The Castelnuovo-Mumford Regularity of Matrix Schubert Varieties

Anna Weigandt (MIT)
weigandt@mit.edu

Fellowship of the Ring
February 1st, 2022

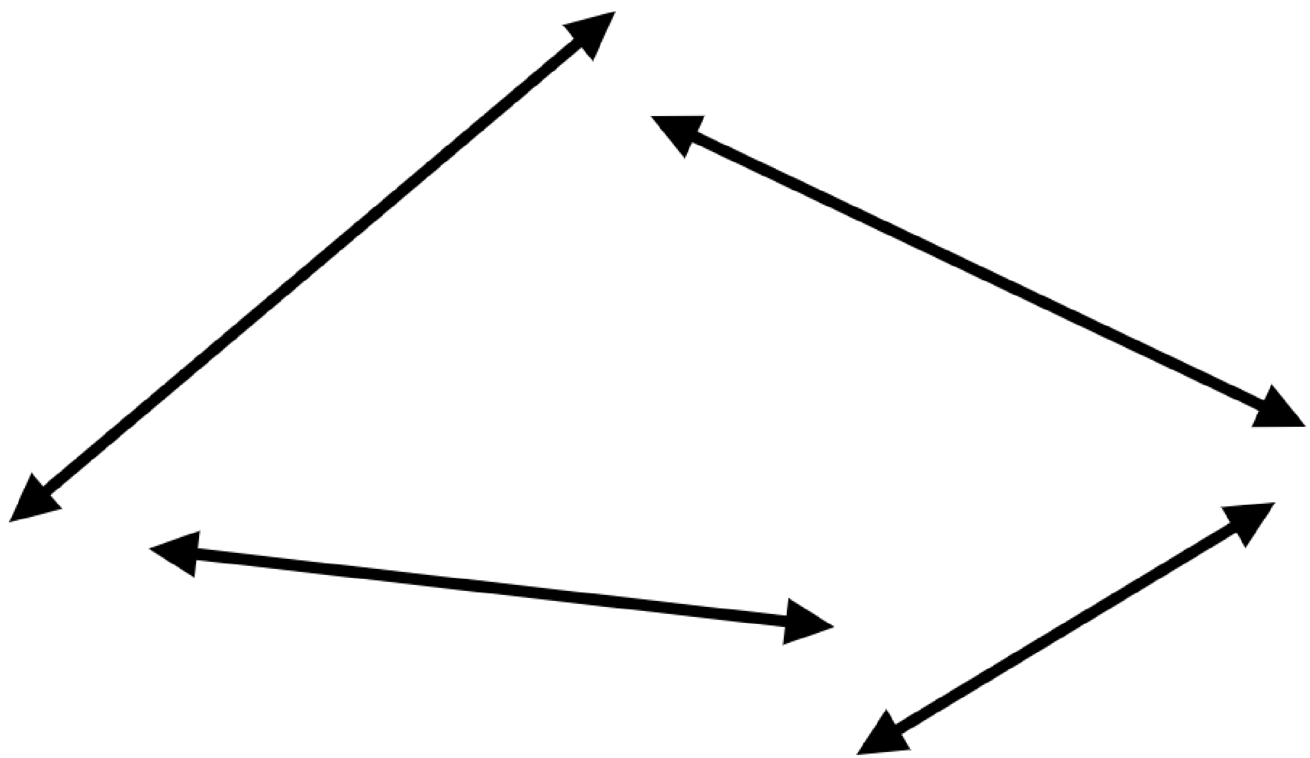
Joint work with
Oliver Pechenik (Waterloo) and David Speyer (Michigan)

Goal:

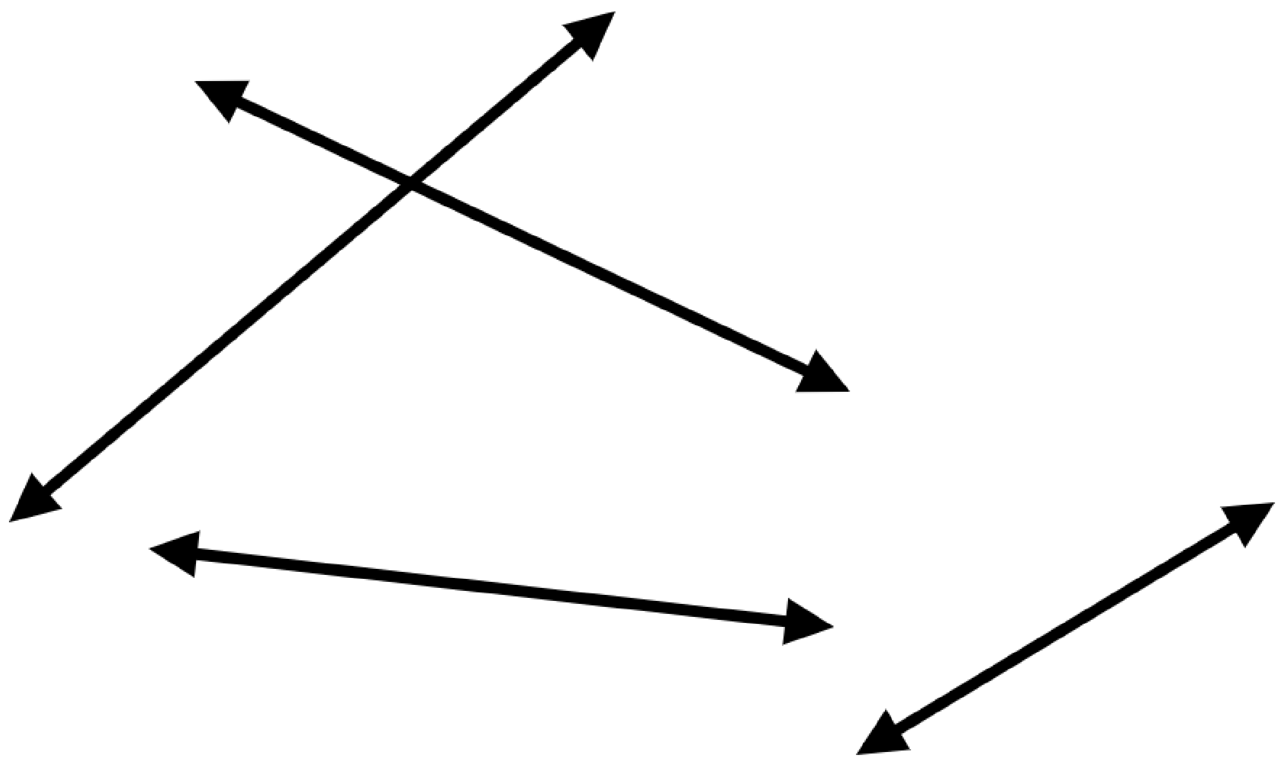
Use **combinatorics** to study questions from **commutative algebra**.

Schubert Calculus

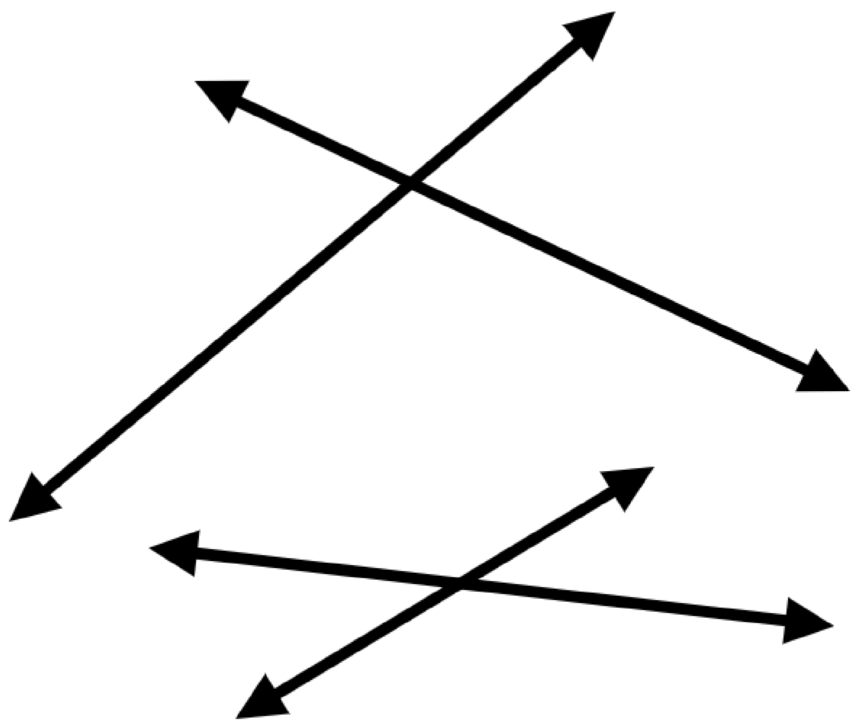
How many lines meet four fixed lines in three space?



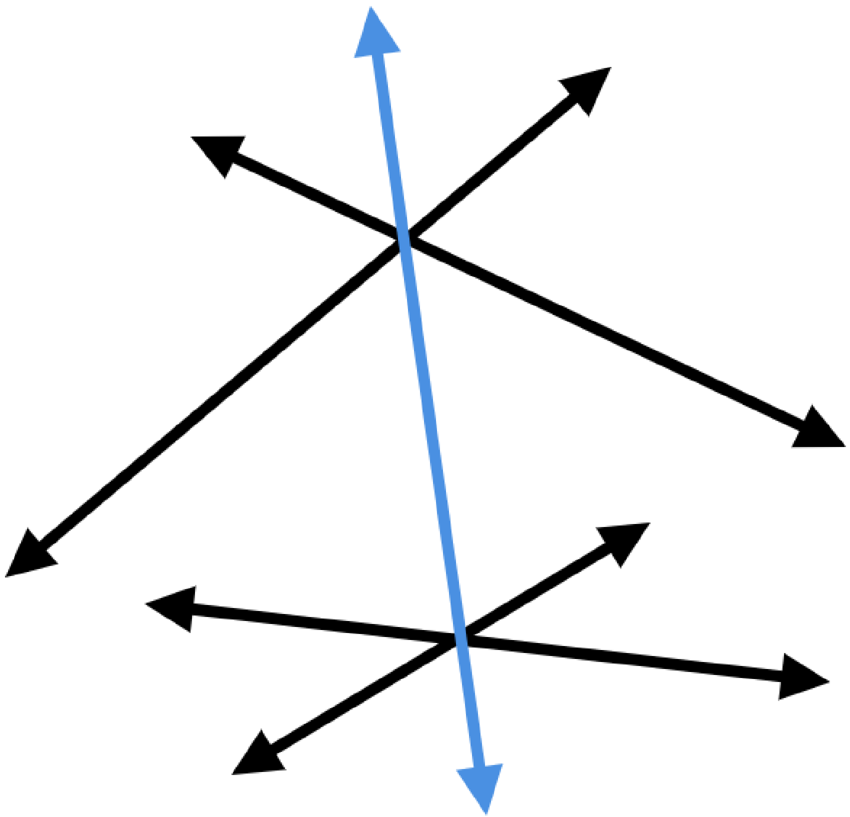
How many lines meet four fixed lines in three space?



