Lozenge tilings and Algebraic Combinatorics

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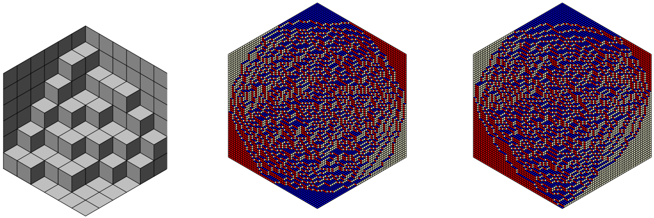
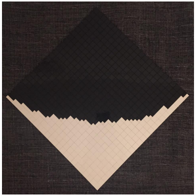
Motivation

1. Lozenge tilings – GUE, LLN, LLT



2. Asymptotic Algebraic Combinatorics and Representation Theory

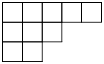


(©Dan Betea, IHP Paris)

Standard Young Tableaux

Integer partitions $\lambda \vdash n$:

$$\lambda = (\lambda_1, \ldots, \lambda_\ell),\ \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_\ell > 0,\ \lambda_1 + \lambda_2 + \cdots = n$$

Young diagram of λ: 

SYT of shape $\lambda = (\lambda_1, \ldots, \lambda_k)$:
$T : \lambda \xrightarrow{\sim} \{1, \ldots, n\}$ and $T_{i,j} < T_{i,j+1}, T_{i+1,j}$

$$T = \wedge \begin{array}{|c|c|c|c|c|} \multicolumn{5}{c}{\qquad\quad <} \\ \hline 1 & 3 & 4 & 7 & 10 \\ \hline 2 & 5 & 8 & \multicolumn{2}{c}{} \\ \cline{1-3} 6 & 9 & \multicolumn{3}{c}{} \\ \cline{1-2} \end{array}$$

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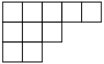
Young diagram of λ: 

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$$T = \wedge \begin{array}{ccccc} & & < & & \\ \hline 1 & 3 & 4 & 7 & 10 \\ 2 & 5 & 8 & & \\ 6 & 9 & & & \end{array}$$

$\lambda = (3, 2, 1)$:

$$\begin{array}{ccc}1&2&3\\4&5&\\6&&\end{array} \quad \begin{array}{ccc}1&2&3\\4&6&\\5&&\end{array} \quad \begin{array}{ccc}1&2&4\\3&5&\\6&&\end{array} \quad \begin{array}{ccc}1&2&4\\3&6&\\5&&\end{array} \quad \begin{array}{ccc}1&2&5\\3&4&\\6&&\end{array} \quad \begin{array}{ccc}1&2&5\\3&6&\\4&&\end{array} \quad \begin{array}{ccc}1&2&6\\3&4&\\5&&\end{array} \quad \begin{array}{ccc}1&2&6\\3&5&\\4&&\end{array} \quad + \textit{all transposed}$$

Standard Young Tableaux

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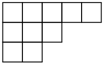
Young diagram of λ: 

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Hook-length formula [Frame-Robinson-Thrall]:

$$f^\lambda := \#\{\text{SYTs of shape } \lambda\} = \frac{|\lambda|!}{\prod_{u \in \lambda} h_u} = \frac{6!}{5 * 3 * 3 * 1 * 1 * 1} = 16$$

Hook length of box $u = (i, j) \in \lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#\blacksquare \in$ 

Schur functions

V_λ – irreducible $GL_N(\mathbb{C})$ module, weight λ

$$s_\lambda(x_1, \ldots, x_N) = \chi_{V_\lambda}\left(\begin{bmatrix} x_1 & 0 & \cdots \\ 0 & x_2 & \cdots \\ \vdots & \ddots & \cdots \end{bmatrix}\right)$$

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Weyl character formula:

$$s_\lambda(x_1,\ldots,x_N) := \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{i<j}(x_i - x_j)}$$

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Jacobi-Trudi identity:

$$s_{\lambda_1,\ldots,\lambda_k} = \det\begin{bmatrix} h_{\lambda_1} & h_{\lambda_1+1} & \cdots & h_{\lambda_1+k-1} \\ h_{\lambda_2-1} & h_{\lambda_2} & \cdots & h_{\lambda_2+k-2} \\ \vdots & \ddots & h_{\lambda_i+k-j} & \vdots \end{bmatrix}_{i,j=1}^k$$

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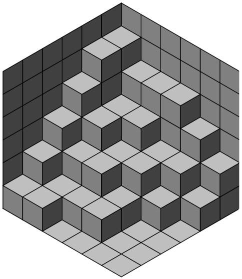
Semi-Standard Young tableaux of shape λ :

$$s_{(2,2)}(x_1,x_2,x_3) = \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&2\\\hline\end{array}}{x_1^2x_2^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 3&3\\\hline\end{array}}{x_1^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 2&2\\\hline 3&3\\\hline\end{array}}{x_2^2x_3^2} + \underset{\begin{array}{|c|c|}\hline 1&1\\\hline 2&3\\\hline\end{array}}{x_1^2x_2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 2&3\\\hline\end{array}}{x_1x_2^2x_3} + \underset{\begin{array}{|c|c|}\hline 1&2\\\hline 3&3\\\hline\end{array}}{x_1x_2x_3^2}.$$

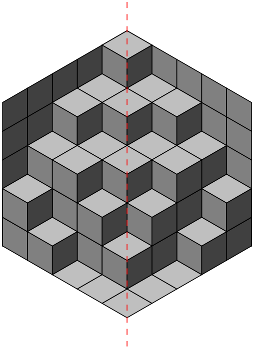
Unrestricted and symmetric lozenge tilings

Tilings of the hexagon $a \times b \times c \times a \times b \times c$, s.t.

