Integrable structure for the multitime distribution of TASEP.

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Overview

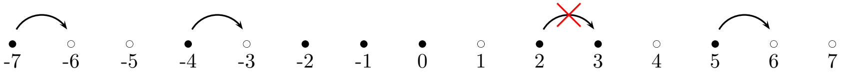
Overview

- Baik, Deift, Johansson, (1999)
- Prähofer, Spohn, (2002)
- Corwin, (2011) (https://arxiv.org/abs/1106.1596)
- Matetski, Quastel, Remenik, (2021)
- Dauvergne, Ortmann, Virag (https://arxiv.org/abs/1812.00309)

TASEP

TASEP

Consider the continuous time totally asymmetric simple exclusion process (TASEP).



[1] $\mathrm{Prob}(t < s) = \int_0^s e^{-x} dx$

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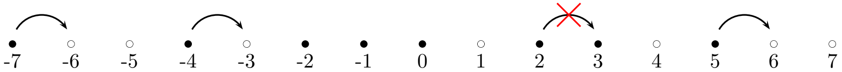

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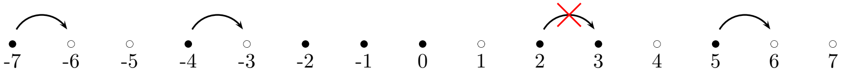

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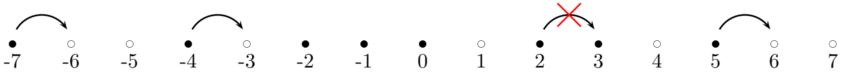

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- Particles only can jump to the right
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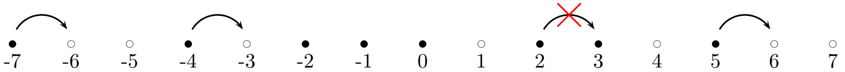

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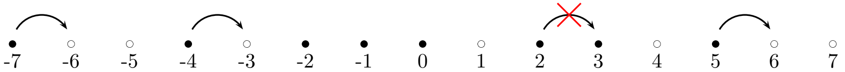
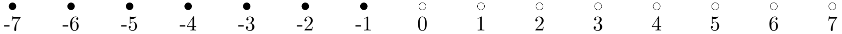


- Each particle is waiting independently exponential time before jumping [1]
- Particles only can jump to the right
- Particles only can jump distance one
- Only one particle can occupy one spot
- There is only a finite number of particles N.

[1] $\mathrm{Prob}(t < s) = \int_0^s e^{-x} dx$

Notation, initial conditions

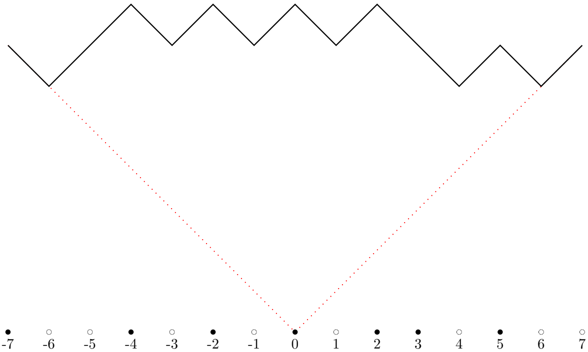
- $x_n(t)$ - locations of the nth particle at the moment t. Numeration of the particles is from right to left.
- Step initial configuration: $x_k(0) = -k$, $k = 1 \dots N$.



Corner growth process

Particle configuration corresponds to the height function.

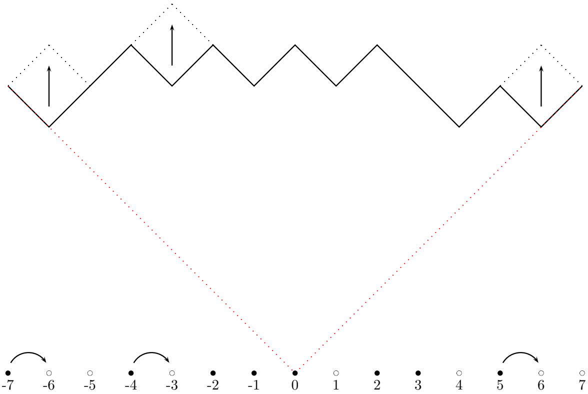
$$H(n,t) \leq a \Leftrightarrow x_{\frac{a-n}{2}}(t) \leq n$$



Vertex j is empty $\Rightarrow$ slope is 1 on the interval $(j, j+1)$.
Vertex j has a particle $\Rightarrow$ slope is -1 on the interval $(j, j+1)$.

Corner growth process

Jumps of particles corresponds to growth of corners. Each corner has independent exponential clock.