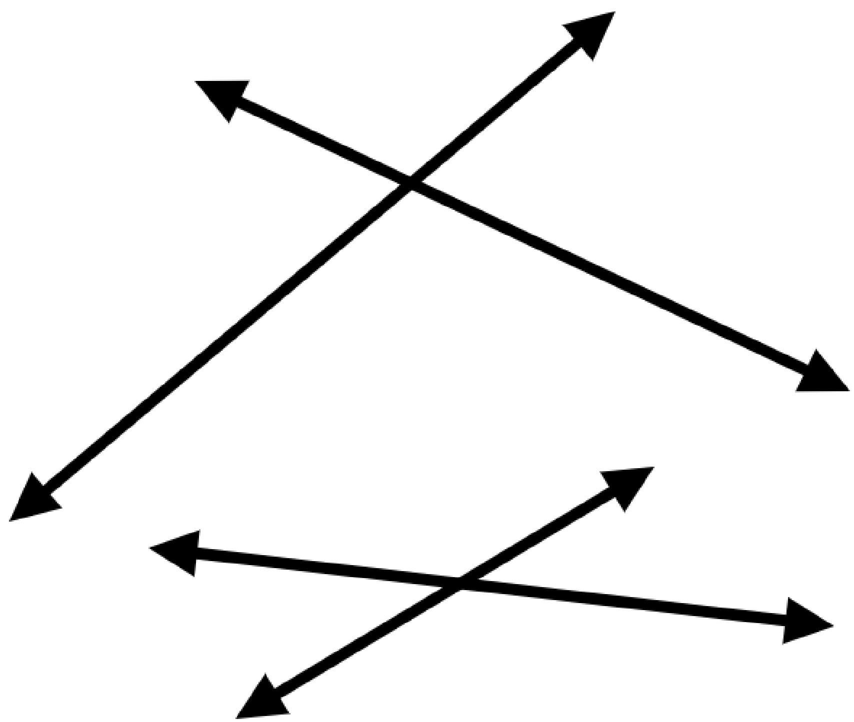
How many lines meet four fixed lines in three space?



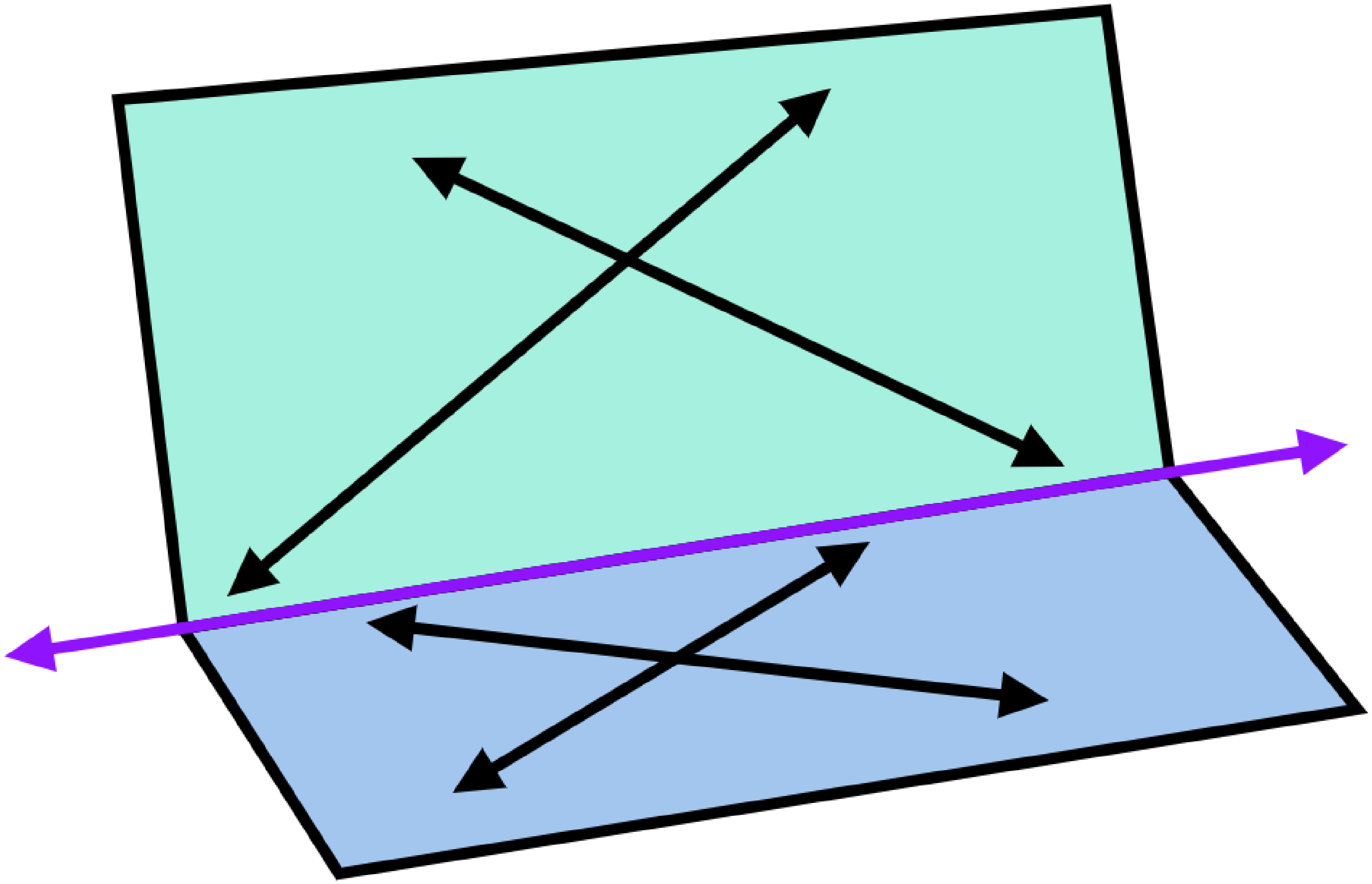
How many lines meet four fixed lines in three space?



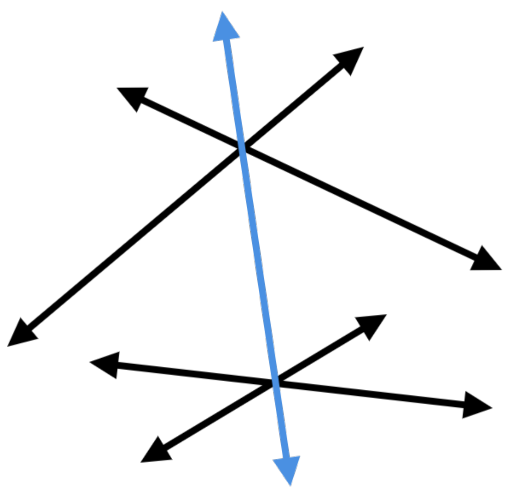
How many lines meet four fixed lines in three space?

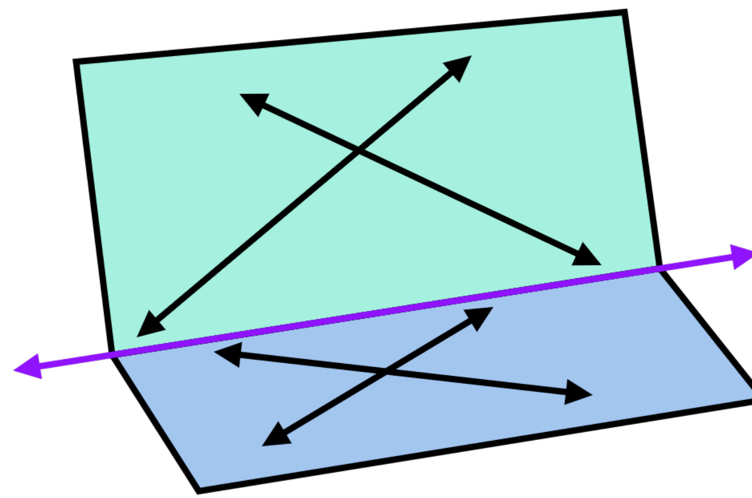


How many lines meet four fixed lines in three space?



In this special case, <u>two lines</u> meet the four fixed lines.





Hermann Schubert (1848-1911) said this holds generally by "conservation of number."

How many twisted cubic curves are tangent to 12 fixed quadric surfaces in three-space?

Quelle für viele andere Anzahlen. Z. B. ergiebt sich *die Zahl N der zwölf gegebene Flächen zweiten Grades berührenden cubischen Raumcurven* vermöge der in § 14 am Schluss von Nr. 1 (pag. 57) angegebenen Formel aus den hier unter Nr. XIV berechneten Anzahlen auf folgende Weise:

$$N = (2 . \nu + 2 . \varrho)^{12} = 2^{12} . (\nu^{12} + 12_1 . \nu^{11}\varrho + 12_2 . \nu^{10}\varrho^2 + \ldots + \varrho^{12})$$
$$= 2^{12} . (80160 + 12_1 . 134400 + 12_2 . 209760 + 12_3 . 297280$$
$$+ 12_4 . 375296 + 12_5 . 415360 + 12_6 . 401920 + 12_7 . 343360$$
$$+ 12_8 . 264320 + 12_9 . 188256 + 12_{10} . 128160$$
$$+ 12_{11} . 85440 + 56960)$$
$$= 5819539783680.$$

Image from Schubert's
"Kalkül der abzähleden Geometry" – 1879

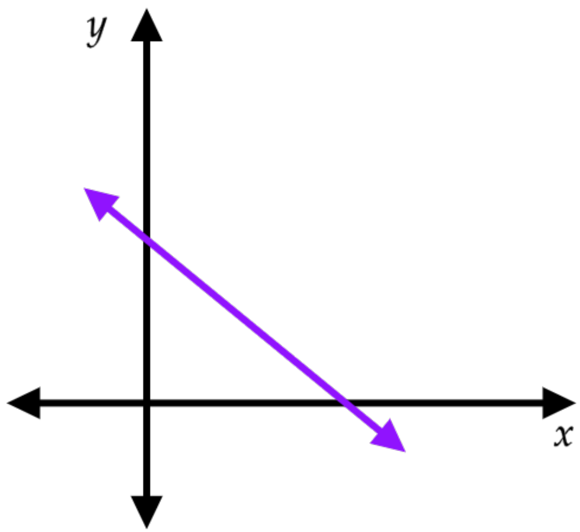
Hilbert's 15th Problem

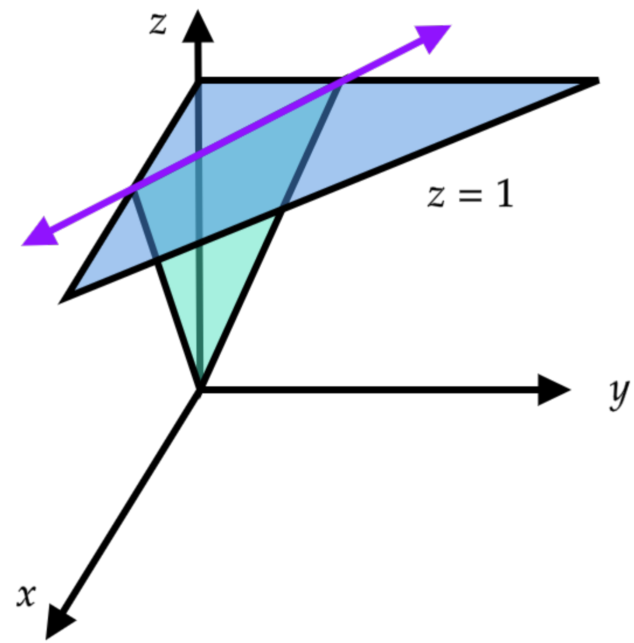
In 1900, Hilbert presented a list of 23 problems.

The 15th problem was:

- establish validity of Schuberts methods of calculating in a rigorous way
- find ways to predict the number of solutions without algebraic elimination of systems of equations.

We can identify affine lines in n space with planes through the origin in $n+1$ space.





For lines in 3-space, we should be thinking about planes through the origin in 4 space, ie the Grassmannian

$$Gr(2,4) = \{ V \subseteq \mathbb{C}^4 : V \text{ is a 2-dim subspace} \}$$

Intersecting lines in 3 space

$$\Updownarrow$$

$$\dim(V \cap W) \geq 1$$

Given $V \in Gr(2,4)$, define

$$X_{\square}(V) = \{ W \in Gr(2,4) : \dim(V \cap W) \geq 1 \}.$$

As a set, $X_{\square}(V)$ encodes "lines meeting a fixed line."

Given $V \in Gr(2,4)$, define

$$X_{\square}(V) = \{ W \in Gr(2,4) : \dim(V \cap W) \geq 1 \}.$$

As a set, $X_{\square}(V)$ encodes "lines meeting a fixed line."

