Multiplicative Statistics for Eigenvalues of Hermitian Matrix Models are (KPZ) Universal

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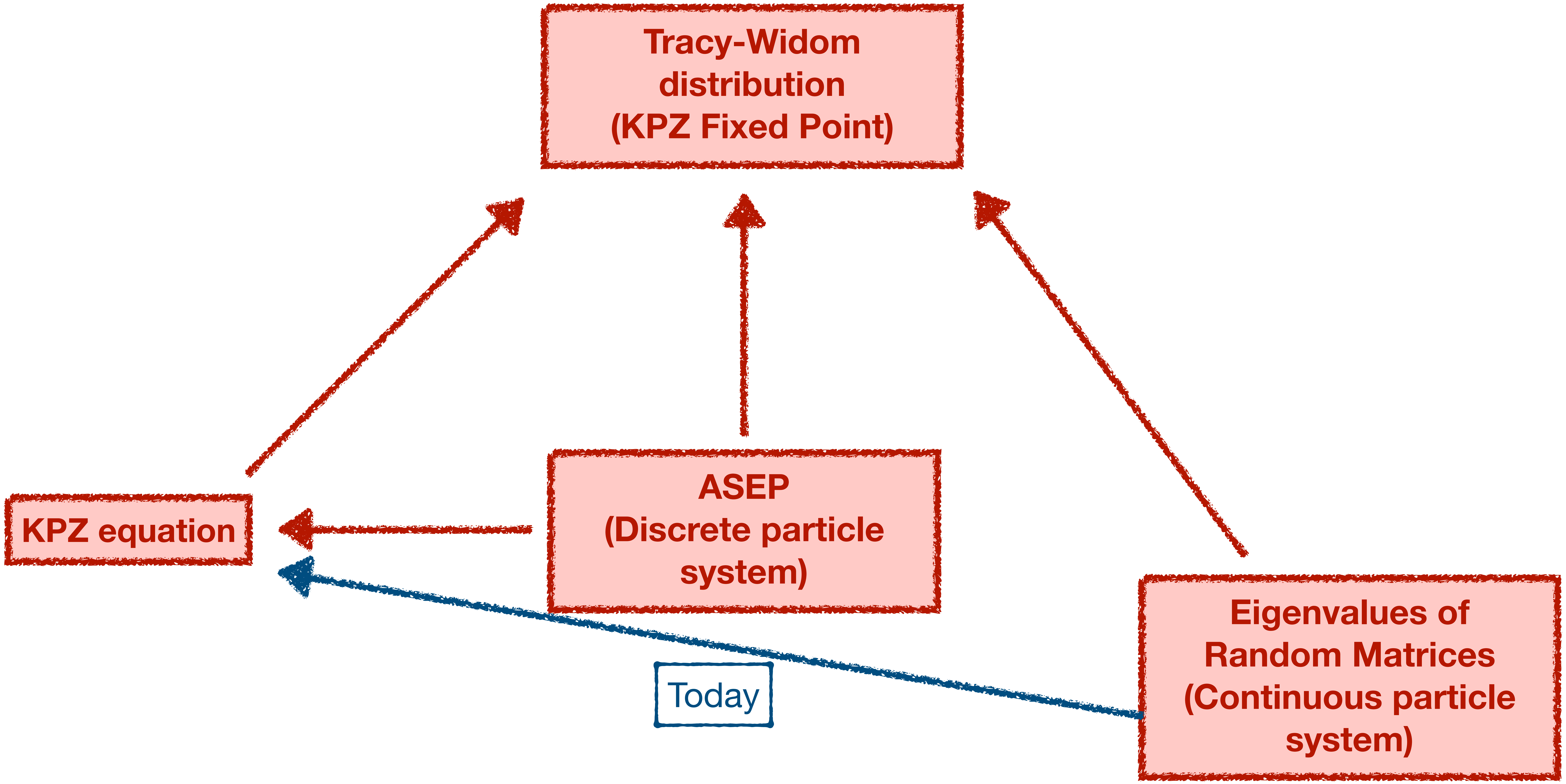
Ongoing work with Promit Ghosal (MIT)

KPZ Universality - one-point distributions



RMT for today

- Hermitian matrices of size N, with distribution

$$P(\mathrm{M})\mathrm{dM} \propto \mathrm{e}^{-V(\mathrm{M})}\mathrm{dM}, \quad V \text{ polynomial}$$

- Eigenvalues have distribution

$$\frac{1}{Z_N}\prod_{j<k}(\lambda_k-\lambda_j)^2\prod_{j=1}^{N}\mathrm{e}^{-V(\lambda_k)}\mathrm{d}\lambda = \det\left(K_N^V(\lambda_j,\lambda_k)\right)_{j,k}\mathrm{d}\lambda$$