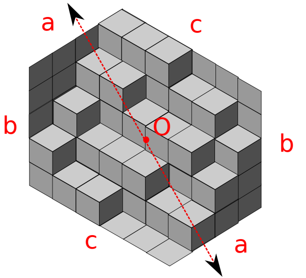
Unrestricted



Vertically symmetric



Centrally symmetric

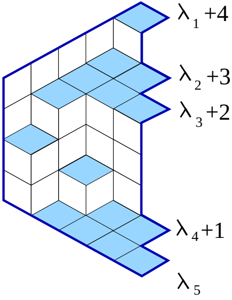


Limit behavior: fluctuations near the boundary (GUE), limit surface, CLT?

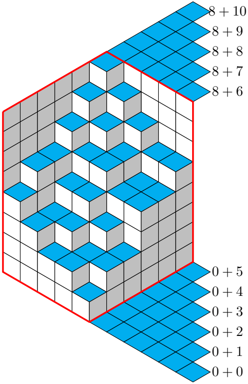
The Schur generating function: domain setup

Domain $\Omega_{\lambda(N)}$:
positions of the N horizontal lozenges on right boundary are:

$$\lambda_1(N) + N - 1 > \lambda_2(N) + N - 2 > \cdots > \lambda_N(N)$$

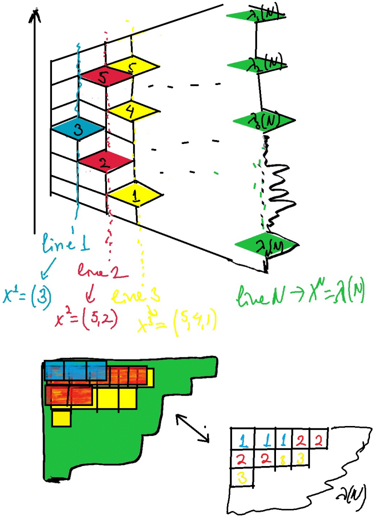


$\lambda(5) = (4, 3, 3, 0, 0)$



$\lambda = (\underbrace{a, \ldots, a}_{c}, \underbrace{0, \ldots, 0}_{b})$
$\leftrightarrow a \times b \times c$... hexagon.

Tilings probability: skew SSYTs



Lozenge tilings with right boundary $\lambda(N)$

$\Longleftrightarrow$

Semi-Standard Young Tableaux T of shape $\lambda(N)$ and entries $1, \ldots, N$.

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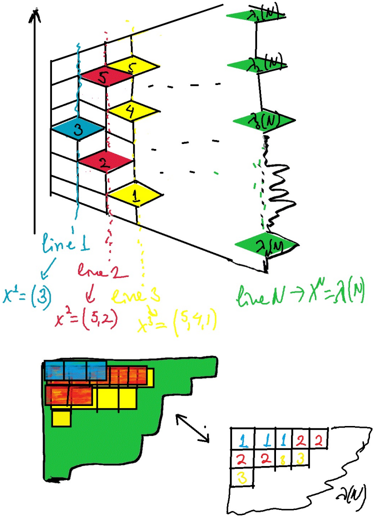

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Tilings with horizontal lozenges on vertical line k at positions $x^k = (\eta_1, \ldots, \eta_k) = \eta$

$\iff$

SSYTs T whose entries $1..k$ have shape η

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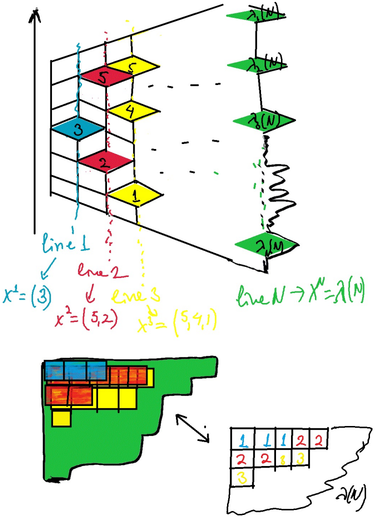

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$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

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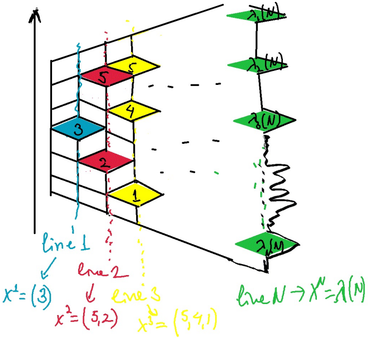

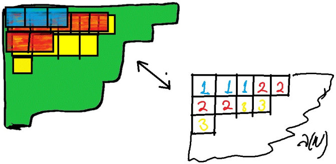

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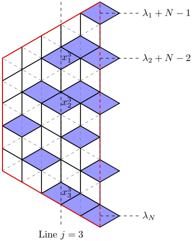
$$\mathrm{Prob}\{x^k(\lambda) = \eta\} = \frac{s_\eta(1^k)s_{\lambda/\eta}(1^{N-k})}{s_\lambda(1^N)},$$

Proposition[Gorin-P'2013] For any variables $y_1, \ldots, y_k$, the **Schur Generating Function** of x^k is $S_\lambda(y_1, \ldots, y_k) :=$

$$\mathbb{E}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right) = \frac{s_\lambda(y_1,\ldots,y_k,\overbrace{1,\ldots,1}^{N-k})}{s_\lambda(\underbrace{1,\ldots,1}_{N})}$$

The explicit Schur Generating Functions[1]



$\mathcal{T}_n$ – set of tilings, $x^j(T)$ – horizontal lozenge positions on line j of $T \in \mathcal{T}_n$

[1] from [Gorin-P'2013], [P, 2014, 2015]

