Projection theorems for linear-fractional families of projections

Joint work with A. Lukyanenko (GMU Fairfax)

AGRS Research Seminar

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Orthogonal projections

Let $L \subset \mathbb{R}^2$ be a line (i.e. one-dimensional linear subspace) and $P_L : \mathbb{R}^2 \to L$ the orthogonal projection onto L.

Given a Borel set $A \subset \mathbb{R}^2$ Borel with $\dim A = s$

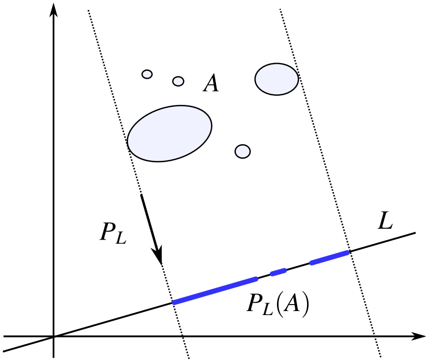
$\to \dim P_L(A) =?$

Trivially:

- $\dim P_L A \leq 1$
- $\dim P_L A \leq s$

Are these good bounds?

$\to$ Yes!



Marstrand's theorem (in $\mathbb{R}^2$)

Theorem [Marstrand 1954].

Given a Borel set $A \subset \mathbb{R}^2$, then for $\mathscr{H}^1$-almost every $\theta \in [0, \pi)$

$$\dim P_\theta A = \min\{1, \dim A\}$$

<u>equivalently...</u>

$$\mathscr{H}^1 \{\underbrace{\theta \in [0,\pi) : \dim P_\theta A < \min\{1, \dim A\}}_{:=E}\} = 0$$

Note the *exceptional set E* depends on the choice of Borel set A.

This result generalizes to higher dimensions...

The Grassmannian of m-planes

Let $G(n,m)$ be the family of m-planes in $\mathbb{R}^n$ (called the *Grassmannian*)

- $G(n,m)$ is a smooth manifold of dimension $k := (n-m)m$.
- The notion of $\mathscr{H}^s$-zero sets on $G(n,m)$ is well-defined (for all $s \geq 0$)
- Hence the notion of Hausdorff dimension is well-defined.
- $\mathscr{H}^k$-zero sets in $G(n,m)$ are also zero sets of the measure $\sigma_{n,m}$ on $G(n,m)$ that is induced by the Haar-measure of $O(n)$, and vice-versa.)

Projection theorems for orthogonal projections

$P_V : \mathbb{R}^n \to V$ orthogonal projection, $V \in G(n,m)$.

Let $A \subset \mathbb{R}^n$ a Borel set, $\dim A = s > 0$.

- Marstrand 1954, Mattila 1975:

  If $s \le m$, $\mathscr{H}^k\{V \in G(n,m) : \dim P_V A < s\} = 0$

  If $s > m$, $\mathscr{H}^k\{V \in G(n,m) : \mathscr{H}^m(P_V(A)) = 0\} = 0$

- Kaufman 1968, Mattila 1975:

  If $s \le m$, $\dim\{V \in G(n,m) : \dim P_V A < s\} < s$.

  (In fact, this works for $0 < \alpha \le s$ by later results.)

- Falconer 1982:

  If $s > m$, $\dim\{V : \mathscr{H}^m(P_L(A)) = 0\} \le (n-m)m + m - s$

- Besicovich 1939, Federer 1945:

  If $\mathscr{H}^m(A) < \infty$, then A is purely m-unrectifiable if and only if $\mathscr{H}^m(P_V(A)) = 0$ for $\mathscr{H}^k$-a.e. $V \in G(n,m)$

Projection theorems for various spaces

There are analogs of the results on the previous slides in various settings...

- horizontal resp. vertical projections in the Heisenberg group
- families of radial or point-source projections
- orthogonal projections in Hyperbolic space and on the (half-) sphere
- closest point projections induced by norms in $\mathbb{R}^n$
- projections theorems for different notions of dimension
- sharpness results for many of the above
- special families of projections (“what is the structure of E?")
- ...

Many(!) authors have worked on the above.

-> see e.g. the great survey articles by Mattila from 2004 and 2015.

Transversality and projections theorems

A more general framework...