Take $V^1, V^2 \in Gr(2,4)$:

$$X_{\square}(V^1) \cap X_{\square}(V^2)$$

encodes lines that meet two fixed lines.

Given $V \in Gr(2,4)$, define

$$X_{\square}(V) = \{ W \in Gr(2,4) : \dim(V \cap W) \geq 1 \}.$$

How many lines meet 4 lines in 3-space is the number of points in:

$$X_{\square}(V^1) \cap X_{\square}(V^2) \cap X_{\square}(V^3) \cap X_{\square}(V^4).$$

Intersection Theory

Fix a non singular variety Z.

- Each subvariety $X \subseteq Z$ defines a class $\sigma_X \in H^*(Z)$
- "Equivalent" subvarieties define the same class.

When $\dim(X \cap Y) = \dim(X) + \dim(Y) - \dim(Z)$,

$$\sigma_X \cdot \sigma_Y = \sum_W \sigma_W$$

where the W's are the irreducible components of $X \cap Y$.

Going back to this example:

$$X_{\square}(V^1) \cap X_{\square}(V^2) \cap X_{\square}(V^3) \cap X_{\square}(V^4)$$

these all correspond to the same

$$\sigma_{\square} \in H^*(Gr(2,4)).$$

We want to compute $\sigma_{\square}^4$.

When you do, you get

$$2 \cdot \sigma_{\{pt\}}.$$

More generally, $H^*(Gr(k,n))$ has a basis of Schubert classes σ_λ indexed by partitions.

Understanding this ring turns into understanding the product

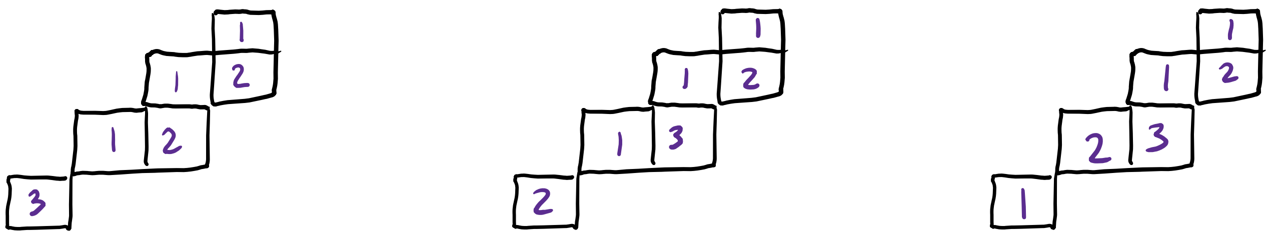
$$\sigma_\mu \cdot \sigma_\nu = \sum_\lambda c_{\mu\nu}^{\lambda} \sigma_\lambda$$

Littlewood-Richardson coefficients

Littlewood-Richardson Tableaux

There are numerous, positive combinatorial formulas for $c_{\mu\nu}^{\lambda}$.

Example: $c_{321,321}^{4431} = 3$



Littlewood-Richardson 1934, Thomas 1974, Schützenberger 1977, Macdonald 1995, ...

Modern Schubert Calculus

In modern Schubert calculus, we study more general spaces

- flag varieties
- Grassmannians in other Lie types
- quotients of algebraic groups
- generalized determinantal varieties

Using various cohomology theories.

The Flag Variety

The complete flag variety $GL(n)/B$ has a special family of subvarieties $\{X_w : w \in S_n\}$ called Schubert varieties.

Each Schubert variety defines a Schubert class $\sigma_w \in H^*(GL(n)/B)$. The Schubert classes are a linear basis for this ring.

$$\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \sigma_w$$

By Borel's isomorphism:

$$H^*(GL(n)/B) \cong \mathbb{Z}[x_1, x_2, \ldots, x_n]/I^{S_n}.$$

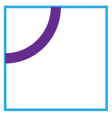
Lascoux and Schützenberger (1982) defined Schubert polynomials $\{\mathfrak{S}_w(x) : w \in S_n\}$ which are a choice of coset representatives for the images of the Schubert classes under Borel's isomorphism.

There's also double Schubert polynomials $\{\mathfrak{S}_w(x;y) : w \in S_n\}$ which are enriched versions of single Schuberts and satisfy

$$\mathfrak{S}_w(x) = \mathfrak{S}_w(x;0).$$

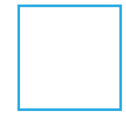
They tell you about T-equi cohomology

Pipe Dreams

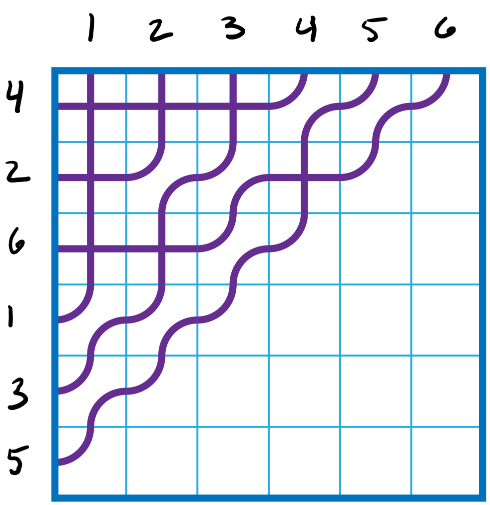


on main antidiagonal

above main antidiagonal



below main antidiagonal

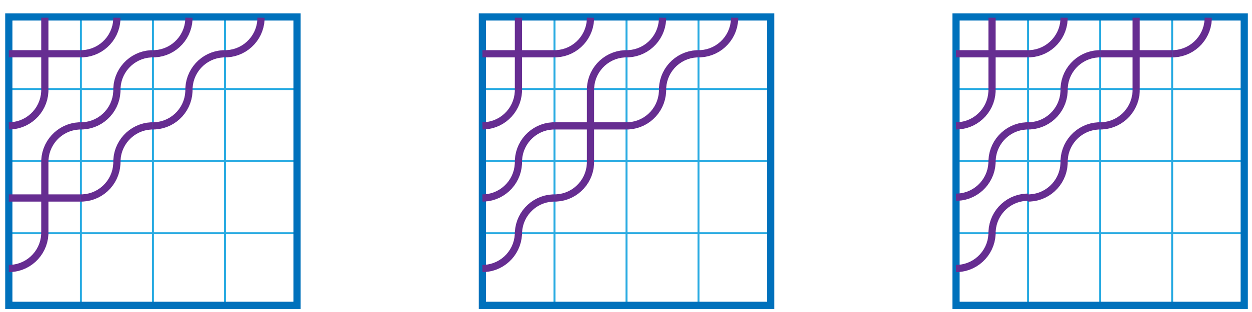


Fill $n \times n$ grid with n pipes that start at the top end at the left and pairwise cross at most one time.

$$Wt(P) = (x_1-y_1)(x_1-y_2)(x_1-y_3)(x_2-y_1)(x_2-y_4)(x_3-y_1)(x_3-y_2)$$

Theorem (Fomin-Kirillov 1996 / Bergeron-Billey 1993): The double Schubert polynomial is

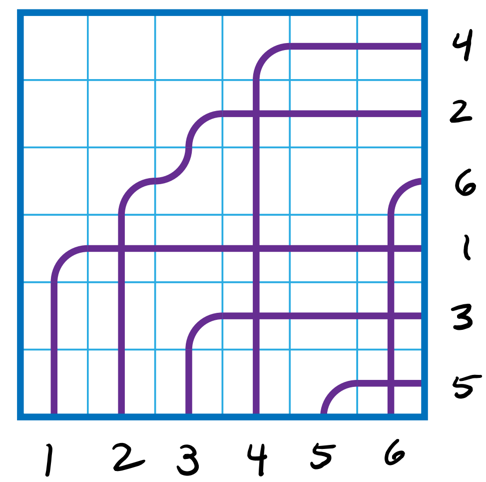
$$\mathfrak{S}_w(x;y) = \sum_{P \in \mathrm{Pipes}(w)} \mathrm{wt}(P).$$

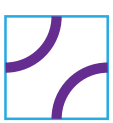


$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

Bumpless Pipe Dreams

Fill $n \times n$ grid with n pipes that start at the bottom, end at the right and pairwise cross at most one time.





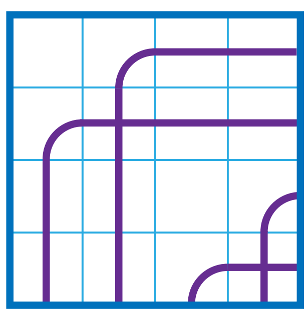
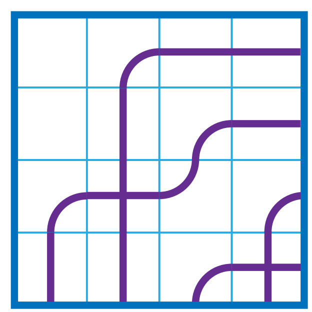
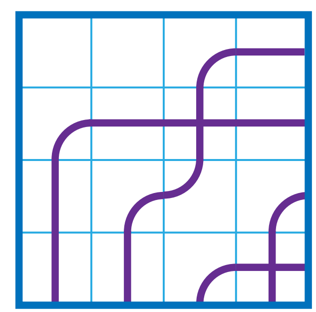
no bumps allowed!

$$wt(P) = (x_1-y_1)(x_1-y_2)(x_1-y_3) \cdot (x_2-y_1)(x_2-y_2)(x_3-y_1)(x_3-y_5)$$

Theorem (Lam-Lee-Shimozono 2018):

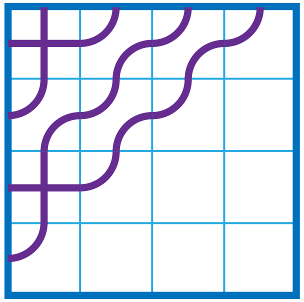
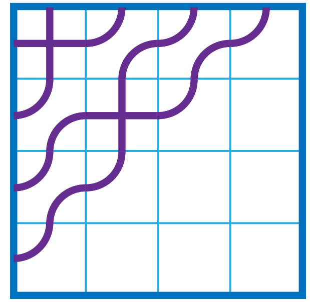
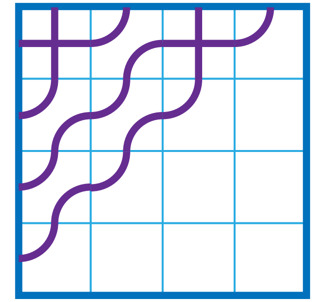
The double Schubert polynomial is

$$\mathfrak{S}_w(x;y) = \sum_{P \in BPD(w)} wt(P).$$

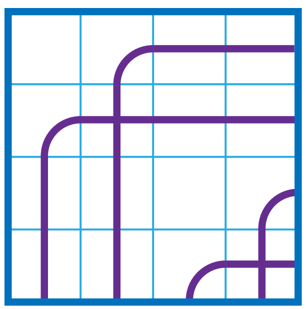
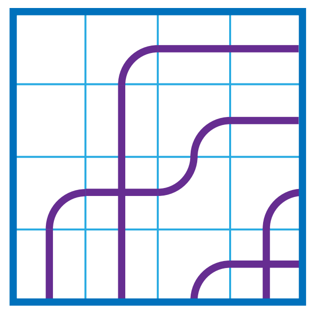
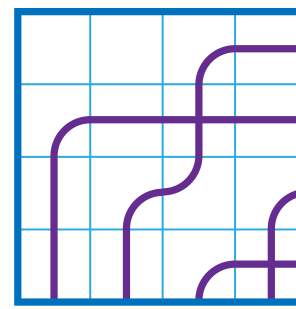
  

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_3) \quad + \quad (x_1-y_1)(x_2-y_1) \quad + \quad (x_1-y_1)(x_1-y_2)$$

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

$$\mathfrak{S}_{2143}(x) = x_1 x_3 \quad + \quad x_1 x_2 \quad + \quad x_1^2$$

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_3) \quad + \quad (x_1-y_1)(x_2-y_1) \quad + \quad (x_1-y_1)(x_1-y_2)$$

Schubert Determinantal Ideals

Let $\mathrm{Mat}(m,n)$ be the space of $m \times n$ matrices and

$$Z_{m,n} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

a generic matrix.