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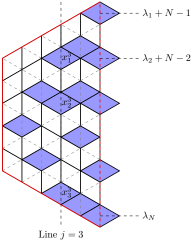

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$$\mathbb{E}\left[\frac{s_{x^k(T)}(y_1,\ldots,y_k)}{s_{x^k(T)}(\underbrace{1,\ldots,1}_{k})} \;\middle|\; T \sim Unif(\mathcal{T}_n) \right]$$

$$= \sum_{\nu} \frac{s_\nu(y_1,\ldots,y_k)}{s_\nu(1^k)} \Pr(x^k(T) = \nu) = \ldots$$

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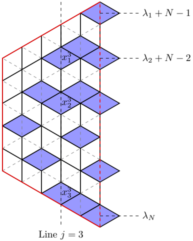

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$$= \sum_{\nu} \frac{s_\nu(y_1, \ldots, y_k)}{s_\nu(1^k)} \Pr(x^k(T) = \nu) = \ldots$$

- $= S_{\lambda(n)}(y_1, \ldots, y_k) = \frac{s_{\lambda(n)}(y_1, \ldots, y_k, 1^{n-k})}{s_{\lambda(n)}(1^n)}$ for $\mathcal{T}_n = \Omega_{\lambda(n)}$.
- $= \prod_i y_i^{m/2} \cdot \frac{\mathfrak{so}_{(\frac{m}{2})^n}(y_1, \ldots, y_k, 1^{n-k})}{\mathfrak{so}_{(\frac{m}{2})^n}(1^n)}$ for $\mathcal{T}_n$ – symmetric tilings of $n \times m \times n$....
- $= S_{(\frac{b}{2})^{a/2}}(y_1, \ldots, y_k)^2$ for $\mathcal{T}_n$ – centrally symmetric tilings of $a \times b \times c$... hexagon.

[1] from [Gorin-P'2013], [P, 2014, 2015]

MGF asymptotics

Proposition (Gorin-P'2013)

$$\mathbb{E}_{\nu\sim\mathbb{GUE}_k}\left[\frac{s_{\nu-\delta_k}(y_1,\ldots,y_k)}{s_{\nu-\delta_k}(\underbrace{1,\ldots,1}_{k})}\right]=\exp\left(\frac{1}{2}(y_1^2+\cdots+y_k^2)\right),$$

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$$\mathbb{E}_{\text{tiling of }\Omega_\lambda(N)}\left(\frac{s_{x^k}(y_1,\ldots,y_k)}{s_{x^k}(\underbrace{1,\ldots,1}_{k})}\right)=\frac{s_{\lambda(N)}(y_1,\ldots,y_k,1^{N-k})}{s_{\lambda(N)}(1^N)}=:S_{\lambda(N)}(y_1,\ldots,y_k)$$

Proposition (Gorin-P'2013)

For any k real numbers $h_1,\ldots,h_k$ and $\lambda(N)/N\to f$ we have:

$$\lim_{N\to\infty} S_{\lambda(N)}\left(e^{\frac{h_1}{\sqrt{NS(f)}}},\ldots,e^{\frac{h_k}{\sqrt{NS(f)}}}\right)e^{\left(-\frac{E(f)}{\sqrt{NS(f)}}\sum_{i=1}^k h_i\right)} \quad = \quad \exp\left(\frac{1}{2}\sum_{i=1}^k h_i^2\right).$$

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Theorem (Gorin-P'2013)

Let $\Upsilon^k_{\lambda(N)} = \{x^k, x^{k-1},\ldots\}$ –collection of positions of the horizontal lozenges on lines $k, k-1,\ldots,1$ of tiling from $\Omega_{\lambda(N)}$, then

$$\frac{\Upsilon^k_{\lambda(N)} - NE(f)}{\sqrt{NS(f)}} \to \mathbb{GUE}_k \text{ (GUE-corners process of rank } k\text{)}.$$

Asymptotics of normalized Schur functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})}$$

Theorem [Gorin-P'2013] For every partition λ and any $x \in \mathbb{C} \setminus \{0,1\}$ we have

$$S_\lambda(x;N,1) = \frac{(N-1)!}{(x-1)^{N-1}} \frac{1}{2\pi\mathbf{i}} \oint_C \frac{x^z}{\prod_{i=1}^N (z-(\lambda_i+N-i))} dz,$$

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Theorem[Gorin-P'2013] If $\frac{\lambda_i(N)}{N} \to f\left(\frac{i}{N}\right)$ [...], for all fixed $y \neq 0$:

$$\lim_{N\to\infty} \frac{1}{N} \ln S_{\lambda(N)}(e^y;N,1) = yw_0 - \mathcal{F}(w_0) - 1 - \ln(e^y-1),$$

where $\mathcal{F}(w;f) = \int_0^1 \ln(w - f(t) - 1 + t)dt$, w_0 – root of $\frac{\partial}{\partial w}\mathcal{F}(w;f) = y$.
If $\frac{\lambda_i(N)}{N} \to f\left(\frac{i}{N}\right)$ [...], for any fixed $h \in \mathbb{R}$:

$$S_{\lambda(N)}(e^{h/\sqrt{N}};N,1) = \exp\left(\sqrt{N}E(f)h + \frac{1}{2}S(f)h^2 + o(1)\right),$$

where $E(f) = \int_0^1 f(t)dt, \quad S(f) = \int_0^1 (f(t)-t+1/2)^2 dt - 1/6 - E(f)^2.$

Asymptotics of normalized Schur functions

$$S_{\lambda(N)}(x_1,\ldots,x_k) := \frac{s_{\lambda(N)}(x_1,\ldots,x_k,\overbrace{1,\ldots,1}^{N-k})}{s_{\lambda(N)}(\underbrace{1,\ldots,1}_{N})}$$