One point distribution for scaling limit

Kardar-Parisi-Zhang (KPZ) scaling limit

Consider the following scaling limit

$$\lim_{\substack{T \to \infty \\ N > 2yT^{\frac{2}{3}} + tT + xT^{\frac{1}{3}}}} Pr\left(\frac{H(2yT^{\frac{2}{3}}, 2tT) - tT}{-T^{\frac{1}{3}}} < x \right) = F(t, y, x)$$

Theorem (Baik, Deift, Johansson(1999))

$$F(t, y, x) = F_{TW}\left(\frac{x}{t^{\frac{1}{3}}} + \frac{y^2}{t^{\frac{4}{3}}} \right)$$

where $F_{TW}(s)$ is the Tracy-Widom distribution.

Airy determinant

$$F_{TW}(s) = \det(1 - \chi_{[s,\infty)} A_1),$$

where $\chi_{[s,\infty)}$ is indicator function and

$$A_1(x, y) = \frac{Ai(x)Ai'(y) - Ai'(x)Ai(x)}{x - y}.$$

and the operator acts in $L_2(\mathbb{R})$.

Integrable operator

Theorem (Bertola,Cafasso, (2012), IMRN)

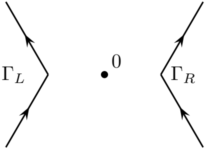
$$\det(1-\chi_{[s,\infty)}A_1)=\det(1-K_{A_1})$$

where

$$K_A(\lambda,\mu)=\frac{\vec{f}^T(\lambda)\vec{g}(\mu)}{\lambda-\mu},$$

$$\vec{f}(\lambda)=\frac{1}{2\pi i}\begin{pmatrix} e^{\frac{\lambda^3}{6}}\chi_{\Gamma_R}(\lambda) \\ e^{-\frac{\lambda^3}{6}+s\lambda}\chi_{\Gamma_L}(\lambda)\end{pmatrix}, \quad \vec{g}(\lambda)=\begin{pmatrix} e^{-\frac{\lambda^3}{6}}\chi_{\Gamma_L}(\lambda) \\ e^{\frac{\lambda^3}{6}-s\lambda}\chi_{\Gamma_R}(\lambda)\end{pmatrix},$$

and the operator acts on $L_2(\Gamma_R\cup\Gamma_L)$.



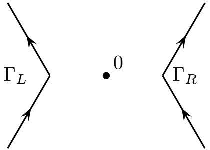
Riemann-Hilbert problem for Painlevé-II equation

1. $Y(\lambda)$ is analytic outside of $\Gamma_L \cup \Gamma_R$.
2. $Y_+(\lambda) = Y_-(\lambda) J(\lambda)$, $\lambda \in \Gamma_L \cup \Gamma_R$, where

$$J(\lambda) = (1 - 2\pi i \vec{f}(\lambda)\vec{g}^T(\lambda)) = \begin{pmatrix} 1 & e^{\frac{\lambda^3}{3} - s\lambda}\chi_{\Gamma_R}(\lambda) \\ e^{-\frac{\lambda^3}{3} + s\lambda}\chi_{\Gamma_L}(\lambda) & 1 \end{pmatrix}$$

3. $Y(\lambda) = I + \frac{Y_1}{\lambda} + \frac{Y_2}{\lambda^2} + O(\lambda^{-3}), \quad \lambda \to \infty$

$$(Y_1)_{11} = \partial_s \ln F_{TW}(s)$$



Isomonodromic deformations

- Consider $\Psi(\lambda) = Y(\lambda)e^{\left(\frac{\lambda^3}{3} - s\lambda\right)E_1}$ where matrix E_j is described by $(E_j)_{mn} = \delta_{mn}\delta_{mj}$.
- It has constant jump and satisfies the Lax pair equations

$$\partial_\lambda \Psi\Psi^{-1} = \lambda^2 E_1 + \lambda[Y_1, E_1] + \partial_s Y_1 - sE_1 \tag{1}$$

$$\partial_s \Psi\Psi^{-1} = -\lambda E_1 - [Y_1, E_1] \tag{2}$$

Painlevé II

- The compatibility condition of equations (1) and (2) provides the isomonodromic deformation equation for Y_1.

$$\partial_s^2 Y_1 - [\partial_s Y_1, [Y_1, E_1]] + s[E_1, [Y_1, E_1]] = 0. \tag{3}$$

- We choose the notation

$$Y_1 = \begin{pmatrix} r & u \\ v & m \end{pmatrix}$$

Using (3) one can get

$$\partial_s r = uv$$
$$\partial_s^2 u = -2u^2 v + su$$
$$\partial_s^2 v = -2v^2 u + sv$$