$R = \mathbb{K}[Z_{m,n}] = \mathbb{K}[z_{11}, z_{12}, \ldots, z_{mn}]$ is the coordinate ring of $\mathrm{Mat}(m,n)$.

Let $I_k(Z_{m,n})$ be the ideal generated by the minors of size k in $Z_{m,n}$.

$I_k(Z_{m,n})$ is a determinantal ideal.

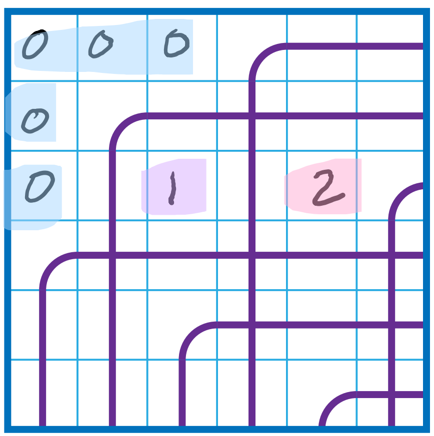
Example:

$$I_2\left(\begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \end{bmatrix}\right) = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \right\rangle$$

$V(R/I_{k+1}(Z_{m,n}))$ is the set of matrices $M \in Mat(m,n)$ such that $\operatorname{rank}(M) \leq k$.

Schubert Determinantal Ideals

Given $w \in S_n$, the Schubert determinantal ideal $I_w \subseteq R = \mathbb{C}[Z_{n,n}]$ is generated by minors of varying size, determined by w.



$$I_{426135} = \langle z_{11}, z_{12}, z_{13}, z_{21}, z_{31} \rangle$$

$$+ \left\langle 2\times 2 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{bmatrix} \right\rangle$$

$$+ \left\langle 3\times 3 \text{ minors in } \begin{bmatrix} z_{11} & z_{12} & z_{13} & z_{14} & z_{15} \\ z_{21} & z_{22} & z_{23} & z_{24} & z_{25} \\ z_{31} & z_{32} & z_{33} & z_{34} & z_{35} \end{bmatrix} \right\rangle$$

Fulton's Generators

Matrix Schubert Varieties

$$\begin{tikzcd} GL(n) \arrow[r, hook, "i"] \arrow[d, two heads, "\pi"'] & Mat(n) \\ GL(n)/B & \end{tikzcd} \qquad \begin{tikzcd} \pi^{-1}(X_w) \arrow[r, "i"] \arrow[d, two heads] & \overline{i(\pi^{-1}(X_w))} \\ X_w & \end{tikzcd}$$

Flag variety

Schubert variety

Matrix Schubert variety

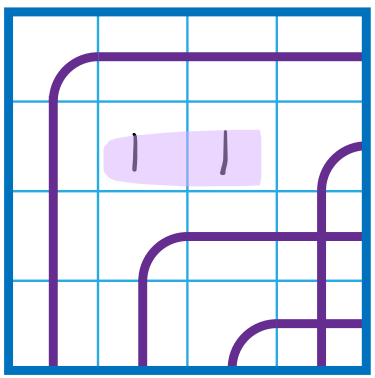
Write $\overline{X_w} := \overline{i(\pi^{-1}(X_w))}$.

Theorem (Fulton 1992): I_w is prime and $\overline{X_w} = V(R/I_w)$.

Determinantal Ideals Revisited

Up to affine factors, determinantal varieties can be realized as matrix Schubert varieties associated to bigrassmannian permutations.

Example:

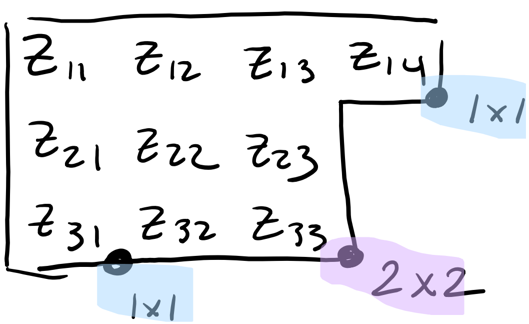


$$I_{1423} = \left\langle \begin{vmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{vmatrix}, \begin{vmatrix} z_{11} & z_{13} \\ z_{21} & z_{23} \end{vmatrix}, \begin{vmatrix} z_{12} & z_{13} \\ z_{22} & z_{23} \end{vmatrix} \right\rangle$$

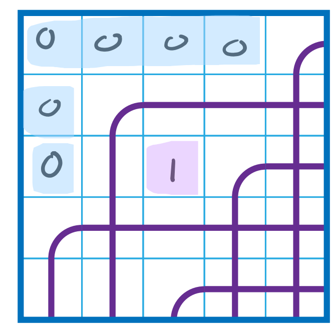
Ladder Determinantal Ideals

One sided mixed ladder ideals are vexillary (2143-pattern avoiding) Schubert determinantal ideals.

Example:



$\longleftrightarrow$



$$I_{52413} = \langle z_{11}, z_{12}, z_{13}, z_{14}, z_{21}, z_{31} \rangle$$

$$+ \langle 2\times 2 \text{ minors in } \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \rangle$$

A Recipe

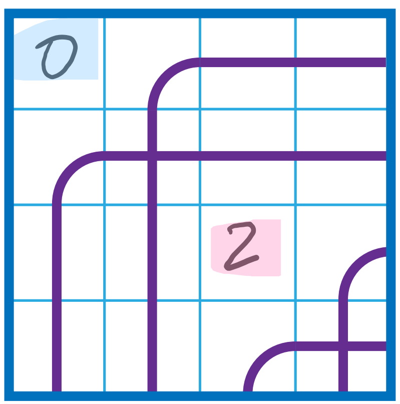
① Fix an antidiagonal term order $<_a$ on R.

② Compute the initial ideal $\mathrm{init}_{<_a}(I_w)$.

③ Take the primary decomposition.

Example:

$$I_{2143} = \left\langle z_{11},\ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$



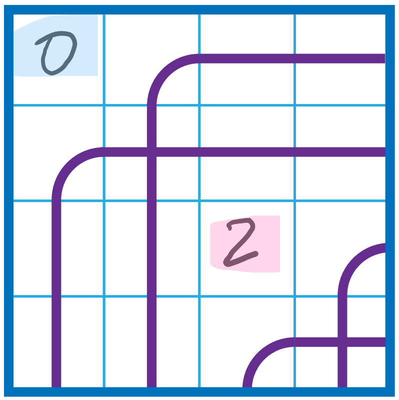
$$\mathrm{init}_{<_a}(I_{2143}) = \langle z_{11},\ z_{31} z_{22} z_{13} \rangle$$

$$= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle$$

$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) + (x_1-y_1)(x_2-y_2) + (x_1-y_1)(x_1-y_3)$$

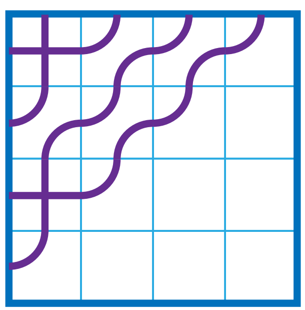
Example:

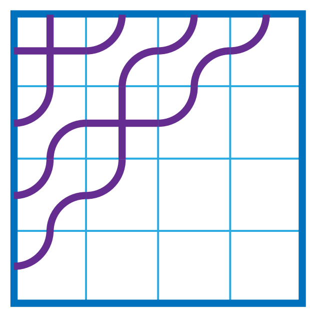
$$I_{2143} = \left\langle z_{11},\ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

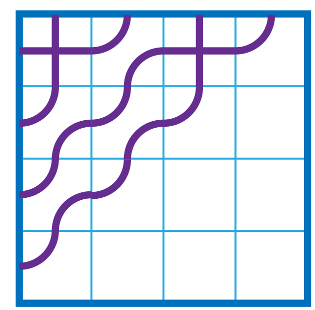


$$\mathrm{init}_{<_a}(I_{2143}) = \langle z_{11},\ z_{31} z_{22} z_{13} \rangle$$

$$= \langle z_{11}, z_{31} \rangle \cap \langle z_{11}, z_{22} \rangle \cap \langle z_{11}, z_{13} \rangle$$







$$\mathfrak{S}_{2143}(x;y) = (x_1-y_1)(x_3-y_1) \quad + \quad (x_1-y_1)(x_2-y_2) \quad + \quad (x_1-y_1)(x_1-y_3)$$

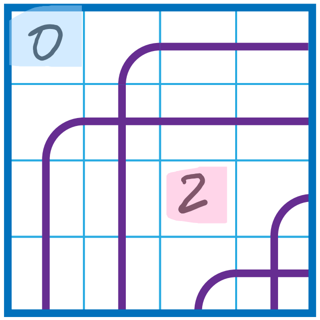
Theorem (Knutson - Miller 2005):

1. Fulton's generators are a Gröbner basis for I_w under any antidiagonal term order.

2. $\mathrm{init}_{<_a}(I_w)$ is radical.

3. $$\mathrm{init}_{<_a}(I_w) = \bigcap_{P \in \mathrm{Pipes}(w)} I_P$$

Example:

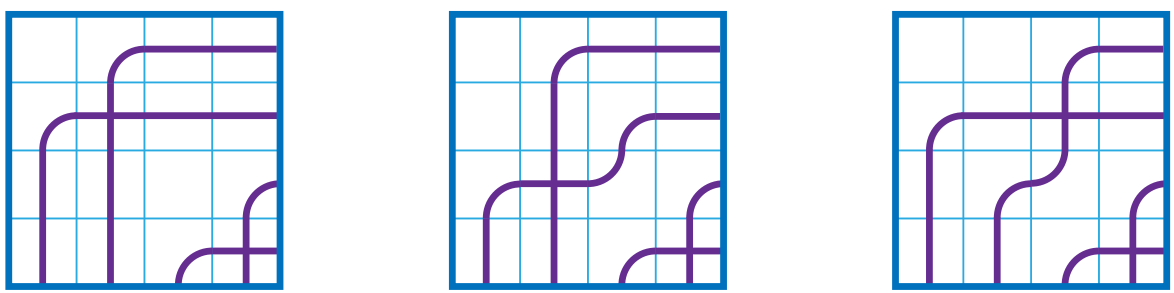


$$I_{2143} = \left\langle z_{11},\ \begin{vmatrix} z_{11} & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

$$= \left\langle z_{11},\ \begin{vmatrix} 0 & z_{12} & z_{13} \\ z_{21} & z_{22} & z_{23} \\ z_{31} & z_{32} & z_{33} \end{vmatrix} \right\rangle$$

$$\mathrm{init}_{<_d}(I_{2143}) = \langle z_{11},\ z_{33} z_{21} z_{12} \rangle$$

$$= \langle z_{11}, z_{33} \rangle \cap \langle z_{11}, z_{21} \rangle \cap \langle z_{11}, z_{12} \rangle$$



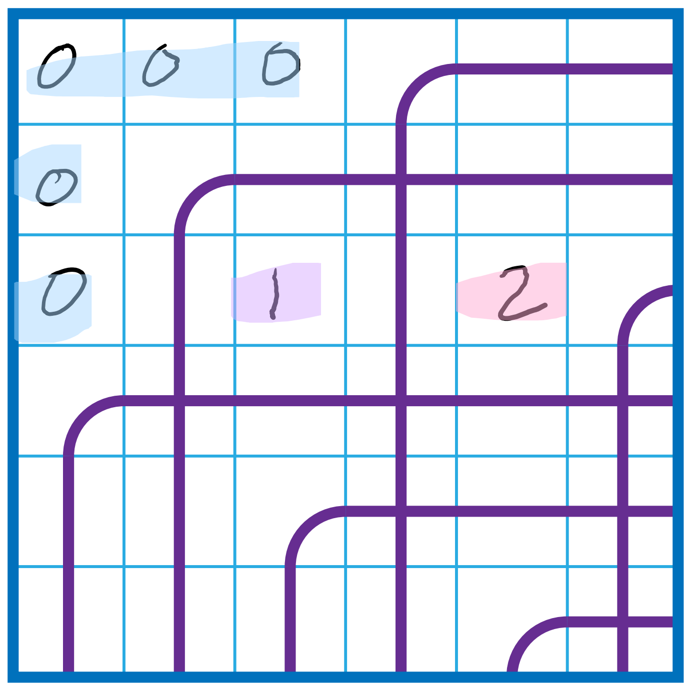
$$\mathfrak{G}_{2143}(x;y) = (x_1 - y_1)(x_3 - y_3) \quad + \quad (x_1 - y_1)(x_2 - y_1) \quad + \quad (x_1 - y_1)(x_1 - y_2)$$

Theorem (Klein - Weigandt 2021):

There is a diagonal term order σ so that the irreducible components of $\mathrm{Spec}(R/\mathrm{init}_\sigma(I_w))$ are labeled (with multiplicity) by elements of $BPD(w)$.