Multivariate: [Gorin-P'2013]

$$S_\lambda(x_1,\ldots,x_k;N) = \prod_{i=1}^{k} \frac{(N-i)!}{(N-1)!(x_i-1)^{N-k}} \times \frac{\det\left[\left(x_i\frac{\partial}{\partial x_i}\right)^{j-1}\right]_{i,j=1}^{k}}{\Delta(x_1,\ldots,x_k)} \prod_{j=1}^{k} S_\lambda(x_j;N,1)(x_j-1)^{N-1}.$$

Corollary[Gorin-P'2013]

If $\dfrac{\ln\left(S_{\lambda(N)}(x;N,1)\right)}{N} \to \Psi(x)$ unif. on a compact $M \subset \mathbb{C}$. Then for any k

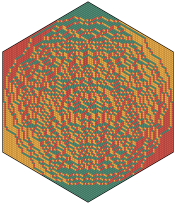
$$\lim_{N\to\infty} \frac{\ln\left(S_{\lambda(N)}(x_1,\ldots,x_k;N,1)\right)}{N} = \Psi(x_1)+\cdots+\Psi(x_k)$$

uniformly on M^k.

More informally, under various regimes of convergence for $\lambda(N)$ and $x_1,\ldots,x_k$ we have

$$S_{\lambda(N)}(x_1,\ldots,x_k) \sim S_{\lambda(N)}(x_1)\cdots S_{\lambda(N)}(x_k).$$

Limit surface for symmetric tilings



Theorem (P, 2014)

Let $n, m \in \mathbb{Z}$, such that $m/n \to a$ as $n \to \infty$, where $a \in (0, +\infty)$. Let $H_n(u,v)$ – height function of a symmetric tiling of $n \times m \times n\ldots$ hexagon,i.e.

$$H_n(u,v) = \frac{1}{n} y^{\lfloor nu \rfloor}_{\lfloor nv \rfloor} - v.$$

For all $1 \geq u \geq v \geq 0$, as $n \to \infty$: $H_n(u,v)$ converges unif. in prob. to a deterministic function $L(u,v)$ ("the limit surface").

For any fixed $u \in (0,1)$, $L(u,v)$ is the distribution function of the measure $\mathbf{m}$, given by its moments:

$$\int_{\mathbb{R}} t^r \mathbf{m}(dt) = \sum_{\ell=0}^{r} \binom{r}{\ell} \frac{1}{(\ell+1)!} u^{-r+\ell} \frac{\partial^\ell}{\partial z^\ell} z^p \Phi_a'(z)^{p-\ell} \Big|_{z=1},$$

where $\Phi_a(e^y) = y\frac{a}{2} + 2\phi(y;a) - 2$ and...

$$h(y) = \frac{1}{4}\left((e^y+1) + \sqrt{(e^y+1)^2 + 4(a^2+a)(e^y-1)^2}\right)$$

$$\phi(y;a) = (\frac{a}{2}+1)\ln\left(h(y) - (\frac{a}{2}+1)(e^y-1)\right) - (\frac{a}{2}+\frac{1}{2})\ln\left(h(y) - (\frac{a}{2}+\frac{1}{2})(e^y-1)\right)$$
$$+ \frac{a}{2}\ln\left(h(y) + \frac{a}{2}(e^y-1)\right) - (\frac{a}{2}-\frac{1}{2})\ln\left(h(y) + (\frac{a}{2}-\frac{1}{2})(e^y-1)\right)$$

Theorem (P, 2015)

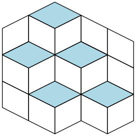
The scaled height function $H_n(u,v)$ of a centrally symmetric tiling of an $a \times b \times c\ldots$ hexagon converges uniformly in probability to a deterministic function $L(u,v)$ – the limit surface, as $n \to \infty$, where $n = \frac{a+c}{2}$ and $a/n, b/n$ – approx constant.
The limit surface coincides with the limit surface for the uniformly random tilings of the hexagon (without symmetry constraints).

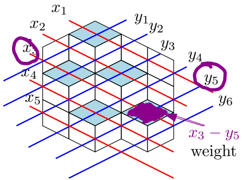
Lozenge tilings with multivariate weights

Plane partitions with base μ, height d

weights of horizontal lozenges $= x_i - y_j$

3	2	1
2	1	





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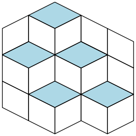
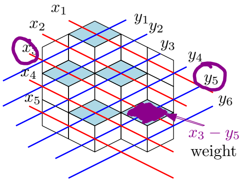




Theorem (Morales-Pak-P'2017)

For tilings with base μ and height d, we have that

$$\sum_{T \in \Omega_{\mu,d}} \prod_{(i,j) \in T} (x_i - y_j) = \det[A_{i,j}(\mu, d)]_{i,j=1}^{d+\ell(\mu)},$$

where

$$A_{i,j}(\mu,d) := \begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{d+\ell(\mu)-j})}{(x_i - x_{i+1})\cdots(x_i - x_{d+\ell(\mu)})}, & \text{when } j = \ell(\mu)+1, \ldots, \ell(\mu)+d, \\ \frac{(x_i - y_1)\cdots(x_i - y_{\mu_j+d})}{(x_i - x_{i+1})\cdots(x_i - x_{d+j})}, & \text{when } j = i-d, \ldots, \ell(\mu), \\ 0, & \text{when } j < i - d. \end{cases}$$

Corollary (Krattenthaler, Stanley etc)

