at rate } 1.$$

2-TASEP transition probability

Coordinates of all particles: $\nu = \{\nu_1, \ldots, \nu_n\} \in \mathbb{Z}^n$
Indices of type 2 particles: $p = \{p_1, \ldots, p_m\} \in \{1, \ldots, n\}^m$.
Free equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbb{P}(\{\mu \to \nu\}, p; t) = \sum_{i=1}^{n} \mathbb{P}(\{\nu_1, \ldots, \nu_i - 1, \ldots, \nu_n\}, p; t) - n\mathbb{P}(\nu, p; t)$$

Boundary conditions (interaction):

1. ...
2. If $i \notin p$ and $i+1 \in p$ then
$$\mathbb{P}(\{\nu_{i+1} = \nu_i\}, p; t) = \mathbb{P}(\{\nu_{i+1} = \nu_i + 1\}, s_i p; t) + \mathbb{P}(\{\nu_{i+1} = \nu_i + 1\}, p; t)$$
3. ...

and initial condition

$$\mathbb{P}(\{\mu \to \nu\}, p; 0) = \prod_{i=1}^{n} \delta_{\nu_i, \mu_i} \prod_{j=1}^{m} \delta_{p_j, p_j^{(0)}}$$

2-TASEP transition probability

Theorem

$$\mathbb{P}(\{\mu \to \nu\}, \{p^{(0)} \to p\}; t) =$$

$$\sum_{\pi \in S_n} (-1)^{|\pi|} \oint \prod_{i=1}^{n} \frac{dz_i}{2\pi \mathrm{i}} \prod_{i=1}^{n} \left(\frac{1 - z_i}{1 - z_{\pi_i}} \right)^i z_{\pi_i}^{\nu_i} z_i^{-\mu_i - 1} e^{(z_i^{-1} - 1)t}$$

$$\times \sum_{\rho \in S_m} (-1)^{|\rho|} \oint \prod_{j=1}^{m} \frac{du_j}{2\pi \mathrm{i}} \prod_{i=1}^{m} \left[\left(\frac{1 - u_i}{1 - u_{\rho_i}} \right)^i \prod_{j=1}^{p_i - 1} (u_{\rho_i} - z_{\pi_j}) \prod_{j=1}^{p_i} \frac{1}{1 - z_{\pi_j}} \right]$$

$$\times \prod_{i=1}^{n} (1 - z_i)^m \prod_{i=1}^{m} \left[\prod_{j=1}^{p_i^{(0)}} \frac{1}{u_i - z_j} \prod_{j=p_i^{(0)}+1}^{n} \frac{1}{1 - z_j} \right]$$

1-TASEP Bethe ansatz
Nested Bethe ansatz
Normalisation

All contours around the origin.

Total crossing

Total crossing: $p_j^{(0)} = j, \quad p_j = n - m + j$

All 2's overtake all 1's

Theorem

The transition probability for total crossing is given by

$$\mathbb{P}_{\text{cross}}(\{\mu \to \nu\}, \{p^{(0)} \to p\}; t) =$$
$$\oint \prod_{i=1}^{m} \frac{dz_i}{2\pi \mathrm{i}} \prod_{j=1}^{n-m} \frac{dw_j}{2\pi \mathrm{i}} \prod_{i=1}^{m} \frac{e^{(z_i^{-1}-1)t}}{(1-z_i)^{n-m}} \prod_{i=1}^{n-m} e^{(w_i^{-1}-1)t} \prod_{i=1}^{m} \prod_{j=1}^{n-m} \left(w_j - z_i\right)$$
$$\det\left(z_i^{\nu_{n-m+j}-\mu_i-1}(1-z_i)^{i-j}\right)_{1\le i,j\le m} \det\left(w_i^{\nu_j-\mu_{m+i}-1}(1-w_i)^{i-j}\right)_{1\le i,j\le n-m}$$

Proof by evaluation of simple poles and determinant calculus.