Painlevé II

- Due to the symmetry $Y(-\lambda) = \sigma_1 Y(\lambda)\sigma_1$ we have $v = -u$ and $u(s)$ satisfies Painlevé-II equation

$$\partial_s^2 u = 2u^3 + su,$$
$$\partial_s^2 \ln F_{TW}(s) = -u^2(s).$$

- One can write the equation for $w(s) = u^2(s) = -\partial_s^2 \ln F_{TW}(s)$. It is called Painlevé-XXXIV equation

$$2w\partial_s^2 w = (\partial_s w)^2 + 8w^3 + 4sw^2.$$

Hamiltonian system

- Painlevé-II equation is Hamiltonian system with Hamiltonian

$$H = \frac{p^2}{4} - sq^2 - q^4, \quad \begin{cases} \partial_s q = \partial_p H \\ \partial_s p = -\partial_q H. \end{cases}$$

- Function $F_{TW}(s)$ can be interpreted as isomonodromic tau function introduced by Jimbo, Miwa, and Ueno in 1980. One of its properties is the equation

$$\partial_s \ln F_{TW}(s) = H(s)$$

- One can write the equation for $H(s) = \partial_s \ln F_{TW}(s)$. It is called σ form of Painlevé-II equation

$$(\partial_s^2 H)^2 + 4(\partial_s H)^3 - 4s(\partial_s H)^2 + 4H\partial_s H = 0.$$

Nonlinear PDEs

- We are interested in

$$F(t,y,x) = F_{TW}\left(\frac{x}{t^{\frac{1}{3}}} + \frac{y^2}{t^{\frac{4}{3}}}\right)$$

- We make transformation of the solution of Riemann-Hilbnert problem $Y(\lambda, s)$

$$X(z,t,y,x) = Y\left(t^{\frac{1}{3}}z - \frac{y}{t^{\frac{2}{3}}}, \frac{x}{t^{\frac{1}{3}}} + \frac{y^2}{t^{\frac{4}{3}}}\right)e^{\left(\frac{2y^3}{3t^2}+\frac{yx}{t}\right)E_1}$$

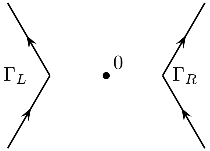
Riemann-Hilbert problem for KP equation

1. $X(z)$ is analytic outside of $\Gamma_L \cup \Gamma_R$.
2. $X_+(z) = X_-(z)J(z)$, $z \in \Gamma_L \cup \Gamma_R$, where

$$J(z) = \begin{pmatrix} 0 & e^{\frac{tz^3}{3} - yz^2 - sz}\chi_{\Gamma_R}(z) \\ e^{-\frac{tz^3}{3} + yz^2 + sz}\chi_{\Gamma_L}(z) & 0 \end{pmatrix}$$

3. $X(z) = I + \frac{X_1}{z} + \frac{X_2}{z^2} + O(z^{-3}), \quad z \to \infty$

$$(X_1)_{11} = \partial_x \ln F(t, y, x)$$



Isomonodromic deformations

- The function $\Phi(z) = X(z)e^{\left(\frac{tz^3}{3} - yz^2 - xz\right)E_1}$ has constant jump and satisfies the Lax equations

$$\partial_t \Phi \Phi^{-1} = \frac{1}{3}(z^3 E_1 + z^2 [Y_1, E_1] - z\partial_x Y_1 - \partial_y Y_1), \tag{4}$$

$$\partial_y \Phi \Phi^{-1} = -z^2 E_1 + z[Y_1, E_1] - \partial_x Y_1, \tag{5}$$

$$\partial_x \Phi \Phi^{-1} = -zE_1 - [Y_1, E_1]. \tag{6}$$

Nonlinear PDEs

- We choose the notation

$$X_1 = \begin{pmatrix} r & u \\ v & m \end{pmatrix}$$

- Using compatibility condition of equations (4) and (5) one can get coupled nonlinear heat equation

$$\partial_y u = \partial_x^2 u + 2u^2 v$$
$$\partial_y v = -\partial_x^2 v - 2v^2 u$$

- Using compatibility condition of equations (4) and (6) one can get coupled mKdV equation

$$3\partial_t u + \partial_x^3 u + 6uv\partial_x u = 0$$
$$3\partial_t v + \partial_x^3 v + 6uv\partial_x v = 0$$

Nonlinear PDEs

- The function $w = uv = \partial_x^2 \ln F(t, y, x)$ satisfies KPII equation

$$3\partial_y^2 w + \partial_x (12\partial_t w + 12w\partial_x w + \partial_x^3 w) = 0.$$

- The function $F(t, y, x)$ is KP tau function and it satisfies Hirota bilinear equation

$$3F\partial_y^2 F - 3(\partial_y F)^2 + 12F\partial_t\partial_x F - 12\partial_t F\partial_x F + F\partial_x^4 F - 4\partial_x^3 F\partial_x F = 0.$$

Two point distribution for the scaling limit

Two point distribution for the scaling limit

Consider for $y_2 > y_1$

$$\lim_{\substack{T \to \infty \\ N > 2y_2 T^{\frac{2}{3}} + tT + xT^{\frac{1}{3}}}} Pr\left(\frac{H(2y_1 T^{\frac{2}{3}}, 2tT) - tT}{-T^{\frac{1}{3}}} < x_1, \frac{H(2y_2 T^{\frac{2}{3}}, 2tT) - tT}{-T^{\frac{1}{3}}} < x_2\right) = F(t, y_1, y_2, x_1, x_2)$$