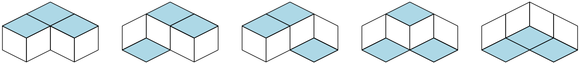
Consider the set $PP(\mu, d)$ of plane partitions of base μ and entries less than or equal to d. Then

$$\sum_{P \in PP(\mu,d)} q^{|P|} = q^{\sum_r r\mu_r} \det[C_{i,j}]_{i,j=1}^{\ell+d},$$

where

$$C_{i,j} = \begin{cases} (-1)^{d+\ell-i} q^{(d-i)(d+\ell-j)-\frac{(d-i+\ell)(d-i-\ell-1)}{2}} \frac{1}{(q;q)_{d+\ell-i}}, & \text{when } j = \ell+1, \ldots, \ell+d, \\ (-1)^{d+j-i} q^{(d-i)(\mu_j+d)-\frac{(d+j-i)(d-i-j-1)}{2}} \frac{1}{(q;q)_{d+j-i}}, & \text{when } j = i-d, \ldots, \ell, \\ 0, & \text{when } j < i-d, \end{cases}$$

E.g. $\mu = (2,1)$, $d = 1$:



$$\sum_{P \in PP((2,1),1)} q^{|P|} = q^0 + q^1 + 2q^2 + q^3$$

Theorem (Morales-Pak-P'2017)

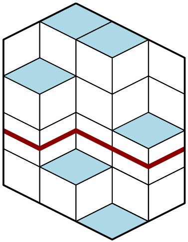
The partition function for tilings of the $a \times b \times c \times a \times b \times c$ ($\mu = a \times b$, $d = c$) hexagon with horizontal lozenges weights $x_i - y_j$ is given by

$$Z(a,b,c) := \sum_{T \in \Omega_{a,b,c}} \prod_{(i,j)\in T} (x_i - y_j) = \det \left[\begin{cases} \frac{(x_i - y_1)\cdots(x_i - y_{c+a-j})}{(x_i - x_{i+1})\cdots(x_i - x_{c+a})} & \text{if } j > a \\ \frac{(x_i - y_1)\cdots(x_i - y_{b+c})}{(x_i - x_{i+1})\ldots(x_i - x_{c+j})} & \text{if } j = i-c, \ldots, a \\ 0, & j < i - c \end{cases} \right]_{i,j=1}^{a+c}$$

The probability that a path through (i, d_i) exists is

$$\mathrm{Prob}(\mathrm{path}) = \frac{\det[A_{i,j}(\mu, d)] \det[\bar{A}_{i,j}(\mu^*, c-d-1)]}{Z}$$

Matrix $\bar{A}$ has $x_i \to x_{a+c+1-i}$ and $y_j \to y_{b+c+1-j}$.

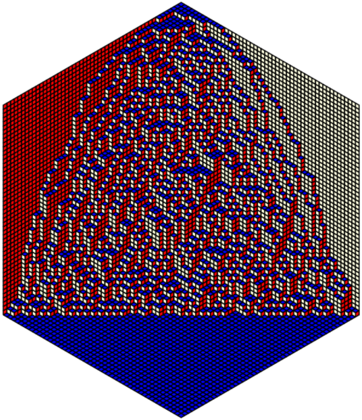


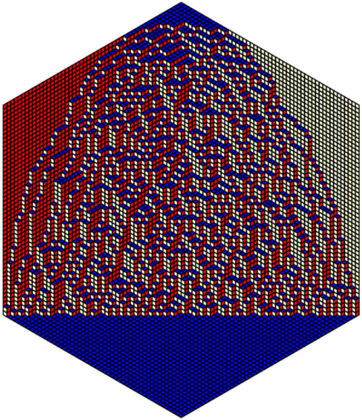
$\mu = 31$

$\mu^* = 20$

Simulation: base $= \delta_n$

Weights: "hook" weights $(4n - i - j)$ versus uniform (i.e. 1).





Factorial Schur functions, multivariate lozenge tilings

$$s_\mu^{(d)}(\mathbf{x}|\mathbf{a}) := \frac{\det\left[(x_j - a_1)\cdots(x_j - a_{\mu_i+d-i})\right]_{i,j=1}^d}{\prod_{1\le i<j\le d}(x_i - x_j)},$$

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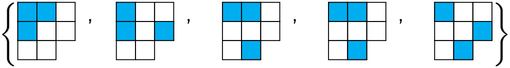
Excited diagrams $\mathcal{E}(\lambda/\mu)$:

$\lambda = (3,3,2)$, $\mu = (2,1)$

$\mathcal{E}(\lambda/\mu) =$



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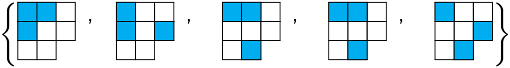
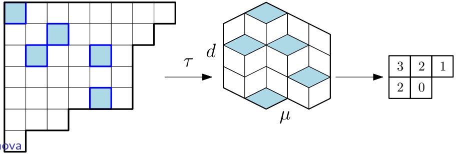
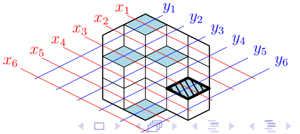
$\mathcal{E}(\lambda/\mu) =$ 

Theorem (Ikeda-Naruse, Kreiman + Knutson-Tao, Lakshmibai-Raghavan-Sankaran)

Let $\mu \subset \lambda \subset d \times (n-d)$. Let $v(n-d+1-i) = \lambda_i + (n-d+1-i)$ and $v(j) = d + j - \lambda'_j$. Then

$$s_\mu^{(d)}(y_{v(1)}, \ldots, y_{v(d)}|y_1, \ldots, y_{n-1}) = \sum_{D\in\mathcal{E}(\lambda/\mu)} \prod_{(i,j)\in D} (y_{v(d-i+1)} - y_{v(d+j)})$$