Generalized work of Knutson - Miller - Yong (2009) who studied the vexillary case.

Break!



Castelnuovo-Mumford Regularity

The Castelnuovo-Mumford Regularity

Measures how complicated minimal free resolutions of graded modules can be.

For us: Ring $R = \mathbb{K}[x_1, \dots, x_n]$

Grading $\deg(x_i) = 1$ ← standard grading

Module R/I

← homogeneous ideal

Example:

$R = \mathbb{k}[x]$ $I = \langle x^2 \rangle$

$$0 \to R \xrightarrow{x^2} R \xrightarrow{1} R/I \to 0$$

Minimal free resolution of R/I

Example:

$R = \mathbb{k}[x]$

$I = \langle x^2 \rangle$

$\boxed{\begin{array}{l} R(-j): \\ \deg(x^d) = d + j \end{array}}$

$$0 \to R(-2) \xrightarrow{x^2} R \xrightarrow{1} R/I \to 0$$

Minimal graded free resolution of R/I

The Castelnuovo-Mumford Regularity

Let M be a finitely generated graded module over $R = \mathbb{k}[x_1, x_2, \ldots, x_n]$.

Take a minimal graded free resolution:

$$0 \to F_\ell \to \cdots \to F_1 \to F_0 \to M \to 0.$$

Then we can write $F_i = \bigoplus_{j \geq 0} R(-j)^{\beta_j}$ and set $b_i = \max\{j : \beta_j \neq 0\}$.

The Castelnuovo-Mumford regularity of M is the minimum integer r such that $b_i - i \leq r$ for all i.

Example:

$R = \mathbb{K}[x, y]$ $\quad\quad$ $I = \langle x^5, y^2, x^2y \rangle$

$$0 \to R(-4) \oplus R(-6) \xrightarrow{\begin{bmatrix} -x^2 & 0 \\ y & -x^3 \\ 0 & y \end{bmatrix}} R(-2) \oplus R(-3) \oplus R(-5) \xrightarrow{\begin{bmatrix} y^2 & x^2y & x^5 \end{bmatrix}} R \to R/I \to 0$$

$$\operatorname{reg}(R/I) = \min\{ r \in \mathbb{Z} : b_i - i \le r \text{ for all } i \}$$

Example:

$R = \mathbb{K}[x, y]$ $\qquad I = \langle x^5, y^2, x^2y \rangle$

$$
\begin{tikzcd}
0 \arrow[r] & R(-4)\oplus R(-6) \arrow[r, "{\begin{bmatrix} -x^2 & 0 \\ y & -x^3 \\ 0 & y \end{bmatrix}}"] & R(-2)\oplus R(-3)\oplus R(-5) \arrow[r, "{\begin{bmatrix} y^2 & x^2y & x^5 \end{bmatrix}}"] & R \arrow[r] & R/I \arrow[r] & 0
\end{tikzcd}
$$

$$\begin{matrix} \| & & \| & & \| \\ F_2 & & F_1 & & F_0 \end{matrix}$$

$b_2 - 2 = 4 \qquad b_1 - 1 = 4 \qquad b_0 - 0 = 0$

$\operatorname{reg}(R/I) = \min\{ r \in \mathbb{Z} : b_i - i \leq r \text{ for all } i \}$

$= 4$

The Hilbert Series

Let $M = \bigoplus_{i \geq 0} M_i$ be a finitely generated graded module over $\mathbb{K}[x_1, \ldots, x_n]$

The Hilbert series of M is

$$H(M) = \sum_{i=0}^{\infty} \dim_{\mathbb{K}}(M_i)\, t^i .$$

We can write $H(M) = \dfrac{\mathbb{K}(M)}{(1-t)^n}$.

$K(M)$ is the K-polynomial.

Example:

$R = \mathbb{K}[x, y]$ $\qquad I = \langle x^5, y^2, x^2 y \rangle$

Generators for R/I as a $\mathbb{K}$-vector space:

$1 \quad x \quad x^2 \quad x^3 \quad x^4$

$y \quad xy$

$$H(R/I) = 1 + 2t + 2t^2 + t^3 + t^4$$

$$= \frac{1 - t^2 - t^3 + t^4 - t^5 + t^6}{(1-t)^2}$$

$$K(R/I) = 1 - t^2 - t^3 + t^4 - t^5 + t^6$$

Lemma: If R/I is Cohen-Macaulay, then

$$\operatorname{reg}(R/I) = \deg K(R/I) - \operatorname{ht}_R(I).$$

(See e.g. Benedetti - Varbaro 2015)

Example: $R = \mathbb{K}[x,y]$ $I = \langle x^5, y^2, x^2y \rangle$

① Check R/I is Cohen-Macaulay

② Compute

$K(R/I) = 1 - t^2 - t^3 + t^4 - t^5 + t^6$ and $\operatorname{ht}_R(I) = 2$

③ Conclude $\operatorname{reg}(R/I) = \deg K(R/I) - \operatorname{ht}_R(I)$

$$= 6 - 2$$

$$= 4.$$

Theorem (Fulton 1992):

1. I_w is radical
2. $V(R/I_w)$ is irreducible
3. $ht_R(I_w) = inv(w)$
4. R/I_w is Cohen-Macaulay

Schubert determinantal ideals generalize lots of nice families of ideals from commutative algebra

Grothendieck Polynomials

Grothendieck Polynomials

Grothendieck polynomials were introduced in 1982 by Lascoux and Schützenberger to study the K-theory of the complete flag variety.

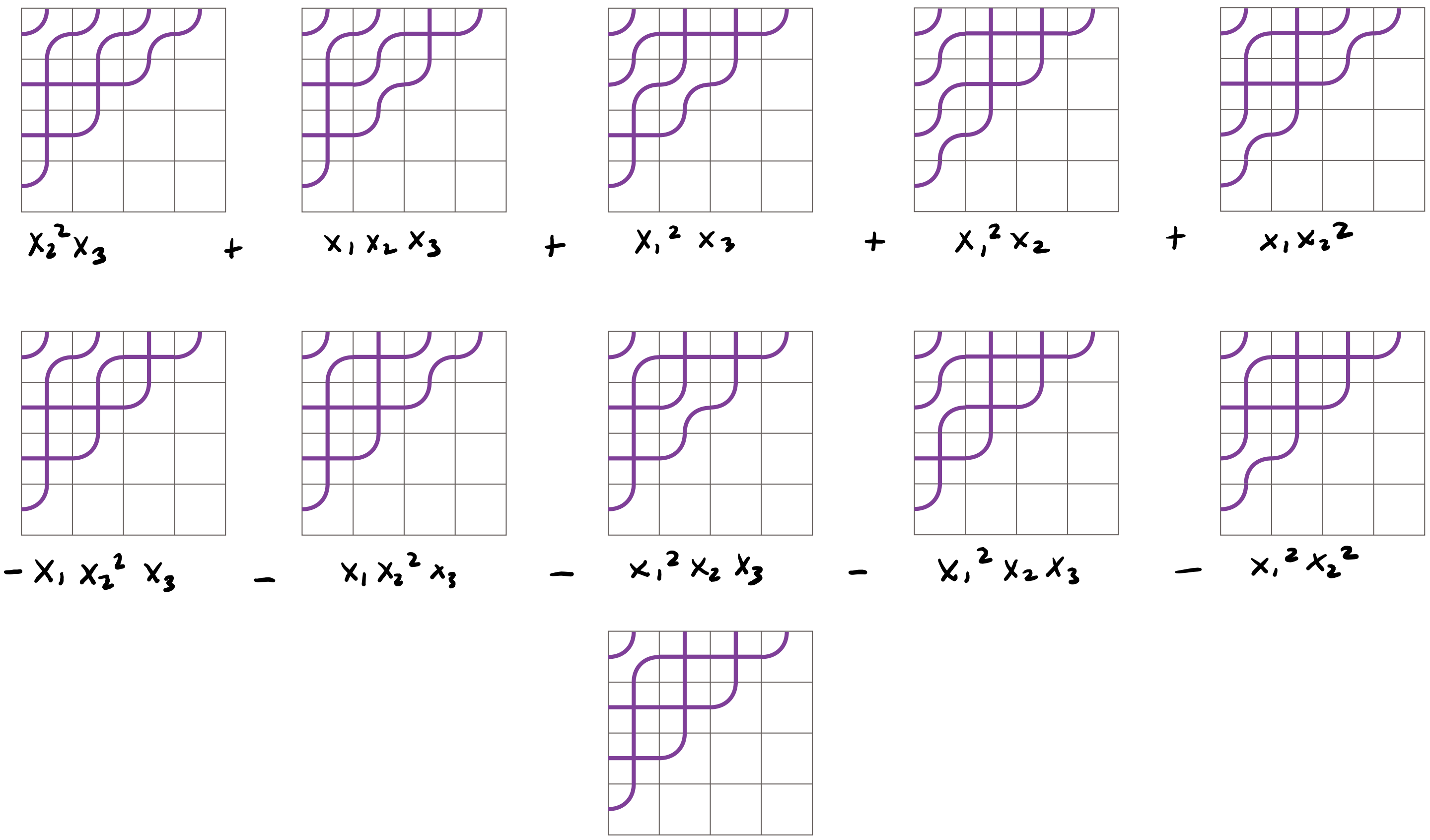
For $w \in S_n$, $G_{n\,n-1\,\cdots\,1} = x_1^{n-1} x_2^{n-2} \cdots x_{n-1}$ and if $w(i) > w(i+1)$, define $G_{ws_i} = \overline{\partial}_i (G_w)$

where $\overline{\partial}_i(f) = \dfrac{(1-x_{i+1})f - (1-x_i)\, s_i \cdot f}{x_i - x_{i+1}}$.

Grothendieck Polynomials for S_3

$$
\begin{tikzcd}
 & G_{321} = x_1^2 x_2 \arrow[dl, dash, "\bar\partial_1"'] \arrow[dr, dash, "\bar\partial_2"] & \\
G_{231} = x_1 x_2 \arrow[d, dash, "\bar\partial_2"'] & & G_{312} = x_1^2 \arrow[d, dash, "\bar\partial_1"] \\
G_{213} = x_1 \arrow[dr, dash, "\bar\partial_1"'] & & G_{132} = x_1 + x_2 - x_1 x_2 \arrow[dl, dash, "\bar\partial_2"] \\
 & G_{123} = 1 &
\end{tikzcd}
$$

Example: w=1432



$x_2^2x_3 + x_1x_2x_3 + x_1^2x_3 + x_1^2x_2 + x_1x_2^2$

$- x_1x_2^2x_3 - x_1x_2^2x_3 - x_1^2x_2x_3 - x_1^2x_2x_3 - x_1^2x_2^2$

$+ x_1^2x_2^2x_3$

Theorem (Buch 2002, Knutson-Miller 2005):

$$K(R/I_w) = \mathfrak{G}_w(1-t, 1-t, \dots, 1-t)$$

Lemma: If R/I is Cohen-Macaulay, then

$$\operatorname{reg}(R/I) = \deg K(R/I) - \operatorname{ht}_R(I).$$

Corollary: $\operatorname{reg}(R/I_w) = \deg \mathfrak{G}_w - \operatorname{inv}(w)$

Upshot: $\deg \mathfrak{G}_w$ tells you $\operatorname{reg}(R/I_w)$.