Let $k, m \in \mathbb{N}$, $k \geq m$.

Let Ω be a compact metric space and $\Lambda \subset \mathbb{R}^k$ a open.

Consider a continuous mapping $\Pi : \Lambda \times \Omega \to \mathbb{R}^m$.

We think of Π as a family of mappings (*projections*)

$$\Pi_\lambda := \Pi(\lambda, \cdot) : \Omega \to \mathbb{R}^m, \quad \lambda \in \Lambda$$

with parameter space Λ.

Example: Orthogonal projections onto m-planes in $\mathbb{R}^n$.

(Choose Λ to be a coordinate chart of $G(n,m)$ and identify m-planes V with $\mathbb{R}^m$ by an isometric isomorphism smootly in $G(n,m)$.)

Transversality and projections theorems

Theorem [Peres and Schlag, 2000].

If the mapping Π is sufficiently *regular* and (locally) *transversal*, then the family of mappings Π_λ, $\lambda \in \Lambda$ satisfies all the projection theorems of the previous slide. (The Bes-Fed theorem is due to Hovila 2014.)

Regularity: We assume that Π is C^L-smooth for some $L \geq 2$.

(This assumption is stronger than necessary but will simplify things.)

Transversality: For all $v \neq w \in \Omega$ and $\lambda \in \Lambda$, define:

$$\Phi(\lambda, v, w) := \frac{\Pi(\lambda, v) - \Pi(\lambda, w)}{|v - w|}.$$

The family Π is transversal if there exists $C > 0$ so that for all λ, v, w:

$$|\Phi(\lambda, v, w)| \leq C \;\Rightarrow\; \left|\det[\mathrm{D}_\lambda \Phi(\lambda, v, w)(\mathrm{D}_\lambda \Phi(\lambda, v, w))^{\mathsf{T}}]\right| \geq C^2$$

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Special case $k = m = 1$

$$|\Phi(\lambda, v, w)| \leq C \;\Rightarrow\; |\mathrm{d}_\lambda \Phi(\lambda, v, w)| \geq C$$

$\rightarrow$ Intuition: if v and w are (close to) collapsed into one point by Π_λ. Then as λ is varied, the images of v and w under Π_λ separate quickly.

Transversality and projections theorems

For a family of projections $\Pi : \Lambda \times \Omega \to \mathbb{R}^m$:

<u>Lemma 1</u>: Local transversality is preserved quantitatively under C^2-smooth change of coordinates:

$$\begin{array}{ccc} \Lambda \times \Omega & \xrightarrow{\ \Pi\ } & U \\ \Big\downarrow{\scriptstyle f\times g} & & \Big\downarrow{\scriptstyle h} \\ \tilde{\Lambda} \times \tilde{\Omega} & \xrightarrow{\ \tilde{\Pi}\ } & \tilde{U} \end{array}$$

(This can be considered a folklore Lemma. We prove a formal version in I.-Lukyanenko, Arxiv2021.)

A different perspective on Marstrand...

Let us restate Marstrand's theorem in $\mathbb{R}^2$:

instead of projecting a set A onto a varying line L, rotate the set A and project onto a fixed line, that is, ...

Let $\pi : \mathbb{C} \to \mathbb{R}$ be defined by $\pi(x) = \mathrm{Re}(x)$ *(the base projection)*

Let $R_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ be the counterclockwise rotation by angle $\lambda \in \Lambda = (0, \pi)$. *(family of motions)*

Define the projection family $\Pi_\lambda : \mathbb{C} \to \mathbb{R}$ by $\Pi_\lambda(x) = \pi(R_\lambda(x))$.

Then Marstrand's theorem states as follows: for every Borel set $A \subset \mathbb{R}^2$,

$$\mathscr{H}^1\{\lambda \in \Lambda : \dim \Pi_\lambda(A) < \min\{1, \dim A\}\}) = 0$$

Question: What if we move A around by a different group of motions that occur naturally in a geometric context?