Theorem (Prähofer, Spohn (2002))

$$F(t, y_1, y_2, x_1, x_2) = G\left(\frac{y_2 - y_1}{t^{\frac{2}{3}}}, \frac{x_1}{t^{\frac{1}{3}}} + \frac{y_1^2}{t^{\frac{4}{3}}}, \frac{x_2}{t^{\frac{1}{3}}} + \frac{y_2^2}{t^{\frac{4}{3}}}\right)$$

where $G(s, u, v)$ is the two point distribution of the Airy_2-process

$$G(s, u, v) = Pr(\mathcal{A}(0) < u, \mathcal{A}(s) < v).$$

Extended Airy kernel

$G(s,u,v) = \det(1 - \chi_{\vec{I}} A_2)$, where the operator acts on $L_2(\mathbb{R}, \mathbb{C}^2) \oplus L_2(\mathbb{R}, \mathbb{C}^2)$ with the matrix kernel

$$(A_2)_{1,1}(x,y) = (A_2)_{2,2}(x,y) = A_1(x,y) = \int_0^\infty \mathrm{Ai}(u+z)\mathrm{Ai}(v+z)dz,$$

$$(A_2)_{1,2}(x,y) = -\int_{-\infty}^{0} e^{zs}\mathrm{Ai}(x+z)\mathrm{Ai}(y+z)dz,$$

$$(A_2)_{2,1}(x,y) = \int_0^\infty e^{-zs}\mathrm{Ai}(x+z)\mathrm{Ai}(x+z)dz$$

and $\chi_{\vec{I}}$ is the indicator function of the multiinterval $\{[u,\infty),[v,\infty)\}$.

Integrable operator

Theorem (Bertola,Cafasso, (2012), Physica D)

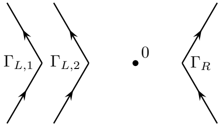
$$\det(1-\chi_{\vec{I}}A_2)=\det(1-K_{A_2})$$

where the operator acts on $L_2(\Gamma_R\cup\Gamma_{L,1}\cup\Gamma_{L,2},\mathbb{C}^2)$ *with matrix kernel*

$$K_{A_2}(\lambda,\mu)=\frac{\vec{f}^T(\lambda)\vec{g}(\mu)}{\lambda-\mu},$$

$$\vec{f}(\lambda)=\frac{1}{2\pi i}\begin{pmatrix} e^{\frac{\lambda^3}{6}}\chi_{\Gamma_R}(\lambda) & e^{\frac{(\lambda-s)^3}{6}}\chi_{\Gamma_R}(\lambda)\\ e^{u\lambda}\chi_{\Gamma_{L,1}}(\lambda) & 0\\ 0 & e^{v(\lambda-s)}\chi_{\Gamma_{L,2}}(\lambda)\end{pmatrix},$$

$$\vec{g}(\lambda)=\begin{pmatrix} e^{-\frac{\lambda^3}{3}}\chi_{\Gamma_{L,1}}(\lambda) & e^{-\frac{(\lambda-s)^3}{3}}\chi_{\Gamma_{L,2}}(\lambda)\\ e^{\frac{\lambda^3}{6}-u\lambda}\chi_{\Gamma_R}(\lambda) & e^{\frac{\lambda^3}{3}-\frac{(\lambda-s)^3}{3}-u\lambda}\chi_{\Gamma_{L,2}}(\lambda)\\ 0 & e^{\frac{(\lambda-s)^3}{6}-v(\lambda-s)}\chi_{\Gamma_R}(\lambda)\end{pmatrix}.$$



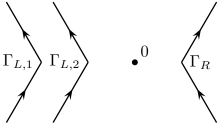
Riemann-Hilbert problem

1. $Y(\lambda)$ is analytic outside of $\Gamma_R \cup \Gamma_{L,1} \cup \Gamma_{L,2}$.
2. $Y(\lambda) = I + \frac{Y_1}{\lambda} + \frac{Y_2}{\lambda^2} + O(\lambda^{-3}), \quad \lambda \to \infty$
3. $Y_+(\lambda) = Y_-(\lambda) J(\lambda)$, $\lambda \in \Gamma_R \cup \Gamma_{L,1} \cup \Gamma_{L,2}$,

where

$$J(\lambda) = \begin{pmatrix} 1 & e^{\frac{\lambda^3}{3} - u\lambda} \chi_{\Gamma_R}(\lambda) & e^{\frac{(\lambda - s)^3}{3} - v(\lambda - s)} \chi_{\Gamma_R}(\lambda) \\ e^{-\frac{\lambda^3}{3} + u\lambda} \chi_{\Gamma_{L,1}}(\lambda) & 1 & 0 \\ e^{-\frac{(\lambda - s)^3}{3} + v(\lambda - s)} \chi_{\Gamma_{L,2}}(\lambda) & 0 & 1 \end{pmatrix}$$

$$(Y_1)_{22} = \partial_u \ln G(s, u, v), \quad (Y_1)_{33} = \partial_v \ln G(s, u, v)$$