- In the weak sense

$$\frac{1}{N}\sum_{k=1}^{N}\delta_{\lambda_j} \to \mu_V$$

Equilibrium measure

- Generically, potential is regular

$$\frac{\mathrm{d}\mu_V}{\mathrm{d}x} = h(x)\prod_{j=1}^{\ell}\sqrt{(b_j-x)(x-a_j)}\chi_{[a_j,b_j]}(x), \quad h(x)>0 \text{ on } \bigcup_{j=1}^{\ell}[a_j,b_j]$$

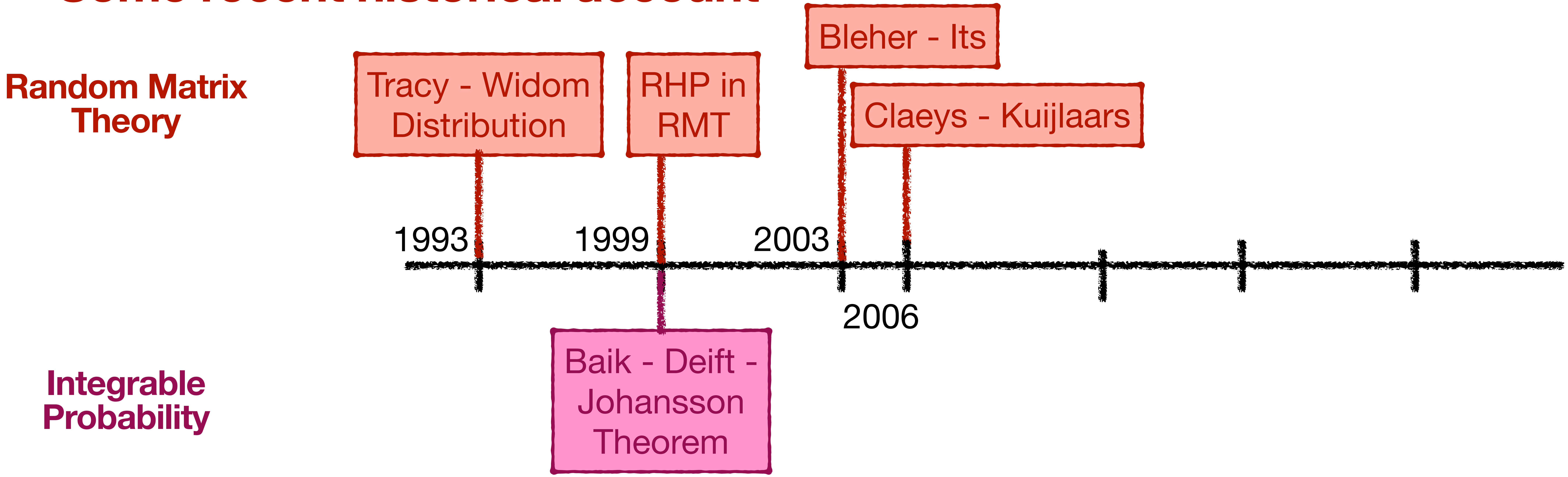
Universality: Tracy-Widom, Sine…

- Critical cases

$$\frac{\mathrm{d}\mu_V}{\mathrm{d}x} = \mathcal{O}\left((x-\alpha)^{p/2}\right), \quad x\to\alpha\in\operatorname{supp}\mu_V, \quad p\geq 2$$