Cumulative total crossing (definitions)

Cumulative total crossing: $p_j^{(0)} = j, \quad p_j = n - m + j$

The cumulative probability of total crossing is given by

$$P_{\text{cross}}(s_1, s_2) = \sum_{s_1 \le \nu_1 < \cdots < \nu_{n-m} < s_2 \le \nu_{n-m+1} < \cdots < \nu_n} \mathbb{P}_{\text{cross}}(\{\mu \to \nu\}, \{p^{(0} \to p\}; t),$$

Under random Bernoulli-step initial conditions with parameter ρ

$$P^{\text{B}}_{\text{cross}}(s_1, s_2) = \sum_{\mu_1 < \mu_2 < \cdots < \mu_m < 0} \mathbb{P}(\mu; 0) P_{\text{cross}}(s_1, s_2)$$

Cumulative total crossing (result)

Theorem

The cumulative total crossing probability under Bernoulli-step initial conditions is given by

$$P^{\mathrm{B}}_{\mathrm{cross}}(s_1, s_2) = \frac{\rho^m}{m!} \oint \prod_{i=1}^{m} \frac{\mathrm{d}z_i}{2\pi \mathrm{i}} \prod_{i=1}^{n-m} \frac{\mathrm{d}w_i}{2\pi \mathrm{i}} \prod_{i=1}^{m} \prod_{j=1}^{n-m} \left(w_j - z_i\right) \prod_{i \neq j} (z_j - z_i)$$

$$\times \prod_{i=1}^{m} \frac{e^{(z_i^{-1}-1)t} z_i^{s_2}}{(1-z_i)^n \, (1-(1-\rho)z_i)} \prod_{i=1}^{n-m} \frac{e^{(w_i^{-1}-1)t} w_i^{s_1 - i}}{(1-w_i)^{n-m-i+1}} \det \left(w_i^{j-1} - w_i^{s_2 - s_1} \right)_{1 \le i,j \le n-m}$$

Proof by geometric series, residue calculus and symmetrisation identities.

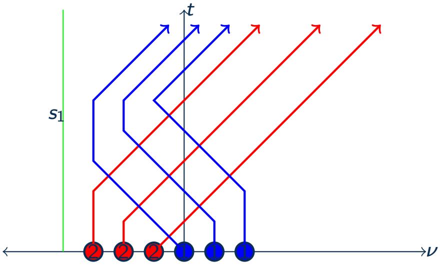


Figure: Space-time diagram total crossing configuration with an irrelevant wall at $s_1 \le -m$.

When $s_1 < -m$:

$$P^{\mathrm{B}}_{\mathrm{cross}}(s_1, s_2) = \rho^m \oint_0 \frac{\mathrm{d}w}{2\pi \mathrm{i}} \frac{e^{(w-1)t} w^{n-2m-s_2-1}}{w-1}$$
$$\det\left(\oint_{0,1,1-\rho} \frac{\mathrm{d}z}{2\pi \mathrm{i}} \frac{e^{(z-1)t} z^{i+j-s_2-m-1}}{(z-1)^{m+1}(z-1+\rho)} (w-z) \right)_{1\le i,j\le m} .$$

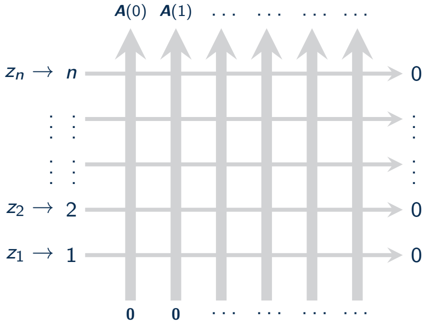
Vertex model approach

With μ and ν compositions, define

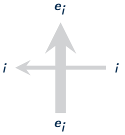
$$\boldsymbol{A}(k) = \sum_{j=1}^{n} \mathbf{1}_{\mu_j = k} \boldsymbol{e}_j, \qquad k \in \mathbb{Z}_{\geq 0}\,,$$

Then

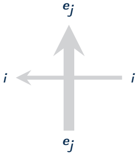
$$f_\mu(z_1, \ldots, z_n) = \qquad (1)$$



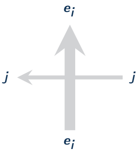
Vertex model approach



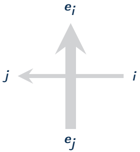
$$1$$



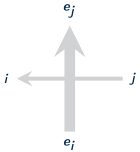
$$\frac{1+z}{\tau+z}$$



$$\frac{1+z}{1+z/\tau}$$



$$\frac{1-\tau}{\tau+z}$$



$$\frac{z(1-\tau)}{\tau+z}$$

Matrix product approach.