Isomonodromic deformations

- Consider $\Psi(\lambda) = Y(\lambda) e^{\left(\frac{\lambda^3}{3} - u\lambda\right) E_2 + \left(\frac{(\lambda - s)^3}{3} - v(\lambda - s)\right) E_3}$.
- It has constant jumps
- Consider the transformation:

$$\Psi(\lambda, s, u, v) = P\Phi(-\lambda, s, u, v - s^2) P^{-1} e^{\left(-\frac{4s^3}{3} + vs\right) E_3} =$$

where

$$P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

- Function $\Phi(\lambda)$ still has constant jumps and it has asymptotic

$$\Phi(\lambda, s, u, w) \simeq e^{\left(-\frac{\lambda^3}{3} + u\lambda\right) E_1 + \left(-\frac{\lambda^3}{3} + s\lambda^2 + w\lambda\right) E_2}, \quad \lambda \to \infty$$

Isomonodromic deformations

- One of the Lax equations for $\Phi(\lambda)$ has form

$$\partial_\lambda \Phi \Phi^{-1} = \lambda^2 R_2 + \lambda R_1 + R_0.$$

- We choose parametrization

$$R_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 0 & 0 & q_2 \\ 0 & 2s & q_3 \\ -q_5 & -q_4 & 0 \end{pmatrix},$$

$$R_0 = \begin{pmatrix} -q_2 q_5 + u & -q_2 q_4 + p_1 & p_5 \\ -2sq_1 - q_5 q_3 & -q_3 q_4 + w & -sq_3 + p_4 \\ p_2 & sq_4 + p_3 & q_2 q_5 + q_3 q_4 \end{pmatrix}$$

Hamiltonians

Introduce polynomial Hamiltonians

$$H^{(u)} = -\frac{p_1p_2q_3}{2s} + \frac{p_1p_4q_5}{2s} - \frac{up_1q_1}{2s} + \frac{wp_1q_1}{2s} - \frac{p_1q_3q_5}{2} + p_2p_5 + p_3q_1q_2$$
$$-p_5q_1q_4 + sq_1q_2q_4 + q_2^2q_5^2 + q_2q_3q_4q_5 - uq_2q_5$$
$$H^{(w)} = \frac{p_1p_2q_3}{2s} - \frac{p_1p_4q_5}{2s} + \frac{up_1q_1}{2s} - \frac{wp_1q_1}{2s} - \frac{p_1q_3q_5}{2} + p_3p_4 - p_3q_1q_2$$
$$+sp_3q_3 - sp_4q_4 + p_5q_1q_4 + sq_1q_2q_4 + q_2q_3q_4q_5 + q_3^2q_4^2 - wq_3q_4 - s^2q_3q_4$$

Hamiltonians

$$H^{(s)} = -p_1q_1q_3q_4 - 2sq_3^2q_4^2 - 2s^2p_3q_3 + \frac{p_1q_1}{2s} + 2s^3q_3q_4 + 2s^2p_4q_4 - 2sp_3p_4 - wp_3q_3 + wp_4q_4 + \frac{q_2p_3q_5q_3}{2} - \frac{q_1p_3q_3p_1}{2s} + \frac{q_1p_4q_4p_1}{2s}$$
$$+ \frac{p_4q_5up_1}{4s^2} - \frac{p_4q_5wp_1}{4s^2} - \frac{p_4q_5q_2p_3}{2s} + \frac{p_4q_5p_5q_4}{2s} - \frac{p_5q_4uq_1}{2s} + \frac{p_5q_4wq_1}{2s} - \frac{p_5q_4q_3p_2}{2s} + \frac{up_1wq_1}{2s^2} - \frac{up_1q_3p_2}{4s^2} - \frac{p_2q_3q_2q_4}{2} + \frac{wp_1q_5q_3}{4s}$$
$$+ \frac{wp_1q_3p_2}{4s^2} + \frac{q_2p_3uq_1}{2s} - \frac{q_2p_3wq_1}{2s} + \frac{q_2p_3q_3p_2}{2s} - \frac{3sq_2q_3q_4q_5}{2} - \frac{p_1q_2q_3q_5^2}{2s} - \frac{p_1q_3^2q_4q_5}{2s} + \frac{up_1q_3q_5}{4s} + \frac{p_1p_2q_1q_2}{2s} - \frac{p_1p_5q_1q_5}{2s}$$
$$- \frac{p_1p_2q_3}{2} + p_1p_4q_5 + p_3p_5q_1 - \frac{u^2p_1q_1}{4s^2} - \frac{w^2p_1q_1}{4s^2} - \frac{uq_1q_2q_4}{2} - 2s^2q_1q_2q_4 + 2swq_3q_4 + sp_1q_3q_5 + 2sp_3q_1q_2 - sp_5q_1q_4 - \frac{p_1p_2p_4}{2s}$$
$$- \frac{wq_1q_2q_4}{2} + q_1q_2^2q_4q_5 + q_1q_2q_3q_4^2 - p_1q_1q_2q_5 - \frac{p_4q_2q_4q_5}{2} + \frac{p_5q_3q_4q_5}{2} - \frac{q_1^2p_1^2}{2s}$$