Integrable hierarchies: PI, PII…

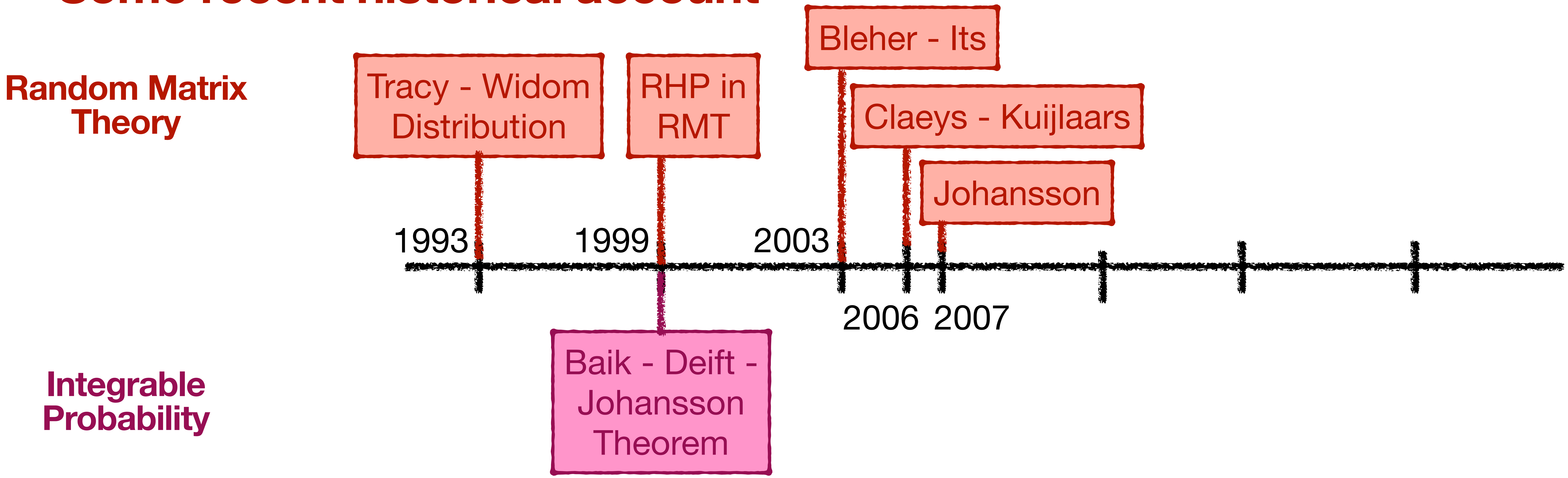
Some recent historical account



- Limiting process (kernel) for critical potentials associated to integrable systems, e.g. of the form

$$\frac{\Phi_1(u)\Phi_2(v) - \Phi_1(v)\Phi_2(v)}{u - v}, \quad \text{where } \Phi_j \text{ solve Lax pair for PII}$$

Some recent historical account

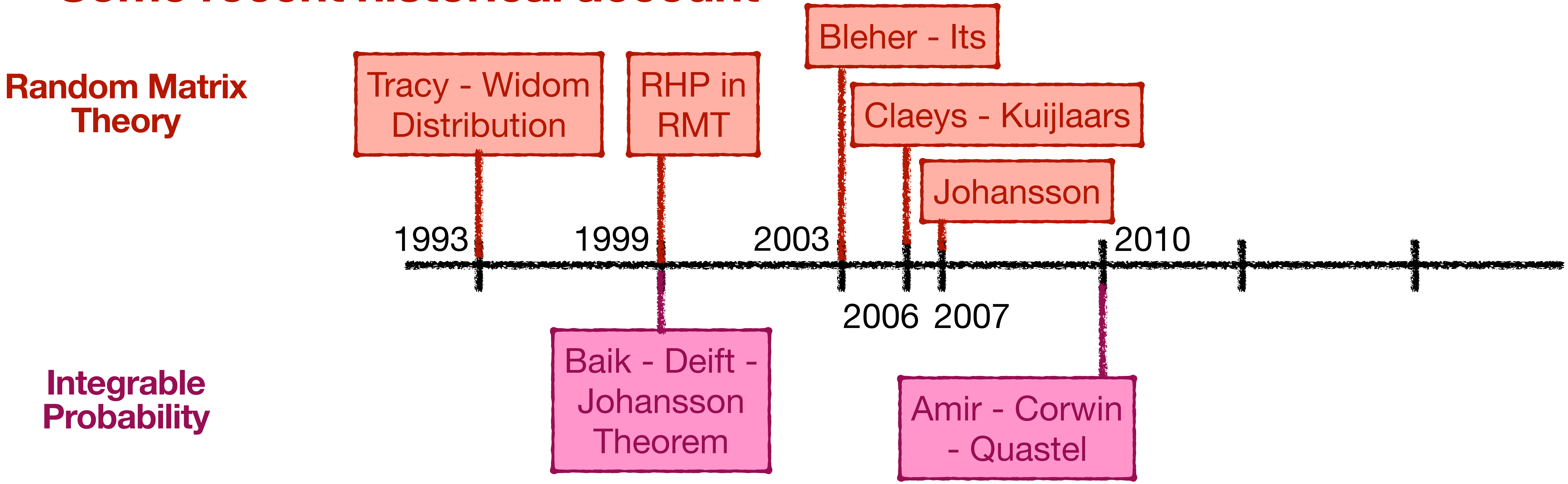


- Deformed Airy kernel

$$K_t(x,y) = \int_{-\infty}^{\infty} \sigma_t(\lambda)\mathrm{Ai}(x+\lambda)\mathrm{Ai}(\lambda+y)\mathrm{d}\lambda, \quad \alpha > 0, \quad \sigma_t(x) := \frac{1}{1+\mathrm{e}^{-tx}}$$