Families induced by Möbius transformations

Consider the family of projections $\Pi : \mathrm{M\ddot{o}b} \times \hat{\mathbb{C}} \to \mathbb{R}$ given by

$$\Pi(g, z) = \mathrm{Re}(g(z)).$$

Its domain is $\mathrm{M\ddot{o}b} \times \hat{\mathbb{C}} \setminus \{(g, g^{-1}(\infty)) : g \in \mathrm{M\ddot{o}b}\}$

Theorem 1. (I.-Lukyanenko, Arxiv2021) $\Pi : \mathrm{M\ddot{o}b} \times \hat{\mathbb{C}} \to \mathbb{R}$ is locally transversal and therefore satisfies projection theorems on its domain.

Lemma 2: transversality is preserved if enlarging the family (as a product)

Proof: write $D_\lambda \Phi$ in the definition of transversality as block matrices.

Theorem 1 now follows from the fact that transversality holds for the family induced by $O(2)$ since $O(2) \subset \mathrm{M\ddot{o}b}$ a Lie subgroup.

Question: What about Lie subgroups $\Gamma \subset \mathrm{M\ddot{o}b}$?

Families induced by Möbius transformations

Let $\Gamma \subset \mathrm{M\ddot{o}b}$ be a Lie subgroup.

Consider the family of projections $\Pi : \Gamma \times \hat{\mathbb{C}} \to \mathbb{R}$ given by

$$\Pi(g, z) = \mathrm{Re}(g(z)).$$

Its domain is $\Gamma \times \hat{\mathbb{C}} \setminus \{(g, g^{-1}(\infty)) : g \in \Gamma\}$

Note: transversality is in general not preserved when passing to smaller families -> We have to actually do some work here!

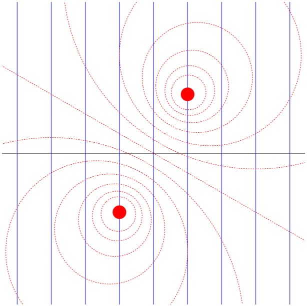
Assume Γ is 1-dimensional.

Let us write Γ in terms of an element $A = (a_{ij})$ in the Lie algebra $\mathfrak{sl}(2, \mathbb{C})$:

$$\Gamma = \{\gamma_t = \exp(At) : t \in \mathbb{R}\}$$

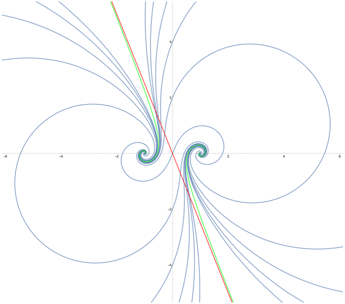
Projections induced by Möb

Two examples



Γ is a compact 1-dim subgroup

$\Gamma \sim_{SL(2,\mathbb{C})} O(2)$



Γ is a loxodromic motion (non-compact)

$\Gamma \sim_{\text{Möb}} \{z \mapsto e^{(a+ib)t} z : t \in \mathbb{R}\}$

Projections induced by Möb

For $\Gamma = \{\gamma_t = \exp(At) : t \in \mathbb{R}\}$ 1-dim Lie subgroup of Möb,

and $\Pi : \Gamma \times \hat{\mathbb{C}} \to \mathbb{R}$ the projection family given by $(\gamma, z) \mapsto \mathrm{Re}(\gamma(z))$ wherever $\gamma(z) \neq \infty$,

Theorem 2. (I.-Lukyanenko, Arxiv2021)

Π is locally transversal precisely away from the closure S of the set

$$\{(\gamma_t, z) : \mathrm{Im}(a_{11} - a_{21}\gamma_t(z)) = 0,\ t \in \mathbb{R},\ z \in \widehat{\mathbb{C}}\},$$

Proof:

- Lemma 3: the nice geometry of our setting implies: it suffices to check transversality for $\gamma_t = id$ (i.e. at $t = 0$).
- Linearize $t \mapsto \Pi(\gamma_t, x)$ using Taylor expansion of $\exp$
- Establish transversality at $t = 0$ (hands-on estimates)

Projections induced by Möb

Question: What about projection theorems?

Combining Theorem 2 with Peres and Schlag's Theorem yields: projection theorems hold wherever we have $\mathrm{Im}(a_{11} - a_{21}\gamma_t(z)) \neq 0$, where $A = (a_{ij})$ is the Lie algebra element generating Γ by $t \mapsto \exp(At)$.