Hamiltonian system

Theorem (Baik,Prokhorov,Silva, in progress)

The equations of isomonodromic deformations form Hamiltonian system

$$\frac{\mathrm{d}p_i}{\mathrm{d}u} = \frac{\mathrm{d}H^{(u)}}{\mathrm{d}q_i}, \quad \frac{\mathrm{d}q_i}{\mathrm{d}u} = -\frac{\mathrm{d}H^{(u)}}{\mathrm{d}p_i},$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}w} = \frac{\mathrm{d}H^{(w)}}{\mathrm{d}q_i}, \quad \frac{\mathrm{d}q_i}{\mathrm{d}w} = -\frac{\mathrm{d}H^{(w)}}{\mathrm{d}p_i}, \quad i = 1 \dots 5.$$
$$\frac{\mathrm{d}p_i}{\mathrm{d}s} = \frac{\mathrm{d}H^{(s)}}{\mathrm{d}q_i}, \quad \frac{\mathrm{d}q_i}{\mathrm{d}s} = -\frac{\mathrm{d}H^{(s)}}{\mathrm{d}p_i}$$

This dynamic has two integrals of motion

$$-p_1 q_1 + q_2 p_2 - q_5 p_5 = 0, \quad p_1 q_1 + p_3 q_3 - p_4 q_4 = 0.$$

Hamiltonian system

Theorem (Baik,Prokhorov,Silva, in progress)

The function $G(s,u,w+s^2)$ *is the isomonodromic tau function and it satisfies the differential identities*

$$\partial_u \ln G(s,u,w+s^2) = -H^{(u)}$$
$$\partial_w \ln G(s,u,w+s^2) = -H^{(w)}$$
$$\partial_s \ln G(s,u,w+s^2) = H^{(s)} + \frac{p_1 q_1}{2s}$$

Nonlinear PDEs

Theorem (Adler, van Moerbeke, 2005)

The function $M = \ln G(s, u, v)$ *satisfies the equation*

$$(v-u)\partial_u\partial_v(\partial_u+\partial_v)M + s\partial_s(\partial_u^2-\partial_v^2)M + s^2\partial_u\partial_v(\partial_u-\partial_v)M$$

$$+\partial_u(\partial_u+\partial_v)M\partial_v(\partial_u+\partial_v)^2M - \partial_v(\partial_u+\partial_v)M\partial_u(\partial_u+\partial_v)^2M = 0$$

Theorem (Quastel, Remenik, 2021)

The function $M = \ln G(s, u, v)$ *satisfies the equation*

$$-4(v\partial_v+u\partial_u)(\partial_u+\partial_v)M - 8s\partial_s(\partial_u+\partial_v)^2M - 12s^2\partial_u\partial_v(\partial_u+\partial_v)M$$

$$+12(\partial_u+\partial_v)^3M(\partial_u+\partial_v)^2M + (\partial_u+\partial_v)^5M - 2(\partial_u+\partial_v)^2M = 0$$

Multipoint distribution for the scaling limit

Multipoint distribution for the scaling limit

Consider for $y_1 < y_2 < \ldots < y_m$

$$\lim_{\substack{T \to \infty \\ N > 2y_m T^{\frac{2}{3}} + tT + xT^{\frac{1}{3}}}} Pr\left(\bigcap_{k=1}^{m}\left(\frac{H(2y_k T^{\frac{2}{3}}, 2tT) - tT}{-T^{\frac{1}{3}}} < x_k\right)\right) = F(t, y_1, \ldots, y_m, x_1, \ldots, x_m)$$

Theorem (Prähofer, Spohn (2002))

$$F(t, y_1, \ldots, y_m, x_1, \ldots, x_m)$$

$$= G\left(\frac{y_2 - y_1}{t^{\frac{2}{3}}}, \ldots, \frac{y_m - y_1}{t^{\frac{2}{3}}}, \frac{x_1}{t^{\frac{1}{3}}} + \frac{y_1^2}{t^{\frac{4}{3}}}, \ldots, \frac{x_m}{t^{\frac{1}{3}}} + \frac{y_m^2}{t^{\frac{4}{3}}}\right)$$

where $G(s_1, \ldots, s_{m-1}, u, v_1, \ldots, v_{m-1})$ *is the multipoint distribution of the* Airy_2*-process*

$$G(s_1, \ldots, s_{m-1}, u, v_1, \ldots, v_{m-1})$$

$$= Pr(\mathcal{A}(0) < u, \mathcal{A}(s_1) < v_1, \ldots, \mathcal{A}(s_{m-1}) < v_{m-1}).$$