as limiting kernel in deformations of Gaussian matrix models

Some recent historical account



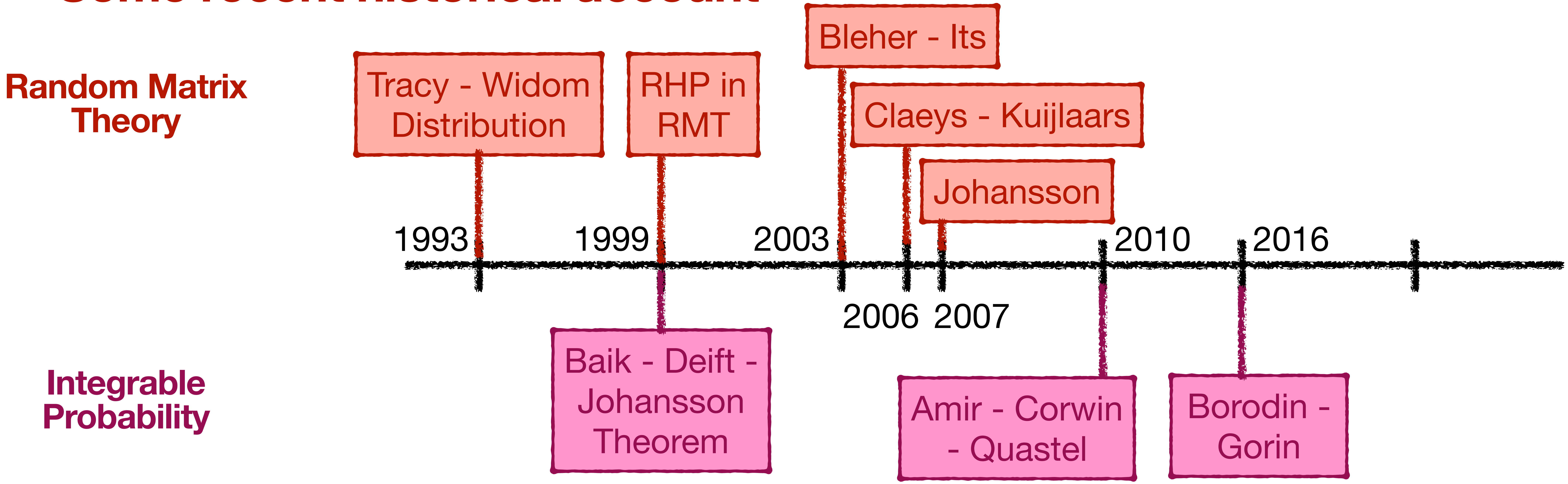
- One-point distribution for narrow wedge solution to KPZ equation given by (rescaled)

$$\int_C \frac{\mathrm{e}^{-u}}{u} \det\left(I - K_{\tilde{t}}\right)_{L^2(\tilde{s},\infty)} \mathrm{d}u, \quad \text{some rescaled } \tilde{t} = \tilde{\alpha}(s,t),\ \tilde{s} = \tilde{s}(s,t)$$

- Tracy-Widom type formula

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} \log\det\left(I - K_t\right)_{L^2(s,\infty)} = \int_{-\infty}^{\infty} q(s \mid u)^2 \, \mathrm{d}\sigma_t(u), \quad q(s \mid t) \text{ solves integro-differential PII}$$

Some recent historical account

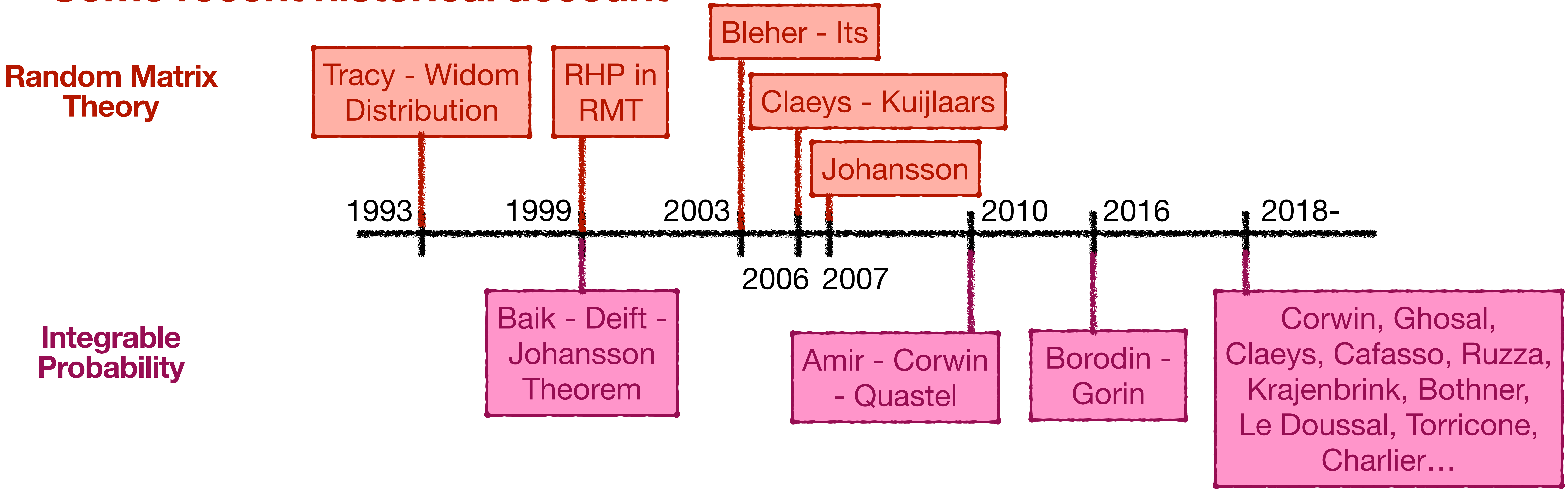


- (After Amir-Corwin-Quastel, Calabrese-Le Doussal-Rosso, Dotsenko, Sasamoto-Spohn) Duality formula between KPZ and Airy2 Point Process