Hook formulas for skew shapes

Standard Young Tableaux of shape λ: $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&5\\\cline{1-2} 6\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&3\\\hline 4&6\\\cline{1-2} 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&5\\\cline{1-2} 6\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&6\\\cline{1-2} 5\\\cline{1-1}\end{array}$ $\begin{array}{|c|c|c|}\hline 1&2&5\\\hline 3&4\\\cline{1-2} 6\\\cline{1-1}\end{array}\ldots$

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Hook-length formula [Frame-Robinson-Thrall]:

$$f^{\lambda} := \#\{\text{SYTs of shape } \lambda\} = \frac{|\lambda|!}{\prod_{u\in\lambda} h_u} = \frac{6!}{5*3*3*1*1*1} = 16$$

Hook length of box $u=(i,j)\in\lambda$: $h_u = \lambda_i - j + \lambda'_j - i + 1 = \#\blacksquare \in$ 

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Standard Young Tableaux of *skew shape* λ/μ: $\begin{array}{cccc}&&3&5\\&1&4&\\2&6&&\end{array}$

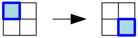
No product formula:

$\lambda/\mu = \delta_{n+2}/\delta_n$: $\begin{array}{cccc}&&3&7\\&1&5&\\2&4&&\\6&&&\end{array}$ $\longleftrightarrow$ $6>2<4>1<5>3<7$ $\quad f^{\delta_{n+2}/\delta_n} = E_{2n+1}$:

$$1 + E_1 x + E_2\frac{x^2}{2!} + E_3\frac{x^3}{3!} + E_4\frac{x^4}{4!} + \ldots = \sec(x) + \tan(x).$$

Euler numbers: $2, 5, 16, 61 \ldots$.

Hook-Length formula for skew shapes

Excited diagrams: $\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

Theorem (Naruse, SLC, September 2014)

$$f^{\lambda/\mu} = |\lambda/\mu|! \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{u \in [\lambda] \setminus D} \frac{1}{h(u)},$$

where $\mathcal{E}(\lambda/\mu)$ *is the set of* excited diagrams *of* λ/μ*.*

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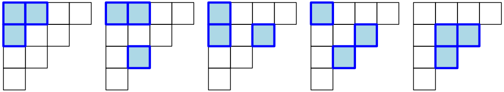

$$f^{(4321/21)} = 7! \left(\frac{1}{1^4 \cdot 3^3} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^3 \cdot 3^3 \cdot 5} + \frac{1}{1^2 \cdot 3^3 \cdot 5^2} + \frac{1}{1^2 \cdot 3^2 \cdot 5^2 \cdot 7} \right) = 61$$

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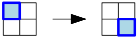
Excited diagrams: $\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

Theorem (Morales-Pak-P'16)

For skew SSYTs, we have that

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(\lambda/\mu)} q^{|T|} = \sum_{D \in \mathcal{E}(\lambda/\mu)} \prod_{(i,j) \in [\lambda] \setminus D} \left[\frac{q^{\lambda'_j - i}}{1 - q^{h(i,j)}} \right].$$

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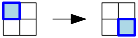
Excited diagrams: $\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

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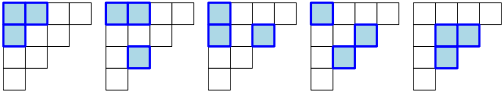

$$s_{\lambda/\mu}(1, q, q^2, \ldots) = \sum_{T \in SSYT(4321/21)} q^{|T|} = \frac{q^3}{(1-q)^4(1-q^3)^3} + 2 \times \frac{q^5}{(1-q)^3(1-q^3)^3(1-q^5)} + \cdots$$

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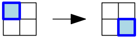
Excited diagrams: $\mathcal{E}(\lambda/\mu) = \{D \subset \lambda :$ obtained from μ via  $\}$

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Theorem (Morales-Pak-P'16)

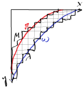
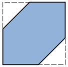
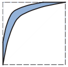
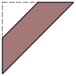
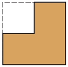
For (reverse) plane partitions of skew shape λ/μ:

$$\sum_{\pi \in RPP(\lambda/\mu)} q^{|\pi|} = \sum_{S \in PD(\lambda/\mu)} \prod_{u \in S} \left[\frac{q^{h(u)}}{1 - q^{h(u)}} \right].$$

where $PD(\lambda/\mu) := \{S \subset [\lambda] : S \subset [\lambda] \setminus D,$ *for some* $D \in \mathcal{E}(\lambda/\mu)\}$*.*

Applications of NHLF

- Asymptotics of $f^{\lambda/\mu}$: $\log f^{\lambda^{(n)}/\mu^{(n)}} \sim \dfrac{1}{2} n \log n.$

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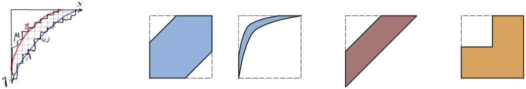
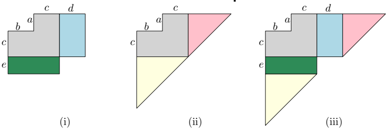


- Product formulas for special $f^{\lambda/\mu}$.



(i) (ii) (iii)

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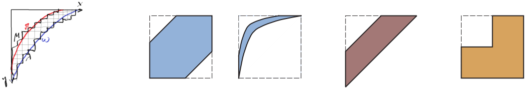

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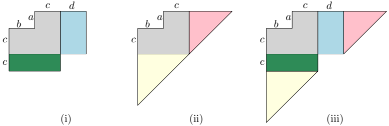
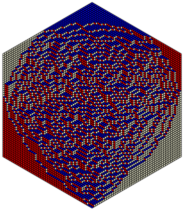


- Weighted lozenge tilings.