ASEP transition probability

Theorem (up to normalisations)

$$\mathbb{P}_{\mathrm{ASEP}}(\{\mu \to \nu\}; t) = \frac{1}{(2\pi \mathrm{i})^n} \oint_{-\tau} \mathrm{d}z_1 \cdots \oint_{-\tau} \mathrm{d}z_n \prod_{1\le i<j\le n} \left(\frac{z_i - z_j}{z_i - \tau z_j} \right) \times \varphi_{z_j}(-\mu_j, t) \frac{1}{\tau + z_j} f_\nu(z_1, \ldots, z_n),$$

Proposition (Rainbow crossing probability)

The probability that all particles have different colours and exchange their order at time t:

$$\mathbb{P}_{\mathrm{ASEP}}(\{\mu \to \nu\}; t) = \frac{1}{(2\pi \mathrm{i})^n} \oint_{-\tau} \mathrm{d}z_1 \cdots \oint_{-\tau} \mathrm{d}z_n \prod_{1\le i<j\le n} \left(\frac{z_i - z_j}{z_i - \tau z_j} \right) \times \prod_{i=1}^{n} \varphi_{z_j}(\nu_j - \mu_j, t) \frac{1}{\tau + z_i}$$

Block symmetrisation leads to r-ASEP and r-TASEP total crossing probabilities.

Rank 2 observable

Higher rank observables can be obtained by degeneration of Macdonald polynomials at $q^k t^\ell = 1$.
Explicit formulas from matrix product expressions.
Let

$$\mu \in \sigma(0^{n-m_1-m_2}, 1^{m_1}, 2^{m_2}), \qquad \nu \in \sigma(0^{n-m_1-m_2-p}, 1^{m_1+2p}, 2^{m_2-p}),$$

and let $\vec{x}$ labels the positions of 1-particles and $\vec{y}$ of 2-particles in μ. The number of "crossings" between the two sets $\vec{x}$ and $\vec{y}$:

$$\chi(\vec{x}, \vec{y}) := \#\{(x_i, y_j) \in (\vec{x}, \vec{y}) \mid x_i > y_j\}.$$

Proposition

The following function $\psi(\nu, \mu)$ satisfies the duality evolution equation for two two-species ASEPs,

$$\psi(\nu, \mu) = \prod_{x \in \vec{x}(\mu)} \prod_{i<x} \left(\tau^{\mathbf{1}_{\nu_i \geq 1}}\right) \cdot \prod_{y \in \vec{y}(\mu)} \prod_{i<y} \left(\tau^{\mathbf{1}_{\nu_i = 1} \mathbf{1}_{\nu_y = 1}}\right) \cdot \tau^{-\chi(\vec{x}, \vec{y})} \cdot I(\mu, \nu),$$

where $I(\mu, \nu)$ denotes the indicator function

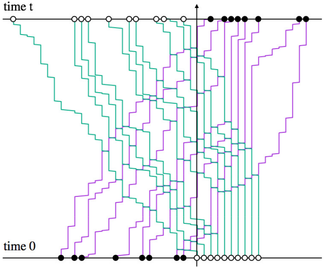
$$I(\mu, \nu) = \begin{cases} 0, & \exists\ k : \mu_k > \nu_k = 0, \ \ \textit{or} \ \ \mu_k < \nu_k = 2, \\ 1, & \textit{otherwise}, \end{cases}$$

Another generalisation

Two TASEPs can be coupled to form another integrable 2-species model.

Introduced by Arndt-Heinzl-Rittenberg (AHR), the transition rates are

$$\begin{aligned} p: &\quad (+,0) \to (0,+) \\ 1-p: &\quad (0,-) \to (-,0) \\ 1: &\quad (+,-) \to (-,+) \end{aligned}$$



Transition probability for total crossing

Initial conditions: assume $x_j^{(0)} < y_k^{(0)}$, i.e. at $t = 0$ all + particles are to the left of all − particles.