$$\mathbb{E}_{\mathrm{KPZ}}\left(\mathrm{e}^{-\mathrm{e}^{T^{1/3}(H_T+S)}}\right) = \mathbb{E}_{\mathrm{Ai}_2}\left(\prod_{k=1}^{\infty}\frac{1}{1+\mathrm{e}^{T^{1/3}(\mathfrak{a}_k+S)}}\right)$$

$$\Longrightarrow L_{\mathrm{Ai}}(S,T) := \mathbb{E}_{\mathrm{Ai}_2}\left(\prod_{k=1}^{\infty}\frac{1}{1+\mathrm{e}^{T^{1/3}(\mathfrak{a}_k+S)}}\right)$$ in terms of int-diff PII

Some recent historical account



- Tail behavior of KPZ solution, multiplicative statistics of Airy2 process, further connections with integrable systems (integro-differential equations)…

Multiplicative statistics for eigenvalues

- Multiplicative statistics

$$L_N(s,Q) = \mathbb{E}\left[\prod_{j=1}^{N} \frac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(\lambda_k)}}\right] = \frac{1}{Z_N}\int_{\mathbb{R}^N} \prod_{j=1}^{N} \frac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(\lambda_k)}} \prod_{j<k}(\lambda_k-\lambda_j)^2 \prod_{j=1}^{N} \mathrm{e}^{-V(\lambda_k)}\mathrm{d}\lambda$$

- Associated deformed point process

$$\frac{1}{Z_N(s,Q)}\prod_{j=1}^{N} \frac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(\lambda_k)}} \prod_{j<k}(\lambda_k-\lambda_j)^2 \prod_{j=1}^{N} \mathrm{e}^{-V(\lambda_k)}\mathrm{d}\lambda = \det\Big(K_N(\lambda_j,\lambda_k \mid s,Q)\Big)_{j,k}\mathrm{d}\lambda$$

so for instance $L_N(s,Q) = \dfrac{Z_N(s,Q)}{Z_N}$

- Correlation kernel and (monic) orthogonal polynomials

$$K_N(x,y \mid s,Q) = \sum_{k=0}^{N-1} \gamma_k(s,Q)^2 P_k(x)P_k(y) \qquad \int P_k(x)P_\ell(x)\mathrm{e}^{-NV(s)}\frac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(x)}}\,\mathrm{d}x = \gamma_k(s,Q)^2\delta_{k,\ell}$$

Multiplicative statistics for eigenvalues - Assumptions

- Multiplicative statistics

$$L_N(s,Q) = \mathbb{E}\left[\prod_{j=1}^{N} \frac{1}{1+e^{-s-N^{2/3}Q(\lambda_k)}}\right] \qquad K_N(x,y \mid s,Q) = \sum_{k=0}^{N-1} \gamma_k(s,Q)^2 P_k(x) P_k(y)$$

- Potential is one-cut regular

$$\frac{d\mu_V}{dx} = h(x)\sqrt{x(x+a)}\chi_{[-a,0]}(x), \quad a>0, \quad h(x)>0 \text{ on } (-a,0)$$

- Statistics is "regular enough"

Q is analytic on neighborhood of $\mathbb{R}$

$Q(x)>0$ on $(-\infty,0)$, $\quad Q(x)<0$ on $(0,\infty)$ $\qquad$ so that as $N\to\infty$, $\displaystyle\frac{1}{1+e^{-s-N^{2/3}Q(x)}} \to \begin{cases} 1, & \text{on } (-\infty,0), \\ 0, & \text{on } (0,\infty) \end{cases}$

$Q(z) = -tz + \mathcal{O}(z^2),\ z\to 0, \qquad t>0$

Multiplicative statistics for eigenvalues - Main results (Ghosal & S.)

$$L_N(s,Q) = \mathbb{E}\left[\prod_{j=1}^{N} \frac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(\lambda_k)}}\right] \qquad K_N(x,y\mid s,Q) = \sum_{k=0}^{N-1} \gamma_k(s,Q)^2 P_k(x)P_k(y) \qquad Q(z) = -tz + \mathcal{O}(z^2),\ z\to 0$$

$$L_N(s,Q) = L_N(s,t), \quad \gamma_k(s,Q) = \gamma_k(s,t), \quad K_N(x,y\mid s,Q) = K_N(x,y\mid s,t),$$