Extended Airy kernel

$G(s_1, \ldots, s_{m-1}, u, v_1, \ldots, v_{m-1}) = \det(1 - \chi_{\vec{I}} A_m)$, where the operator acts on $\bigoplus_{k=1}^{m} L_2(\mathbb{R}, \mathbb{C}^m)$ with the matrix kernel

$$(A_m)_{i,j}(x,y) = \begin{cases} \int_0^{\infty} e^{-z(s_i - s_j)} \mathrm{Ai}(u+z)\mathrm{Ai}(v+z)dz, & i < j \\ -\int_{-\infty}^{0} e^{-z(s_i - s_j)} \mathrm{Ai}(u+z)\mathrm{Ai}(v+z)dz, & i > j \end{cases}$$

$$i, j = 1 \ldots m.$$

and $\chi_{\vec{I}}$ is the indicator function of the multiinterval $\{[u, \infty), [v_1, \infty), \ldots, [v_m, \infty)\}$.

Integrable operator

Theorem (Bertola,Cafasso, (2012), Physica D)

$$\det(1 - \chi_{\vec{I}} A_m) = \det(1 - K_{A_m})$$

where the operator acts on $L_2(\Gamma_R \cup \Gamma_{L,1} \cup \ldots \cup \Gamma_{L,m}, \mathbb{C}^m)$ *with matrix kernel*

$$K_{A_m}(\lambda, \mu) = \frac{\vec{f}^T(\lambda)\vec{g}(\mu)}{\lambda - \mu},$$

Riemann-Hilbert problem for Painlevé-II equation

1. $Y(\lambda)$ is analytic outside of $\Gamma_R \cup \Gamma_{L,1} \cup \Gamma_{L,2}$.
2. $Y(\lambda) = I + \frac{Y_1}{\lambda} + \frac{Y_2}{\lambda^2} + O(\lambda^{-3}), \quad \lambda \to \infty$
3. $Y_+(\lambda) = Y_-(\lambda)J(\lambda)$, $\lambda \in \Gamma_{L,1} \cup \ldots \cup \Gamma_{L,m}$,

$$(Y_1)_{22} = \partial_u \ln G(s_1, \ldots, s_{m-1}, u, v_1, \ldots, v_{m-1}),$$

$$(Y_1)_{2+k,2+k} = \partial_{v_k} \ln G(s_1, \ldots, s_{m-1}, u, v_1, \ldots, v_{m-1})$$

Equations

- Tracy,Widom,2003 - Matrix Painlevé-II
- Quastel, Remenik, 2019 - Matrix KP
- Wang, 2009 - Nonlinear PDE for $G(s_1, \ldots, s_{m-1}, u, v_1, \ldots, v_{m-1})$

Multitime distribution for the scaling limit

Multitime distribution for the scaling limit

Consider for $y_1 < y_2 < \ldots < y_m$, $t_1 < t_2 < \ldots < t_m$

$$\lim_{\substack{T \to \infty \\ N > 2y_m T^{\frac{2}{3}} + t_m T + xT^{\frac{1}{3}}}} Pr\left(\bigcap_{k=1}^{m} \left(\frac{H(2y_k T^{\frac{2}{3}}, 2t_k T) - t_k T}{-T^{\frac{1}{3}}} < x_k \right) \right) = F(t_1, \ldots, t_m, y_1, \ldots, y_m, x_1, \ldots, x_m)$$

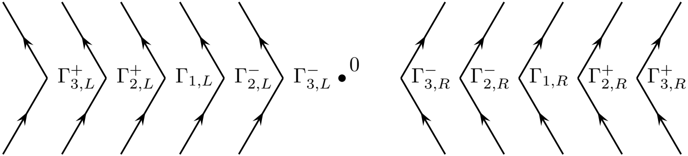
Theorem (Liu,2019)

$$F(t_1, \ldots, t_m, y_1, \ldots, y_m, x_1, \ldots, x_m)$$
$$= \oint_{|\zeta_1|=\varepsilon} \cdots \oint_{|\zeta_{m-1}|=\varepsilon} \mathrm{D}(\zeta) \left(\prod_{i=1}^{m-1} \frac{1}{1-\zeta_i} \right) \prod_{i=1}^{m-1} \frac{d\zeta_i}{2\pi i \zeta_i}$$