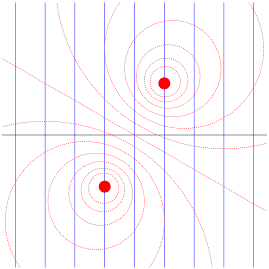
If $\underline{a_{21} = 0}$, then Γ preserves ∞ and:

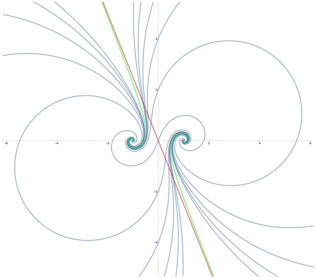
- if $\mathrm{Im}(a_{11}) = 0$, then ($S = \Gamma \times \widehat{C}$ and more importantly) Γ consists of translations and dilations. projection theorems fail completely.
- if $\mathrm{Im}(a_{11}) \neq 0$, then $\Gamma = O(2)$ and projection theorems hold.

If $\underline{a_{21} \neq 0}$, consider the compactified line $L = \{z : \mathrm{Im}(a_{11} - a_{21}z) = 0\}$ in $\widehat{\mathbb{C}}$, that is, the portion of the closure or S that comes from $t = 0$.

Note that $\gamma_t(\infty) \in L$ and L is tangent to the orbit $\Gamma(\infty) = \{\gamma_t(\infty) : t \in \mathbb{R}\}$ at $\gamma_0(\infty) = \infty$.

Projections induced by Möb





There are three cases:

- (bad case) L is a vertical line and $L = \Gamma(\infty)$: projection thms fail for subsets of L (L is projected to a single point) but hold elsewhere.
- (good case) L is non-vertical and equals the orbit $\Gamma(\infty)$: the restriction of the projection to L is a similarity mapping, so Hausdorff measure and dimension is preserved along L.
- (artifact case) $L \neq \Gamma(\infty)$: any sufficiently small set inside L will be moved away from L by γ_t after some time. Thus projection thms hold.

Projections induced by Möb

This proves the following theorem...

Theorem 3. (I.-Lukyanenko, Arxiv2021)

Let $\Gamma \subset \mathrm{M\ddot{o}b}$ be a one-dimensional Lie subgroup, and $\Pi : \Gamma \times \hat{\mathbb{C}} \to \mathbb{R}$ the family given by $\Pi(\gamma, z) = \mathrm{Re}(\gamma(z))$. Then Π satisfies projection theorems, with the following natural exceptions:

- If Γ consists of Euclidean dilations and translations, then projection theorems fail globally.
- If the orbit $\Gamma(\infty)$ is a vertical line, then projection theorems fail along this line.

More results and final remarks

In our paper (I.-Lukyanenko, Arxiv2021), we also cover:

- Analogs of Theorems 1, 2, and 3 for real-linear fractional transformations: $\Pi : \mathrm{PSL}(3, \mathbb{R}) \times \mathbb{RP}^2 \to \mathbb{RP}^1$
- Lemmas 2 and 3 hold in a more general framework: $\Pi : G \times M \to N$ (M, N manifolds, G a Lie group acting on M).
- Transversality and projection thms for closest point projections in hyperbolic n-space and n-sphere (improving results by Balogh-I.)

Further directions:

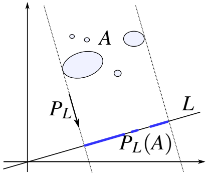
- 2-dim subgroups? Respectively, are 1-dim subgroups the only minimal transversal subgroups?
- Other groups acting on surfaces? E.g. focussing on properties of group actions.
- Higher dimensions?
- Similar considerations in Sub-Riemannian setting?

Thank you for listening!

Geometric measure theory

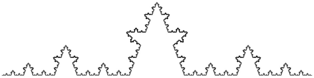
Projection theorems

How well are sets generically preserved under families of projections?



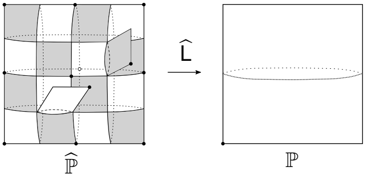
IFS fractal curves

When are two IFS curves bi-Lipschitz or quasisymmetrically equivalent?



The dynamics of Thurston maps

- Thurston's characterizations of rational
- The curve attractor problem



- Visual spheres of Thurston maps