Non-explicit formulas:

$$\deg \mathfrak{G}_w = \max\{\#\text{crossings in } P : P \in \text{Pipes}(w)\}$$

Theorem (Lenart 1999): $\mathfrak{G}_w = \sum a_v^w \mathfrak{S}_w$

where $|a_v^w|$ counts certain 0-1 arrays.

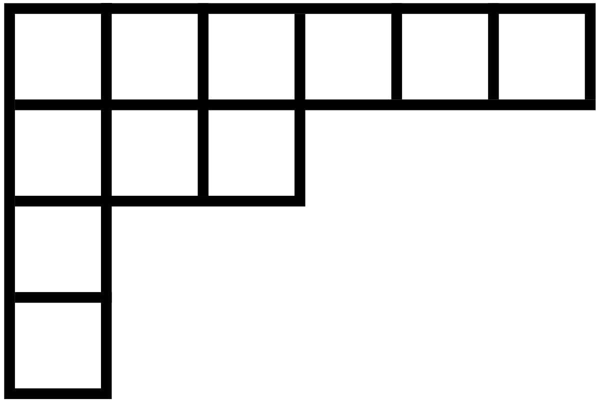
Problem:

Find a **simple and explicit** formula for the degree of a Grothendieck polynomial.

Symmetric Grothendieck Polynomials

A *partition* is a weakly decreasing sequence of non-negative integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$

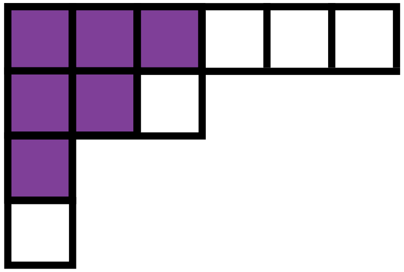
Example: $\lambda = (6, 3, 1, 1, 0)$



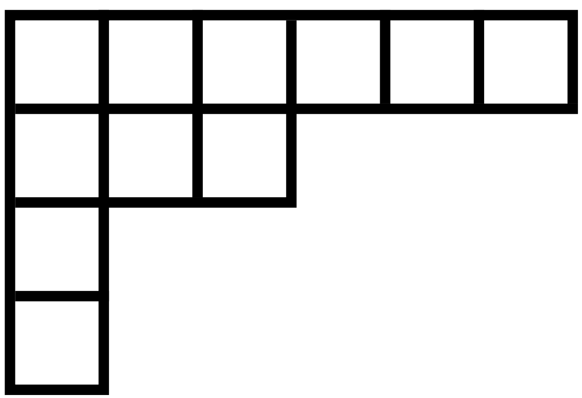
Let $\delta^{(k)} = (k, k-1, \dots, 1)$ be the **staircase** of size k.

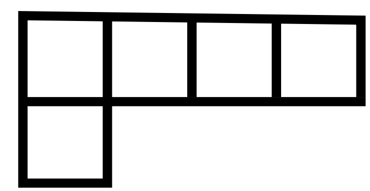
The **Sylvester triangle** of λ is the largest $\delta^{(k)}$ such that $\delta^{(k)} \subseteq \lambda$.

We write $sv(\lambda) = \max\{k : \delta^{(k)} \subseteq \lambda\}$.



Let $\mu^{(i)}$ be the partition obtained from λ by throwing away the first i columns of the Young diagram of λ

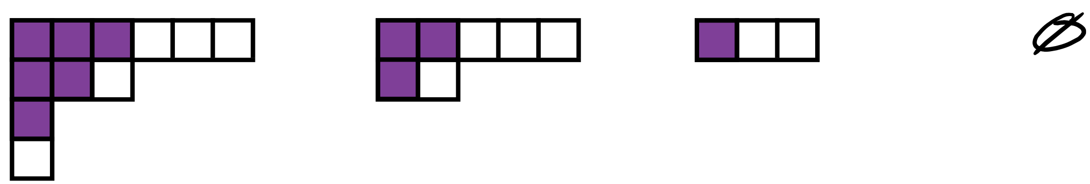
$\lambda =$ 

$\mu^{(2)} =$ 

Theorem (Rajchgot–Ren–Robichaux–St. Dizier–W. 2021):

$$\deg G_\lambda(x_1, \ldots, x_k) = |\lambda| + \sum_{i=1}^{k} sv(u^{(\lambda_i)})$$

Example: $\lambda = (6, 3, 1, 1, 0)$



$sv(u^{(0)}) = 3$ $\quad sv(u^{(1)}) = 2$ $\quad sv(u^{(3)}) = 1$ $\quad sv(u^{(6)}) = 0$

$$\deg G_\lambda = 11 + 0 + 1 + 2 + 2 + 3 = 19$$

In work in progress with Rajchgot and Robichaux, we

1. extend this formula to the vexillary setting (2143-avoiding)
2. give a similar formula for 1432 avoiding permutations, and
3. use the vexillary formula to give regularity formulas for certain Kazhdan-Lusztig varieties.

The Rajchgot Code

The Rajchgot Code

We define the Rajchgot code of $w \in S_n$

$\mathrm{rajcode}(w) = (r_1, \ldots, r_n)$ where r_i is the size of the complement of a maximal length increasing subsequence in $w(i)\ w(i+1)\ \ldots\ w(n)$ which uses $w(i)$.

Let $\mathrm{raj}(w) = r_1 + \cdots + r_n$.

Example:

Let $w = 7\ 1\ 6\ 4\ 5\ 8\ 2\ 3$.

$$
\begin{array}{rl}
\boxed{7}\ 1\ 6\ 4\ 5\ \boxed{8}\ 2\ 3 & 6 \\
\boxed{1}\ 6\ \boxed{4}\ \boxed{5}\ \boxed{8}\ 2\ 3 & 3 \\
\boxed{6}\ 4\ 5\ \boxed{8}\ 2\ 3 & 4 \\
\boxed{4}\ \boxed{5}\ \boxed{8}\ 2\ 3 & 2 \\
\boxed{5}\ \boxed{8}\ 2\ 3 & 2 \\
\boxed{8}\ 2\ 3 & 2 \\
\boxed{2}\ \boxed{3} & 0 \\
\boxed{3} & 0
\end{array}
$$

$\mathrm{rajcode}(w) = (6, 3, 4, 2, 2, 2, 0, 0)$

$\mathrm{raj}(w) = 19$

Theorem (Pechenik-Speyer-W. 2021):

Given $w \in S_n$, $\deg \mathfrak{G}_w = \mathrm{raj}(w)$.

Upshot: $\mathrm{inv}(w)$ is lowest degree terms,

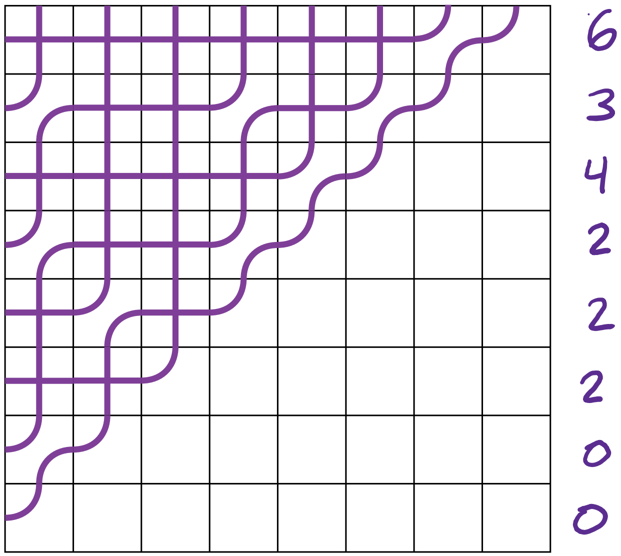
$\mathrm{raj}(w)$ highest.

$\mathrm{raj}(w) - \mathrm{inv}(w)$ is regularity!

Example: $w = 7\ 1\ 6\ 4\ 5\ 8\ 2\ 3$.

$\mathrm{raj}(w) = 19 = \deg \mathfrak{G}_w$

Bonus: lex last top degree monomial is $x^{\mathrm{rajcode}(w)}$



$x_1^6 x_2^3 x_3^4 x_4^2 x_5^2 x_6^2$

The Lehmer Code

The Lehmer code of $w \in S_n$ is

$\mathrm{invcode}(w) = (l_1, l_2, \ldots, l_n)$

where $l_i = \#\{j \geq i : w(j) < w(i)\}$.

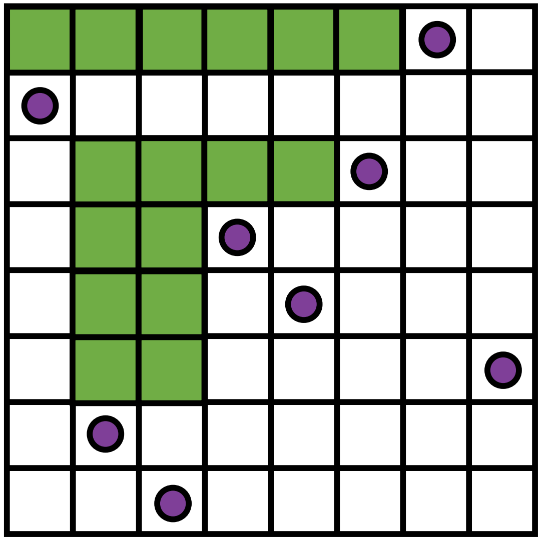
We write $\mathrm{inv}(w) = l_1 + \ldots + l_n$ for the Coxeter length of w.

Fact: $\mathrm{inv}(w)$ tells you the minimal degree of the monomials in G_w.

Example:

Let $w = 7\ 1\ 6\ 4\ 5\ 8\ 2\ 3$.

7	1	6	4	5	8	2	3	6
	1	6	4	5	8	2	3	0
		6	4	5	8	2	3	4
			4	5	8	2	3	2
				5	8	2	3	2
					8	2	3	2
						2	3	0
							3	0



$\mathrm{invcode}(w) = (6, 0, 4, 2, 2, 2, 0, 0)$

$\mathrm{inv}(w) = 16$

Lemma (Pechenik–Speyer–W. 2021):

$$\mathrm{rajcode}(w) \geq \mathrm{invcode}(w) \quad \text{(entrywise)}.$$

Example:

Let $w = 7\,1\,6\,4\,5\,8\,2\,3$.

$\boxed{7}\ 1\ 6\ 4\ 5\ \boxed{8}\ 2\ 3$	6	6
$\boxed{1}\ 6\ \boxed{4}\ \boxed{5}\ \boxed{8}\ 2\ 3$	3	0
$\boxed{6}\ 4\ 5\ \boxed{8}\ 2\ 3$	4	4
$\boxed{4}\ \boxed{5}\ \boxed{8}\ 2\ 3$	2	2
$\boxed{5}\ \boxed{8}\ 2\ 3$	2	2
$\boxed{8}\ 2\ 3$	2	2
$\boxed{2}\ \boxed{3}$	0	0
$\boxed{3}$	0	0

Relation with Major Index

The major index of w is $\operatorname{maj}(w) = \sum_{w(i) > w(i+1)} i$.

Example: $w = 7\ 1\ 6\ 4\ 5\ 8\ 2\ 3$.