$\mathrm{D}(\zeta)$ - *Fredholm determinant.*

Riemann-Hilbert problem

1. $X(z)$ is $(m+1)\times(m+1)$ matrix valued function analytic outside of Σ.
2. $X_+(z) = X_-(z)J(z)$, $z \in \Sigma$.
3. $X(z) = I + \frac{X_1}{z} + O(z^{-2}), \quad z \to \infty$



Contour Σ.

$$\partial_{x_i} \ln D(\zeta) = -(X_1)_{ii}$$

Riemann-Hilbert problem

Lemma

Matrix $\Delta(z)^{-1}J(z)\Delta(z)$ *is independent on* z. *Here we used*

$$\begin{aligned}\Delta(z) &= \operatorname{diag}\begin{pmatrix}\mathsf{F}_1(z) & \mathsf{F}_2(z) & \cdots & \mathsf{F}_m(z) & 1\end{pmatrix} \\ &= \operatorname{diag}\begin{pmatrix}e^{\mathsf{t}_1 z^3+\mathsf{y}_2 z^2+\mathsf{x}_1 z} & e^{\mathsf{t}_2 z^3+\mathsf{y}_2 z^2+\mathsf{x}_2 z} & \dots & e^{\mathsf{t}_m z^3+\mathsf{y}_m z^2+\mathsf{x}_m z} & 1\end{pmatrix}\end{aligned}$$

Matrix KP

Introduce operators

$$D_t{}^{(s)} = \sum_{j=1}^{s} \partial_{t_j}, \quad D_x{}^{(s)} = \sum_{j=1}^{s} \partial_{x_j} \quad D_y{}^{(s)} = \sum_{j=1}^{s} \partial_{y_j}, \quad s = 1 \dots m$$

Theorem (Baik, Prokhorov, Silva (in preparation))

$$3D_y{}^{(s)} D_y{}^{(s)} \mathrm{u}^{(s)} + D_x{}^{(s)} \Big(-4D_t \mathrm{u}^{(s)} + D_x{}^{(s)} D_x{}^{(s)} D_x{}^{(s)} \mathrm{u}^{(s)} \\ +6(D_x{}^{(s)} \mathrm{u}^{(s)})\mathrm{u}^{(s)} + 6\mathrm{u}^{(s)}(D_x{}^{(s)} \mathrm{u}^{(s)}) + 6[\mathrm{u}^{(s)}, D_y{}^{(s)} \mathrm{q}^{(s)}] \Big) = 0$$

where

$$D_x{}^{(s)}(\ln D(\zeta)) = -\mathrm{Tr}(q^{(s)}), \quad D_x{}^{(s)} q^{(s)} = -u^{(s)},$$

$$\mathrm{q}^{(s)}, \mathrm{u}^{(s)} - (s \times s) \textit{ matrices.}$$

Matrix Painlevé II

Introduce operator $\mathrm{f}' = \sum_{j=1}^{m} \mathrm{t}_j \partial_{\mathrm{x}_j} \mathrm{f}$.

Theorem (Baik, Prokhorov, Silva (in preparation))

$$
\begin{aligned}
&3\mathrm{q}'' - 2[\mathrm{q}', \mathrm{y}] - 3([[\mathrm{q}, \mathrm{t}], \mathrm{q}'] - \mathrm{tpr}' - \mathrm{p}'\mathrm{rt}) \\
&+ 2([[\mathrm{q}, \mathrm{t}], [\mathrm{q}, \mathrm{y}]] - \mathrm{tpry} + \mathrm{yprt}) + [[\mathrm{q}, \mathrm{t}], \mathrm{x}] = 0 \\
&3\mathrm{p}'' + 2\mathrm{yp}' - 3([\mathrm{q}, \mathrm{t}]\mathrm{p}' - \mathrm{tprtp} + \mathrm{q}'\mathrm{tp}) + 2(-[\mathrm{q}, \mathrm{t}]\mathrm{yp} + [\mathrm{q}, \mathrm{y}]\mathrm{tp}) \\
&+ \mathrm{xtp} = 0 \\
&3\mathrm{r}'' - 2\mathrm{r}'\mathrm{y} - 3(\mathrm{rtq}' - \mathrm{r}'[\mathrm{q}, \mathrm{t}] - \mathrm{rtprt}) + 2(\mathrm{rt}[\mathrm{q}, \mathrm{y}] - \mathrm{ry}[\mathrm{q}, \mathrm{t}]) + \mathrm{rtx} = 0
\end{aligned}
$$

where

$$(\ln D(\zeta))' = -\mathrm{Tr}(\mathrm{t}q), \quad \mathrm{q}, \mathrm{p}, \mathrm{r} - (m \times m), (m \times 1), (1 \times m) \textit{matrices.}$$

Multicomponent KP tau function

Multicomponent KP tau function can be found in (Kac, Van-De Leur (2003)).

Theorem (Baik, Prokhorov, Silva (in preparation))

The Fredholm determinant $D(\zeta)$ coincides up to a constant factor with multicomponent KP tau function.

Thank you!