Final condition: $x_j > y_k$, i.e. at time t all + particles have passed all − particles

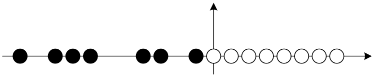
Then

$$\mathbb{P}(x_j, y_k, t) = \oint \prod_{j=1}^{n} \mathrm{d}z_j \prod_{k=1}^{m} \mathrm{d}w_k \; \mathrm{e}^{\Lambda t} \prod_{k=1}^{m} \prod_{j=1}^{n} \frac{1}{qz_j + pw_k}$$
$$\times \det\left(\left(\frac{z_j - 1}{z_i - 1}\right)^{j-1} z_i^{x_j} \right) z_j^{-x_j^{(0)}-1}$$
$$\times \det\left(\left(\frac{w_k - 1}{w_\ell - 1}\right)^{m-k} w_\ell^{-y_k} \right) w_k^{y_k^{(0)}-1},$$

with all contours around the origin, and with eigenvalue

$$\Lambda = p \sum_{j=1}^{n} (z_j^{-1} - 1) + q \sum_{k=1}^{m} (w_k^{-1} - 1).$$

Step-Bernoulli condition



Proposition

The total exchange probability $\mathbb{P}_{n,m,\rho}(t)$ with Bernoulli initial data is given by

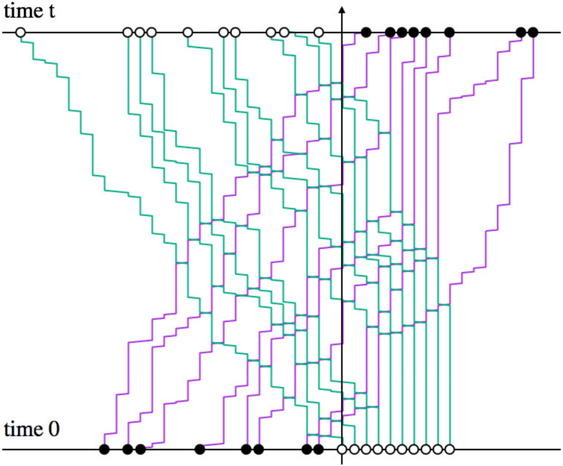
$$\mathbb{P}_{n,m,\rho}(t) = \oint \prod_{j=1}^{n} \mathrm{d}\, z_j \prod_{k=1}^{m} \mathrm{d}\, w_k \; \mathrm{e}^{\Lambda_{n,m} t} \times$$

$$\frac{\rho^n \displaystyle\prod_{1\le i<j\le n} (z_i - z_j) \prod_{1\le k<l\le m} (w_l - w_k) \prod_{j=1}^{n} z_j^{n-j} \prod_{k=1}^{m} w_k^{k-1}}{\displaystyle\prod_{j=1}^{n} (z_j - 1)^{n+1-j} \left(1 - \rho z_j\right) \prod_{k=1}^{m} (w_k - 1)^k \prod_{j=1}^{n} \prod_{k=1}^{m} \left(q z_j + p w_k\right)},$$

with all contours around the origin.

The w-contours can be readily evaluated if $n > m$ but not when $n < m$

Exchange



Asymptotics

Non-linear fluctuating hydrodynamics (KPZ formalism) suggests a scaling limit of the form

$$n = j_1 t + \alpha t^{1/3} + \beta t^{1/2},$$
$$m = j_2 t + \gamma t^{1/3} + \delta t^{1/2},$$

where $j_{1,2}$, $\alpha, \beta, \gamma, \delta$ are known functions of ρ', and $n < m$.

Need to analyse

$$\mathbb{P}_{n,m,\rho}(t) = \underbrace{\oint \dots \oint}_{n \times m} \text{factorised integrand}$$

where n, m, t are large.

Trick: Convert to Fredholm determinant:

$$\mathbb{P}_{n,m,\rho}(t) = \det(\mathbb{I} - AB)_{m \times m} = \det(\mathbb{I} - BA)_{L^2(\mathbb{R})},$$

where n, m, t all occur as parameters in BA.

Asymptotics

Need to calculate integrals like

$$\mathcal{I}_2 = \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \det(\mathbb{I} - K(\vec{z}))_{\ell^2(\mathbb{N})}$$

with

$$K(x, y; \vec{z}) = \oint_1 \frac{\mathrm{d}\zeta}{2\pi\mathrm{i}} F(\zeta, x) \prod_{j=1}^{n-1} \frac{1 + z_j\zeta}{1 + \zeta} \oint_C \frac{\mathrm{d}w}{2\pi\mathrm{i}} G(w, y) \prod_{j=1}^{n-1} \frac{1 + w}{1 + z_j w} \frac{1}{w - z},$$