$\operatorname{maj}(w) = 1 + 3 + 6 = 10$

Theorem (Pechenik-Speyer-W. 2021):

$$\operatorname{raj}(w) = \max\{\operatorname{maj}(v) : v \leq_R w\}$$
$$= \max\{\operatorname{maj}(u^{-1}) : u \leq_L w\},$$

$$
\begin{tikzcd}[column sep=small]
 & \mathrm{maj}(3\,2\,1)=3 \arrow[dl, dash, "\cdot s_1"'] \arrow[dr, dash, "\cdot s_2"] & \\
\mathrm{maj}(2\,3\,1)=2 \arrow[d, dash, "\cdot s_2"'] & & \mathrm{maj}(3\,1\,2)=1 \arrow[d, dash, "\cdot s_1"] \\
\mathrm{maj}(2\,1\,3)=1 \arrow[dr, dash, "\cdot s_1"'] & & \mathrm{maj}(1\,3\,2)=2 \arrow[dl, dash, "\cdot s_2"] \\
 & \mathrm{maj}(1\,2\,3)=0 &
\end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small]
 & G_{321}=x_1^2x_2 \arrow[dl, dash, "\bar{\partial}_1"'] \arrow[dr, dash, "\bar{\partial}_2"] & \\
G_{231}=x_1x_2 \arrow[d, dash, "\bar{\partial}_2"'] & & G_{312}=x_1^2 \arrow[d, dash, "\bar{\partial}_1"] \\
G_{213}=x_1 \arrow[dr, dash, "\bar{\partial}_1"'] & & G_{132}=x_1+x_2-x_1x_2 \arrow[dl, dash, "\bar{\partial}_2"] \\
 & G_{123}=1 &
\end{tikzcd}
$$

Theorem (Pechenik–Speyer–W. 2021):

$\mathrm{raj}(w) = \mathrm{maj}(w)$ if and only if w is a fireworks (3-12 avoiding) permutation.

Example: $v = 24753186$ $\quad \mathrm{maj}(v) = 3+4+5+7 = 19$

$$
\begin{array}{rl}
\boxed{2}\,\boxed{4}\,7\,\boxed{5}\,3\,1\,\boxed{8}\,6 & 4 \\
\boxed{4}\,\boxed{7}\,5\,3\,1\,\boxed{8}\,6 & 4 \\
\boxed{7}\,5\,3\,1\,\boxed{8}\,6 & 4 \\
\boxed{5}\,3\,1\,\boxed{8}\,6 & 3 \\
\boxed{3}\,1\,\boxed{8}\,6 & 2 \\
\boxed{1}\,\boxed{8}\,6 & 1 \\
\boxed{8}\,6 & 1 \\
\boxed{6} & 0
\end{array}
$$

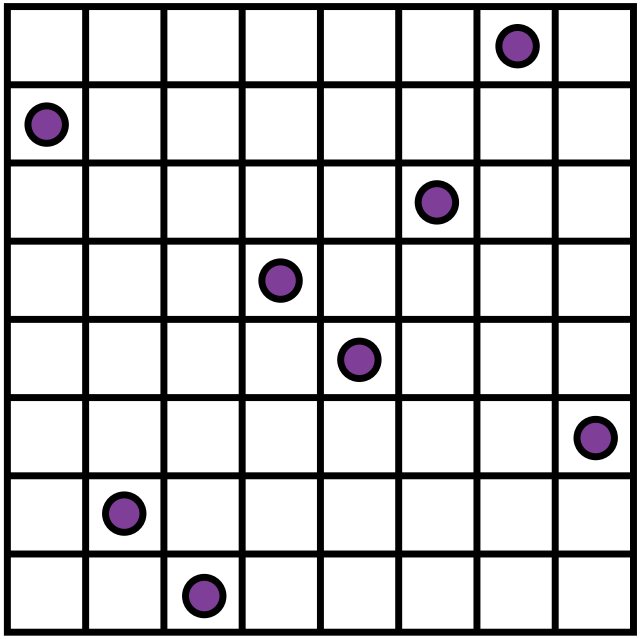
$$\mathrm{raj}(v) = 4+4+4+3+2+1+1 = 19$$

Proof of Main Theorem

Theorem (Pechenik–Speyer–W. 2021):

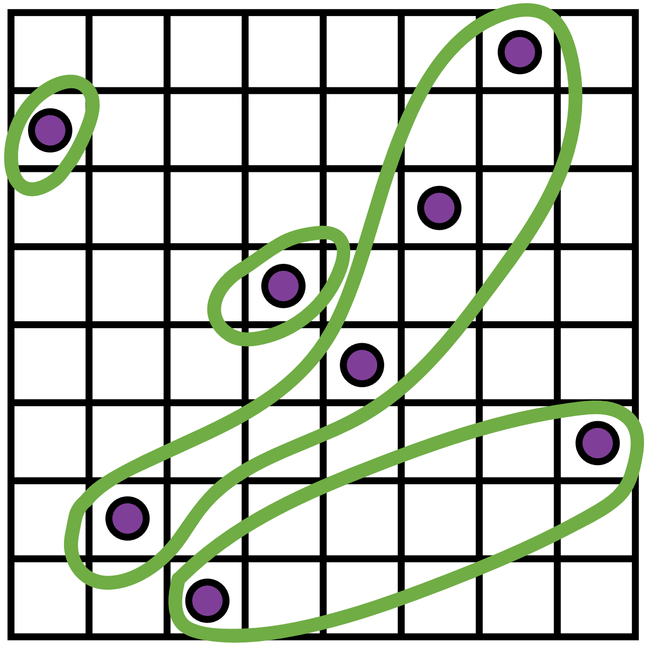
Given $w \in S_n$, $\deg \mathfrak{G}_w = \mathrm{raj}(w)$.

Blob Diagrams



$w = 7\ 1\ 6\ 4\ 5\ 8\ 2\ 3$

Blob Diagrams



$w = 7\ 1\ 6\ 4\ 5\ 8\ 2\ 3$

$\mathrm{shape}(w) = (1, 1, 4, 2)$

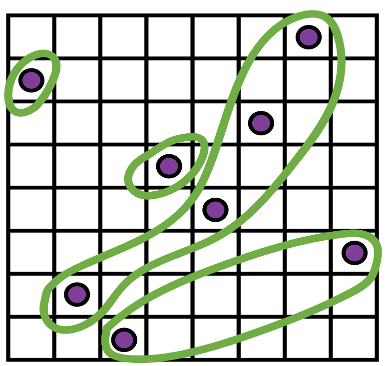
Lemma (Pechenik–Speyer–W. 2021):

raj(w) only depends on shape(w):

if $\mathrm{shape}(w) = (d_k, d_{k-1}, \ldots, d_n)$ then

$$\mathrm{raj}(w) = \sum_{i=k}^{n} i\, d_i - \binom{n+1}{2}.$$

Example: $w = 7\ 1\ 6\ 4\ 5\ 8\ 2\ 3$.



$\mathrm{shape}(w) = (1, 1, 4, 2)$

$$\mathrm{raj}(w) = 5\cdot 1 + 6\cdot 1 + 7\cdot 4 + 8\cdot 2 - \binom{9}{2}$$
$$= 19$$

By symmetry, $\mathrm{raj}(w) = \mathrm{raj}(w^{-1})$.

Dominant Permutations

Lemma (Pechenik–Speyer–W. 2021):

If w is dominant (132-avoiding) then $\mathrm{invcode}(w) = \mathrm{rajcode}(w)$. In particular, $\mathrm{inv}(w) = \mathrm{raj}(w)$.

Example: $w = 7\,6\,3\,4\,5\,8\,1\,2$

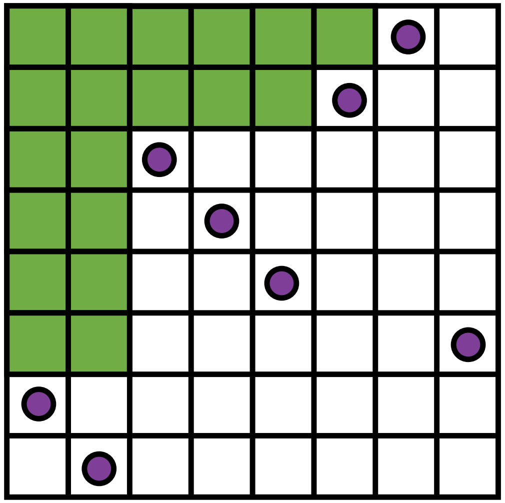
$$\mathrm{rajcode}(w) = (6,5,2,2,2,2,0,0)$$
$$= \mathrm{invcode}(w)$$

$$\boxed{7}\ 6\ 3\ 4\ 5\ \boxed{8}\ 1\ 2$$
$$\boxed{6}\ 3\ 4\ 5\ \boxed{8}\ 1\ 2$$
$$\boxed{3}\ \boxed{4}\ \boxed{5}\ \boxed{8}\ 1\ 2$$
$$\boxed{4}\ \boxed{5}\ \boxed{8}\ 1\ 2$$
$$\boxed{5}\ \boxed{8}\ 1\ 2$$
$$\boxed{8}\ 1\ 2$$
$$\boxed{1}\ \boxed{2}$$
$$\boxed{2}$$

Corollary: If w is dominant, $\deg \mathfrak{G}_w = \mathrm{raj}(w)$.

Proof: If w is dominant, $\mathfrak{G}_w = x^{\mathrm{inv\,code}(w)}$.

Example: $w = 7\,6\,3\,4\,5\,8\,1\,2$



$$\mathfrak{G}_w = x_1^6 x_2^5 x_3^2 x_4^2 x_5^2 x_6^2$$

Layered Permutations

Write $w_0^{(j)} = j\ j{-}1 \cdots 1$ for the longest permutation in S_j.

Given a composition $\alpha = (\alpha_k, \ldots, \alpha_n)$, let

$$e_\alpha = w_0^{(\alpha_k)} \times w_0^{(\alpha_{k+1})} \times \cdots \times w_0^{(\alpha_n)}.$$

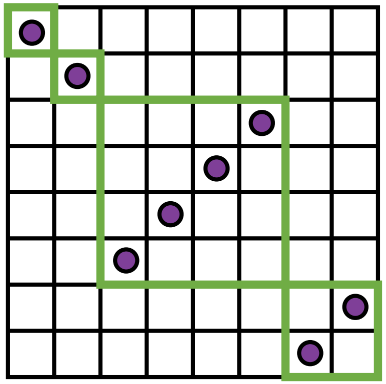
We call this a layered permutation.

Example:

$\alpha = (1, 1, 4, 2)$

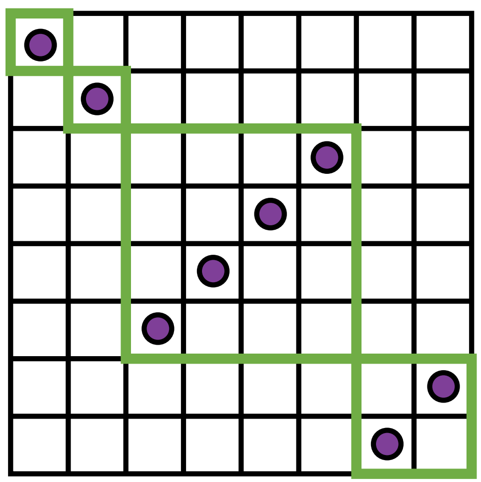
$e_\alpha = 1 \times 1 \times 4321 \times 21$

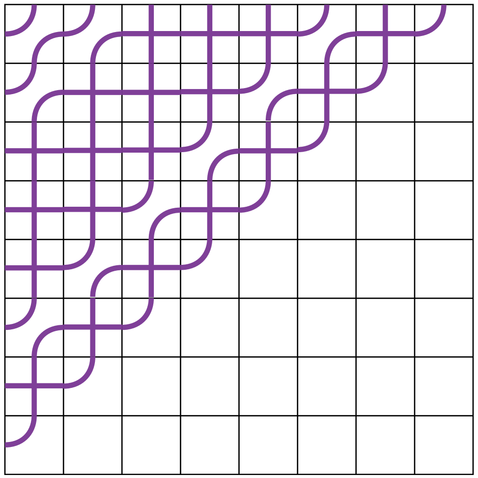
$= 1\ 2\ 6543\ 87$



Lemma (Pechenik–Speyer–W. 2021):
e_α has a unique top degree pipe dream and it has $\mathrm{raj}(e_\alpha)$ crossings. Thus $\deg G_{e_\alpha} = \mathrm{raj}(e_\alpha)$.