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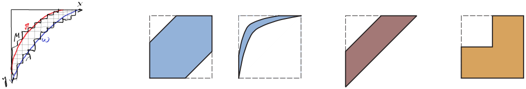

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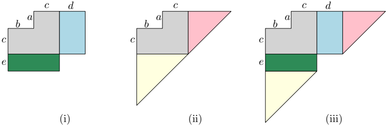

(i) (ii) (iii)

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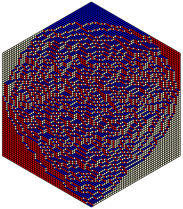
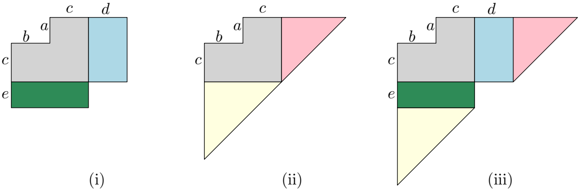


- Principle evaluations of Schubert polynomials and asymptotics.

Product formulas



(i) (ii) (iii)

$\Phi(n) := 1! \cdot 2! \cdots (n-1)!$, $\Psi(n) := 1!! \cdot 3!! \cdots (2n-3)!!$,
$\Psi(n;k) := (k+1)!! \cdot (k+3)!! \cdots (k+2n-3)!!$, $\Lambda(n) := (n-2)!(n-4)! \cdots$

Theorem (Morales-Pak-P'17)

The number of skew SYTs of shapes as in (i), (ii), (iii) above, with side lengths parametrized by a, b, c, d, e, are:

$$f^{sh(i)} = n! \, \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(d)\Phi(e)\Phi(a+b+c)\Phi(c+d+e)\Phi(a+b+c+e+d)}{\Phi(a+b)\Phi(e+d)\Phi(a+c+d)\Phi(b+c+e)\Phi(a+b+2c+e+d)},$$

$$f^{sh(ii)} = n! \, \frac{\Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)} \, \frac{\Psi(c)\Psi(a+b+c)}{\Psi(a+c)\Psi(b+c)\Psi(a+b+2c)},$$

$$f^{Sh(iii)} = \frac{n! \quad \Phi(a)\Phi(b)\Phi(c)\Phi(a+b+c)\,\Psi(c;d+e)\Psi(a+b+c;d+e)\,\Lambda(2a+2c)\Lambda(2b+2c)}{\Phi(a+b)\Phi(b+c)\Phi(a+c)\Psi(a+c)\Psi(b+c)\Psi(a+b+2c;d+e)\Lambda(2a+2c+d)\Lambda(2b+2c+e)}$$

Increasing Tableaux

$$\begin{array}{c c} & < \\ \wedge & \begin{array}{|c|c|c|c|} \hline \ \ & \ \ & \ \ & \ \ \\ \hline & & & \\ \cline{1-3} & & & \\ \cline{1-1} \end{array} \end{array}$$

Increasing Tableau

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 6 & 8 \\ \hline 2 & 3 & 8 \\ \cline{1-3} 6 \\ \cline{1-1} \end{array}$$

Standard Increasing Tableau

$$\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 5 \\ \hline 2 & 3 & 4 \\ \cline{1-3} 5 \\ \cline{1-1} \end{array}$$

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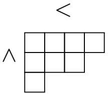

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Theorem (Morales-Pak-Panova'21+)

Fix $d \geq 1$, $\beta \in \mathbb{R}$. For every $\lambda \vdash n$ with $\ell(\lambda) \leq d$, we have:

$$\sum_{T \in \mathrm{SIT}(\lambda)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\left([T_{<k}]_i\right)+d-i+1)}{1+\beta\left(\lambda_i+d-i+1\right)} \right] - 1 \right)^{-1}$$

$$= \frac{1}{(-\beta)^n} \prod_{i=1}^{\ell(\lambda)} \left(1+\beta(\lambda_i+d-i+1)\right)^{\lambda_i} \prod_{(i,j)\in\lambda} \frac{1}{h(i,j)}. \tag{K-HLF}$$

Theorem (Multivariate K-HLF, Morales-Pak-Panova'21+)

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Corollary:

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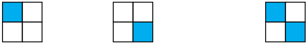
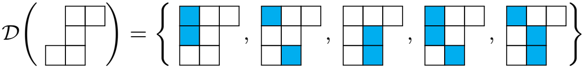
$$\underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \\ \cline{1-1} \end{array}}{q^5 \frac{1}{(1-q^3)(1-q^2)}} + \underset{\begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \\ \cline{1-1} \end{array}}{q^6 \frac{1}{(1-q^3)(1-q^2)(1-q)}} + \underset{\begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \\ \cline{1-1} \end{array}}{q^6 \frac{1}{(1-q^3)(1-q^2)(1-q)}}$$

$$= q^5 \frac{1}{(1-q^3)(1-q)^2}$$

Generalized Excited Diagrams $\mathcal{D}(\lambda/\mu)$

Type I move Type II move





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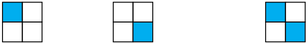
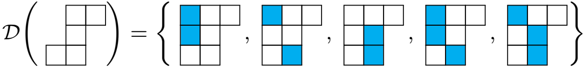
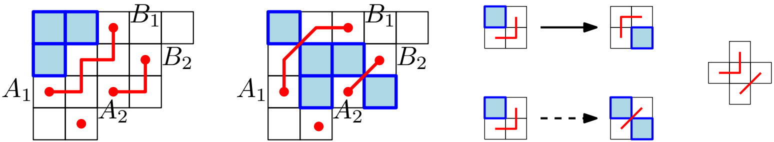




Non-intersecting Delannoy paths[MPP] with forbidden configuration:



Skew K-HLF

Theorem (Morales-Pak-Panova'21+)

Fix $d \geq 1$, $\beta \in \mathbb{R}$. For every $\mu \subset \lambda$ with $\ell(\lambda) \leq d$, we have:

$$\sum_{T \in \mathrm{SIT}(\lambda/\mu)} \prod_{k=1}^{m(T)} \left(\left[\prod_{i=1}^{d} \frac{1+\beta\big([T_{<k}]_i + d - i + 1\big)}{1+\beta(\lambda_i + d - i + 1)} \right] - 1 \right)^{-1}$$
$$= \sum_{D \in \mathcal{D}(\lambda/\mu)} (-\beta)^{|D| - |\lambda|} \prod_{(i,j) \in \lambda \setminus D} \frac{\beta(\lambda_i + d - i + 1) + 1}{h(i,j)} . \qquad \text{(K-NHLF)}$$

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Factorial Grothendieck polynomials

(double Grothendieck polynomials for Grassmannian permutations)

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$$x \oplus y \;:=\; x + y + \beta xy\,, \qquad x \ominus y \;:=\; \frac{(x-y)}{(1+\beta y)}\,, \qquad \ominus x \;:=\; \frac{-x}{1+\beta x}\,,$$

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[McNamara]: The **Factorial Grothendieck polynomials** are given by:

$$G_\mu(x_1,\ldots,x_d\,|\,\mathbf{y}) = \;\det\Big([x_i\,|\,\mathbf{y}]^{\mu_j+d-j}(1+\beta x_i)^{j-1}\Big)_{i,j=1}^{d} \prod_{1\le i<j\le d} \frac{1}{(x_i - x_j)}\,.$$

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Vanishing property:
evaluation at $\mathbf{y}_\lambda := (\ominus y_{\lambda_1 + d}, \ominus y_{\lambda_2 + d - 1}, \ldots, \ominus y_{\lambda_d + 1})$ for $\ell(\lambda) \le d$,

$$G_\mu(\mathbf{y}_\lambda \,|\, \mathbf{y}) \ = \ \begin{cases} 0 & \text{if } \mu \not\subseteq \lambda, \\ \prod_{(i,j) \in \lambda} (y_{d+j-\lambda'_j} \ominus y_{\lambda_i + d - i + 1}) & \text{if } \mu = \lambda. \end{cases}$$

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Pieri rule:

$$G_\mu(\mathbf{x} \,|\, \mathbf{y})\big(1 + \beta G_1(\mathbf{x} \,|\, \mathbf{y})\big) \ = \ \big(1 + \beta G_1(\mathbf{y}_\mu \,|\, \mathbf{y})\big) \sum_{\nu \mapsto \mu} \beta^{|\nu/\mu|} G_\nu(\mathbf{x} \,|\, \mathbf{y})\,.$$

Skew K-HLF

Evaluations of factorial Grothendieck polynomials $G_\mu(\mathbf{y}_\lambda|\mathbf{y})$ for $\mu \subset \lambda$
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$$K_{\mu\lambda}^{\lambda} = \sum_{D\in\mathcal{D}(\lambda/\mu)} (-1)^{|D|-|\mu|} \prod_{(i,j)\in D} \frac{y_{d+j-\lambda'_j} - y_{\lambda_i+d+1-i}}{1 - y_{\lambda_i+d+1-i}}.$$

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$$K^\lambda_{\mu\lambda}\left(\frac{K^\lambda_{1\lambda} - 1 + wt'(\mu)}{wt'(\mu)}\right) = \sum_{\nu \mapsto \mu} (-1)^{|\nu/\mu|-1} K^\lambda_{\nu\lambda},$$

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Compare with Chevalley formula for factorial Grothendiecks at $\beta = -1$:

$$G_\mu(\mathbf{y}_\lambda \mid \mathbf{y}) \left(\frac{G_1(\mathbf{y}_\lambda|\mathbf{y}) - G_1(\mathbf{y}_\mu \mid \mathbf{y})}{1+\beta G_1(\mathbf{y}_\mu \mid \mathbf{y})}\right) = \sum_{\nu\supsetneq\mu} \beta^{|\nu/\mu|-1} G_\nu(\mathbf{y}_\lambda \mid \mathbf{y}).$$

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General β: substitute $y_i \leftarrow -\beta y_i$

Theorem (**Skew K-HLF**, Morales-Pak-Panova'21+)

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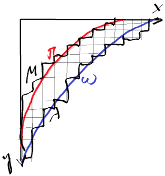
Asymptotic Algebraic Combinatorics

- **Asymptotics of $f^{\lambda/\mu}$ when $\lambda, \mu \to$ limit shape?**



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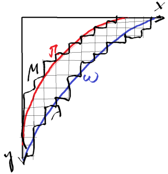

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- **Asymptotics of structure constants:**

  Littlewood-Richardson $c^{\lambda}_{\mu\nu}$

$$s_\mu(x)s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(x)$$

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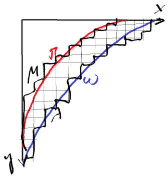

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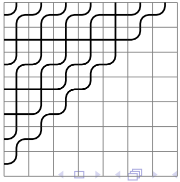
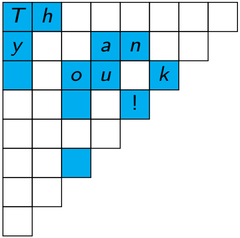
- **Asymptotics of Schubert polynomial evaluations, pipe dreams:**





$$\left|\mathcal{D}(\delta_{n+2k}/\delta_n)\right| = 2^{-\binom{k}{2}} \det\left[s_{n-2+i+j}\right]_{i,j=1}^{k}$$