Proposition

For any $(x_1, x_2, \ldots, x_k) \in \mathbb{N}^k$, $\rho \in (0, 1)$, $t > 0$ and $n, m \in \mathbb{N}$, the following equality holds:

$$\oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \det\left[K(x_i, x_j, \vec{z})\right]_{1\le i,j\le k}$$

$$= \oint_1 \mathrm{d}^{n-1} z\, L(\vec{z}) \det\left\{K_{\mathrm{W}}(x_i, x_j) - \left[\sum_{l=1}^{n-1} \prod_{k=1}^{l} (z_k - 1) A_l(x_i)\right] B(x_j)\right\}_{1\le i,j\le k} .$$

Asymptotics

$$\mathcal{I}_2 = \mathcal{I}_z \det\left(\mathbb{I} - K(\vec{z}=\vec{1})\right)_{\ell^2(\mathbb{N})} + \text{ lower order}$$

In order to perform asymptotic analysis, we define the rescaled functions

$$\xi = x/\lambda_c t^{1/3}, \qquad \zeta = y/\lambda_c t^{1/3}$$

such that

$$\mathcal{K}(\xi,\zeta) = (w_c + c)^{\lambda_c t^{1/3}(\xi-\zeta)} \lambda_c t^{1/3} K(\lambda_c t^{1/3}\xi, \lambda_c t^{1/3}\zeta),$$

The rescaled kernel is explicitly described as

$$\mathcal{K}(\xi,\zeta) = \lambda_c t^{1/3} \oint_1 \frac{\mathrm{d}\,z}{2\pi \mathrm{i}} \left(\frac{z+\rho'}{z+1}\right) \mathrm{e}^{f(z,t,\xi)-f(w_c,t,\xi)} \times$$
$$\oint_{0,-\rho',-1} \frac{\mathrm{d}\,w}{2\pi \mathrm{i}} \left(\frac{w+1}{w+\rho'}\right) \mathrm{e}^{-f(w,t,\zeta)+f(w_c,t,\zeta)+g(w)} \frac{1}{w-z},$$

Theorem

$$\lim_{t\to\infty} \det(1-\mathcal{K})_{\ell^2(\mathbb{N}/(\lambda_c t^{1/3}))} = \lim_{t\to\infty} \det(1-\mathcal{K})_{L^2(0,\infty)} = \det(1-A)_{L^2(s,\infty)} = F_2(s)$$

$$A(x,y) = \int_0^\infty \mathrm{Ai}(x+\lambda)\,\mathrm{Ai}(y+\lambda)\,\mathrm{d}\,\lambda$$

and

$$s = \frac{1}{c_2 t^{1/3}}\Big((1+\rho)n - (3-\rho)m + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t\Big)$$

Recall

$$\mathcal{I}_2 = \mathcal{I}_z \det\Big(\mathbb{I} - K(\vec{z}=\vec{1})\Big)_{\ell^2(\mathbb{N})} + \text{ lower order}$$

The integral $\mathcal{I}_z$ converges to a Gaussian

$$\mathcal{I}_2 \to \Big(1 - F_G(s')\Big)F_2(s) \quad \text{as } t\to\infty$$

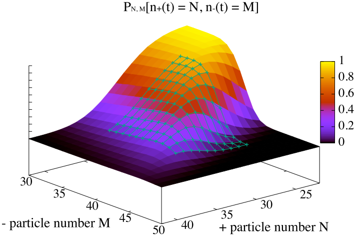
Final result

Theorem

In the appropriate scaling limit

$$\lim_{t\to\infty} \mathbb{P}_{n,m,\rho}(t) = F_{\mathrm{GUE}}(s) F_{\mathrm{Gauss}}(s'),$$

$$s(n,m;t) =: \frac{1}{c_2 t^{1/3}} \Big((1+\rho)n - (3-\rho)m + \tfrac{1}{2}(1-\rho)(1-(1-\rho)^2/4)t \Big),$$
$$s'(n,m;t) =: \frac{1}{c_g t^{1/2}} \Big(-2(2-\rho)n + 2\rho m + (2-\rho)(1-\rho)\rho t \Big),$$



Conclusion

- Explicit transition probabilities for two-TASEP and two-species AHR model
- Vertex model expressions for r-ASEP
- Explicit total crossing probabilities
- (First) proof of Nonlinear Fluctuating Hydrodynamics for a two-component mixture
- Mix of Gaussian and KPZ modes
- How to deal with dynamic poles in integrand (random matrix interpretation?)