Example: $\alpha = (1,1,4,2)$

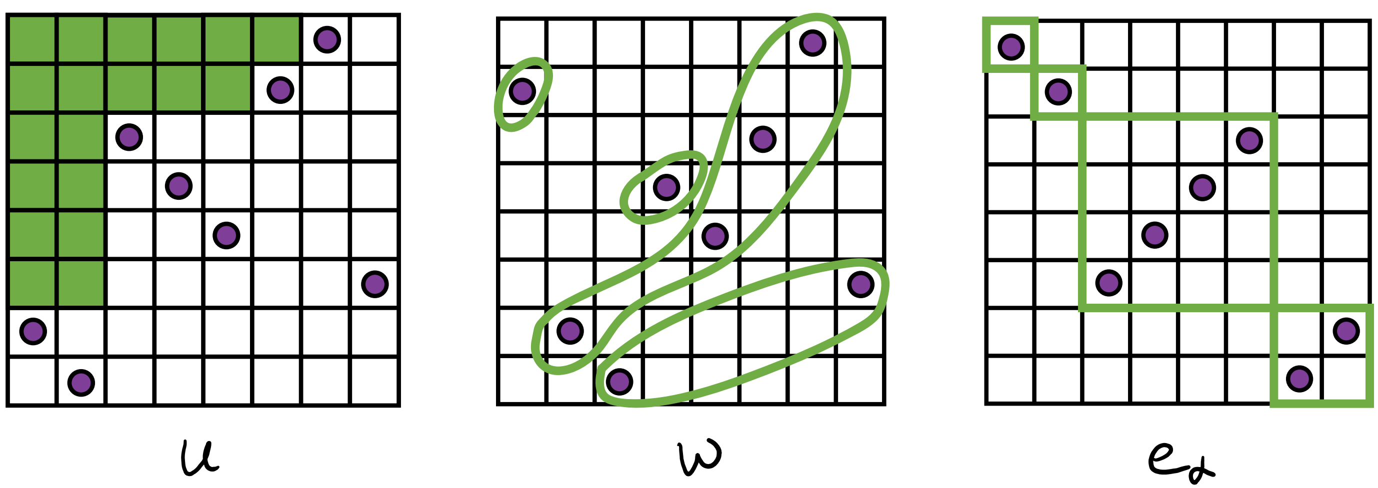




Lemma: Fix $w \in S_n$ with $\mathrm{shape}(w) = \alpha$.

① $w \geq_{LR} e_\alpha$.

② There exists a dominant permutation u so that $\mathrm{Shape}(u) = \alpha$ and $u \geq_{LR} w$.



Facts: ① $\deg G_w = \deg G_{w^{-1}}$.

② If $v <_{LR} w$ then $\deg G_v \leq \deg G_w$.

Proof: ① Follows from symmetry of pipe dreams.

② Grothendieck polynomials are defined recursively by $G_{ws_i} = \bar{\partial}_i(G_w)$ where

$$\bar{\partial}_i(f) = \frac{(1-x_{i+1})f - (1-x_i)\, s_i \cdot f}{x_i - x_{i+1}}.$$

Proof of main theorem:

Keeping the notation from before:

$$\mathrm{raj}(u) \underset{\substack{\uparrow \\ u \text{ is dominant}}}{=} \deg \mathfrak{G}_u \geq \deg \mathfrak{G}_w \geq \deg \mathfrak{G}_{e_\alpha} \underset{\substack{\uparrow \\ e_\alpha \text{ is layered}}}{=} \mathrm{raj}(e_\alpha)$$

Since $\mathrm{shape}(u) = \mathrm{shape}(w) = \mathrm{shape}(e_\alpha)$,

$$\mathrm{raj}(u) = \mathrm{raj}(w) = \mathrm{raj}(e_\alpha).$$

Thus $\mathrm{raj}(w) = \deg \mathfrak{G}_w$. $\blacksquare$

Castelnuovo-Mumford Polynomials

We define the Castelnuovo-Mumford polynomial $CM_w(x)$ to be the top degree part of $G_w(x)$ (times $(-1)^{\deg(G_w) - inv(w)}$).

$$G_{321} = \boxed{x_1^2 x_2}$$

$$G_{231} = \boxed{x_1 x_2} \qquad G_{312} = \boxed{x_1^2}$$

$$G_{213} = \boxed{x_1} \qquad G_{132} = x_1 + x_2 - \boxed{x_1 x_2}$$

$$G_{123} = \boxed{1}$$

We define the Castelnuovo-Mumford polynomial $CM_w(x)$ to be the top degree part of $G_w(x)$ (times $(-1)^{\deg(G_w)-\operatorname{inv}(w)}$).

$$\begin{tikzcd}
 & G_{321} = \boxed{x_1^2 x_2} \arrow[dl, dash] \arrow[dr, dash] & 210 \\
110 \quad G_{231} = \boxed{x_1 x_2} \arrow[d, dash] & & G_{312} = \boxed{x_1^2} \quad 200 \arrow[d, dash] \\
100 \quad G_{213} = \boxed{x_1} \arrow[dr, dash] & & G_{132} = x_1 + x_2 - \boxed{x_1 x_2} \quad 110 \arrow[dl, dash] \\
 & G_{123} = \boxed{1} & 000
\end{tikzcd}$$

Double Grothendieck polynomials were introduced in 1982 by Lascoux and Schützenberger to study the K-theory of the complete flag variety.

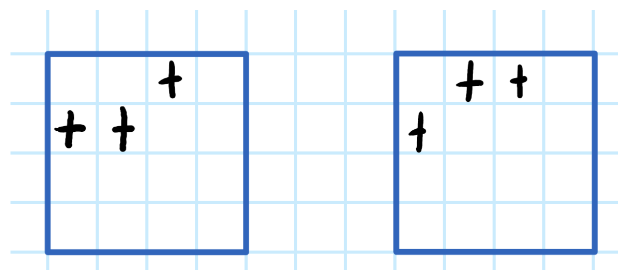
For $w \in S_n$, $\mathcal{G}_{w_0}(x;y) = \prod_{i+j \le n} (x_i + y_j - x_i y_j)$ and if $w(i) > w(i+1)$, define $\mathcal{G}_{ws_i}(x;y) = \bar{\partial}_i (\mathcal{G}_w(x;y))$

where $\bar{\partial}_i(f) = \dfrac{(1-x_{i+1}) f - (1-x_i)\, s_i \cdot f}{x_i - x_{i+1}}$.

only acts on x's

We define the double Castelnuovo-Mumford polynomial $CM_w(x;y)$ to be the top degree monomials of $G_w(x;y)$ (again normalized).

Example: $w = 1423.$

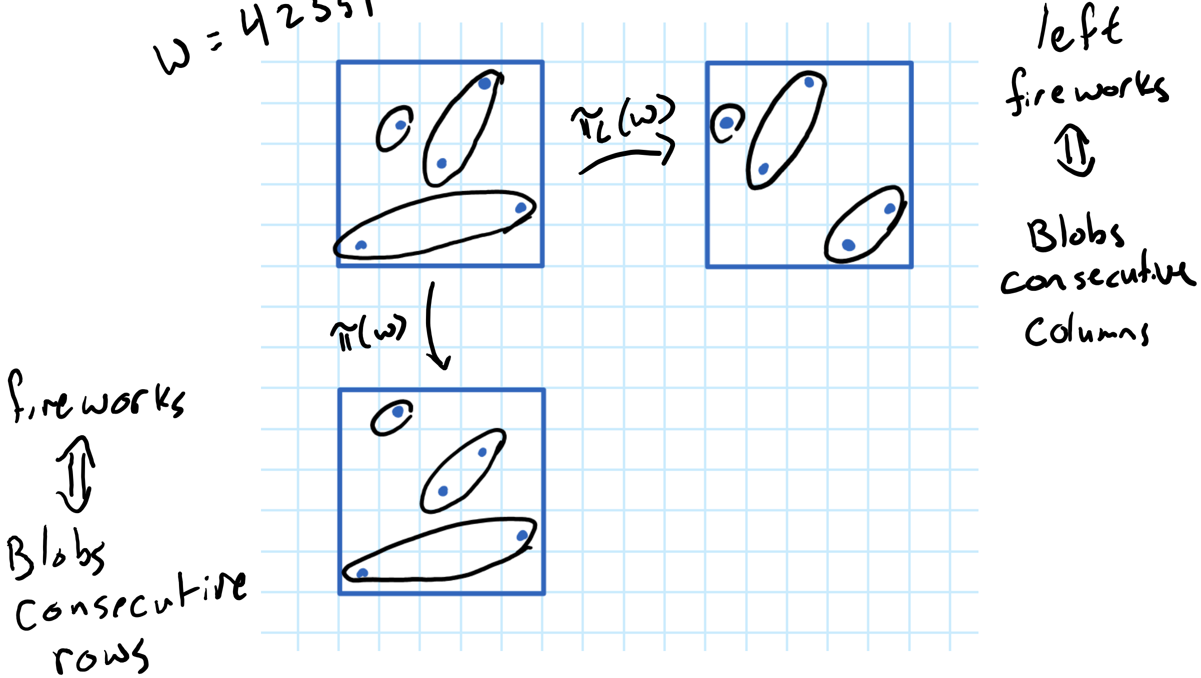


$$CM_w(x;y) = x_1 x_2^2 y_1 y_2 y_3 + x_1^2 x_2 y_1 y_2 y_3$$

$$= (x_1 x_2^2 + x_1^2 x_2)\, y_1 y_2 y_3$$

We can project $w \in S_n$ to a fireworks permutation with the same rajcode.

$w = 42351$

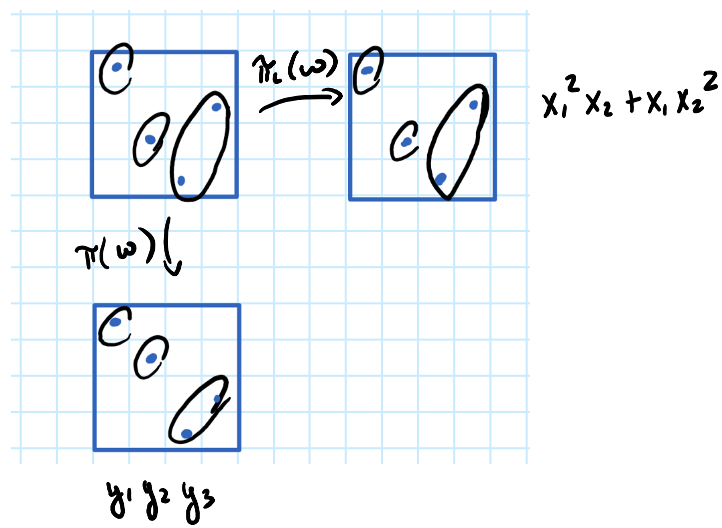


Theorem (Pechenik–Speyer–W. 2021):

$$CM_w(x;y) = CM_{\pi_L(w)}(x)\, CM_{\pi(w)^{-1}}(y)$$

Example: $w = 1432$

$$\begin{aligned} CM_w(x;y) &= x_1 x_2^2 y_1 y_2 y_3 + x_1^2 x_2 y_1 y_2 y_3 \\ &= (x_1 x_2^2 + x_1^2 x_2)\, y_1 y_2 y_3 \\ &= CM_{1423}(x) \cdot CM_{1243}(y) \end{aligned}$$



Proof Idea

This follows from the Cauchy identity for Grothendieck polynomials:

$$G_\omega(x;y) = \sum_{q*p} (-1)^{inv(\omega)-inv(p)-inv(q)} G_p(x)\, G_{q^{-1}}(y).$$

Lemma: Let $p \in [\pi_L(\omega), \omega]_L$ and $q \in [\pi(\omega), \omega]_R$. Then $q*p = \omega$ if and only if $p = \pi_L(\omega)$ and $q = \pi(\omega)$.

Corollary: (Pechenik–Speyer–W. 2021):

$$CM_w(x) = CM_{\pi_L(w)}(x)\, CM_{\pi(w)^{-1}}(1)$$

Thanks!