Asymptotics for multiplicative statistics

$$\lim_{N\to\infty} L_N(s,t) = 1 - L_{\mathrm{Ai}}(-s/t, t^3)$$

Multiplicative statistics for eigenvalues - Main results (Ghosal & S.)

$$L_N(s,Q) = \mathbb{E}\left[\prod_{j=1}^{N} \frac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(\lambda_k)}}\right] \qquad K_N(x,y\mid s,Q) = \sum_{k=0}^{N-1} \gamma_k(s,Q)^2 P_k(x)P_k(y) \qquad Q(z) = -tz + \mathcal{O}(z^2),\ z\to 0$$

$$L_N(s,Q) = L_N(s,t), \quad \gamma_k(s,Q) = \gamma_k(s,t), \quad K_N(x,y\mid s,Q) = K_N(x,y\mid s,t), \qquad T = t^{-3/2}, \quad S = \frac{s}{t^{3/2}}$$

Asymptotics for correlation kernel

$$\lim_{N\to\infty} \omega_N(x)\omega_N(y) K_N\left(\frac{x}{c_N N^{2/3}}, \frac{y}{c_N N^{2/3}} \mid s,t\right) = \frac{\Phi(y)\partial_S\Phi(x) - \Phi(x)\partial_S\Phi(y)}{x-y}$$

where $\Phi = \Phi(\xi \mid S,T)$ solves int-diff PII

$$\partial_S^2\Phi(\xi\mid S,T) = \left(\xi + \frac{S}{T} + \frac{2}{T}\int_{-\infty}^{\infty} \Phi(r\mid S,T)\frac{\mathrm{e}^{-r}}{(1+\mathrm{e}^{-r})^2}\mathrm{d}r\right)\Phi(\xi\mid S,T)$$

Multiplicative statistics for eigenvalues - Main results (Ghosal & S.)

$$L_N(s,Q) = \mathbb{E}\left[\prod_{j=1}^{N} \frac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(\lambda_k)}}\right] \qquad K_N(x,y\mid s,Q) = \sum_{k=0}^{N-1} \gamma_k(s,Q)^2 P_k(x)P_k(y) \qquad Q(z) = -tz + \mathcal{O}(z^2),\ z\to 0$$

$$L_N(s,Q) = L_N(s,t), \quad \gamma_k(s,Q) = \gamma_k(s,t), \quad K_N(x,y\mid s,Q) = K_N(x,y\mid s,t), \qquad T = t^{-3/2}, \quad S = \frac{s}{t^{3/2}}$$

Asymptotics for norming constants

$$\gamma_N(s,t)^2 = \frac{1}{2\pi}\mathrm{e}^{-N\ell_V}\left(\frac{a}{4} + \frac{1}{N^{2/3}}\frac{1}{t^{1/2}}\left(p(s,t) + \frac{s^2}{4t^{3/2}}\right) + \mathcal{O}(N^{-2/3-\varepsilon})\right), \quad N\to\infty,$$

where $p(s,t) = u(S,T)$ satisfies

$$\partial_S u(S,T) = \frac{S}{2T} + \frac{1}{T}\int_{-\infty}^{\infty} \Phi(r\mid S,T)\frac{\mathrm{e}^{-r}}{(1+\mathrm{e}^{-r})^2}\mathrm{d}r$$

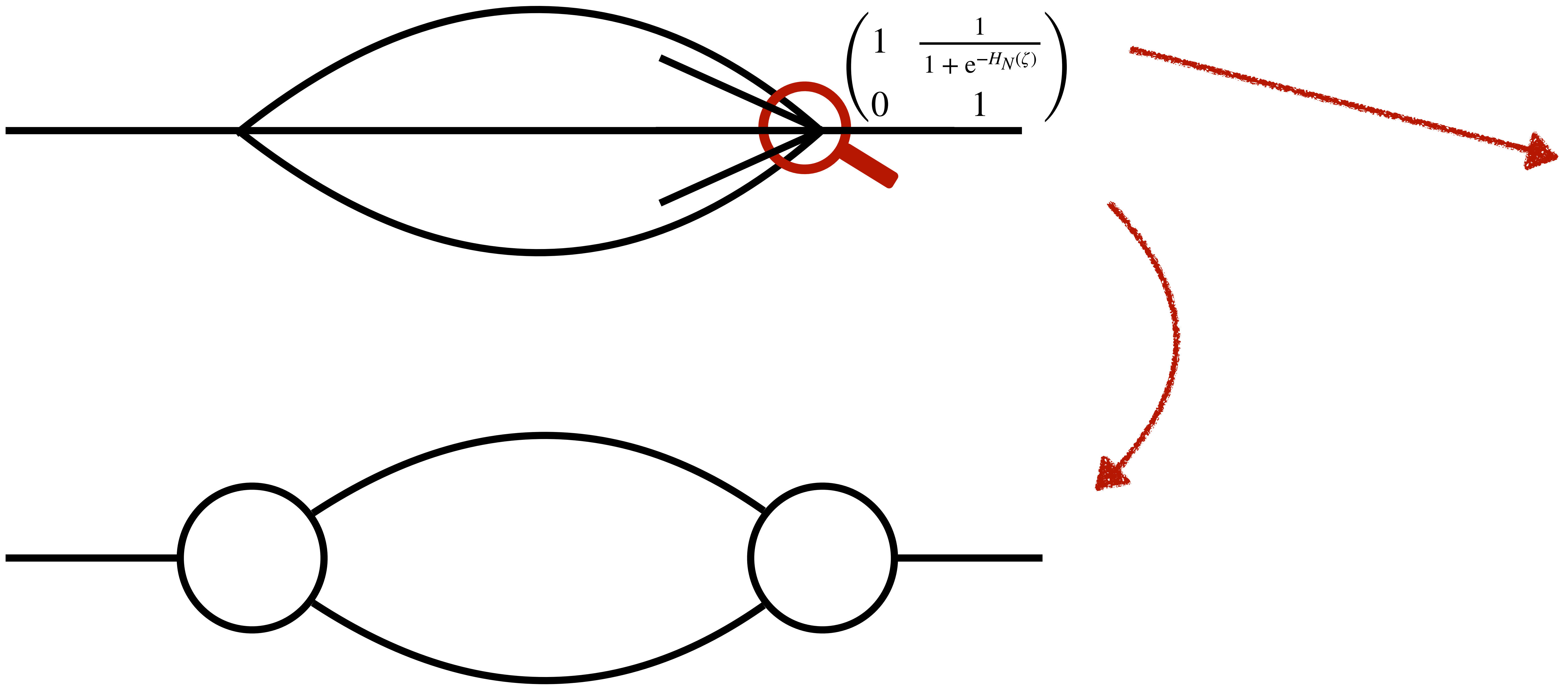
Comments

- Results are valid uniformly for $s \geq -M, \quad \delta \leq t \leq 1/\delta$
- Asymptotic analysis based on RHP for OPs with weight $\mathrm{e}^{-NV(s)}\dfrac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(x)}}\,\mathrm{d}x$
- Int-diff PII is reduction of KdV, boundary condition explicit [Cafasso - Claeys - Ruzza (2021)]
- From OPs to linear statistics via a new exact formula (no deformation!)
- Multiplicative statistics useful in applications (e.g. lead to tails of KPZ solution)
- Discrete OPs models/duality formulas (under investigation)
- Potential applications to rigidity of eigenvalues (under investigation)
- Critical potential, underlying integrable system?

Comments on RHP analysis

- Asymptotic analysis based on RHP for OPs with weight $\mathrm{e}^{-NV(s)} \dfrac{1}{1+\mathrm{e}^{-s-N^{2/3}Q(x)}}\,\mathrm{d}x$
- Main issue: local parametrix

$\Phi_N(\zeta)$



so $\quad P_N(z) = E_N(z)\Phi_N(\varphi(z)/N^{2/3})$

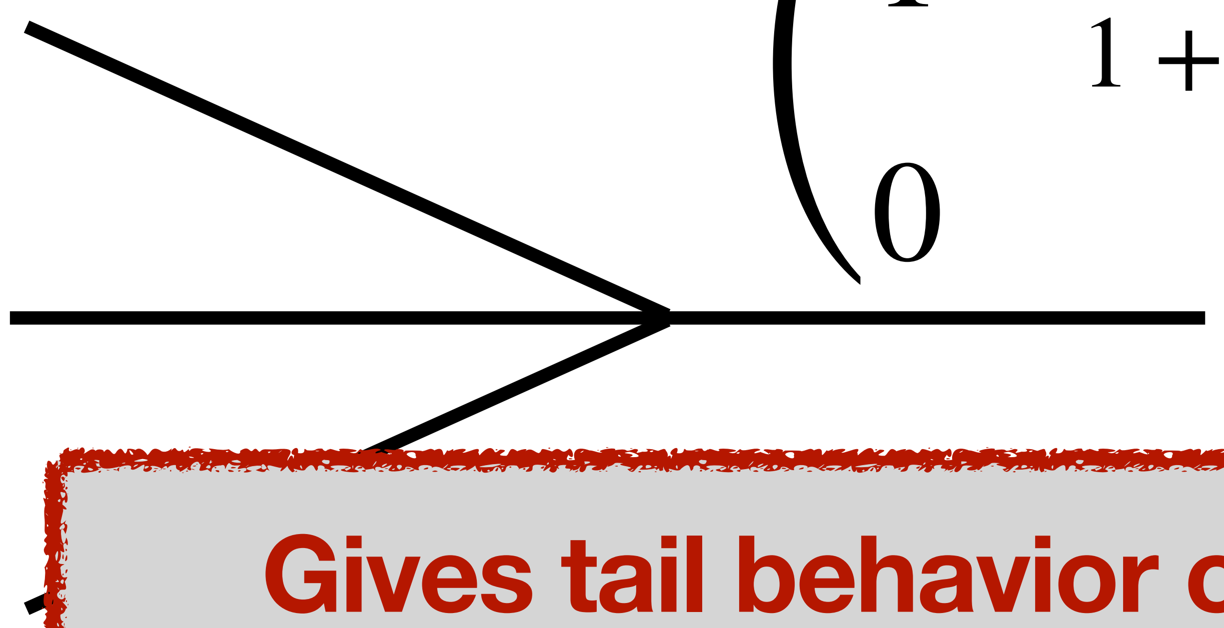
- If $\Phi_N(\zeta)$ exists, conclude asymptotic analysis

Comments on RHP analysis

Recall $Q(z) = -tz + \mathcal{O}(z^2),\ z \to 0$

- If $\Phi_N(\zeta)$ exists, conclude asymptotic analysis

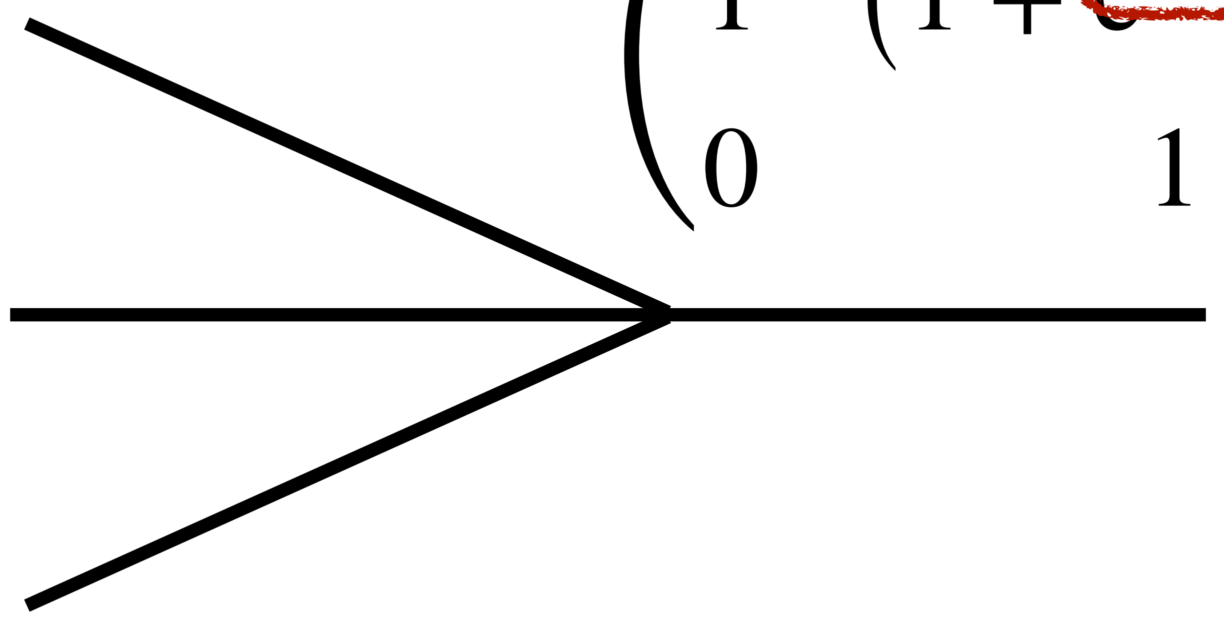
$\Phi_N(\zeta)$



$$\begin{pmatrix} 1 & \frac{1}{1+e^{-H_N(\zeta)}} \\ 0 & 1 \end{pmatrix}$$

$\Phi_N(\zeta) \to \Phi_{KPZ}(\zeta)$

$N \to \infty$



$$\begin{pmatrix} 1 & (1+e^{-s+t\zeta})^{-1} \\ 0 & 1 \end{pmatrix}$$

Gives tail behavior of KPZ
[Cafasso - Claeys, 2020]
[Cafasso - Claeys - Ruzza, 2021]

$\Phi_{KPZ}(\zeta) \sim (\cdots)\, e^{-\frac{2}{3}\zeta^{3/2}\sigma_3}, \quad \zeta \to \infty$

$s \geq -M, \quad \delta \leq t \leq 1/\delta$ is important

Comes from $\mu_V'(x) = \mathcal{O}(x^{1/2}), x \nearrow 0$

- Critical potentials/bulk points/different Q's

  ⟶ other (reductions of) integrable systems?

Time to wrap up!

- Limit of multiplicative statistics are universal in V and Q (and connect with KPZ equation), natural because of Airy2 point process limit

- Integro-differential PII appears at different levels (OPs, correlation kernel, norming constants)

- Multiplicative statistics may provide playing field for new families of (integro-differential?) integrable systems to arise

Thank you!