Conformal walk dimension

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Based on joint works with
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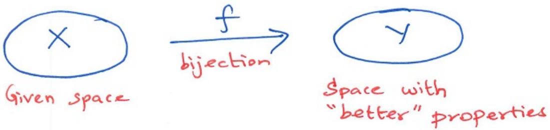
Context

- Question: What is the 'best' way to parameterize a space X?



Examples

- Riemann mapping theorem(or uniformization theorem): X is a simply connected region (or simply connected Riemann surface), f is a conformal map, $Y = \mathbb{D}, \mathbb{C}$ or $\mathbb{S}^2$.
- Ricci flow: X is a manifold, f is a diffeomorphism, Y manifold with constant Ricci curvature.
- Goal: Describe such a parametrization for a metric space with a symmetric diffusion process.

Outline

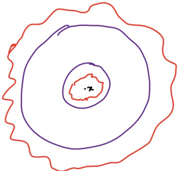
- Quasisymmetry, Ahlfors regular conformal dimension.
- Diffusions on fractals: walk dimension.
- Dirichlet forms: energy measure, intrinsic metric and harmonic functions.
- Conformal walk dimension.
- Connections between conformal dimensions.

Quasisymmetry and Conformal gauge

- Quasisymmetry (QS): A notion of 'conformal maps' on metric spaces (Ahlfors-Beurling '56, Tukia-Väisälä '80).
  $f : (X_1, d_1) \to (X_2, d_2)$ is a homeomorphism.
  $\eta : [0, \infty) \to [0, \infty)$ is a self-homeomorphism on $[0, \infty)$.
  Def. f is η-QS

$$\frac{d_2(f(x), f(y))}{d_2(f(x), f(z))} \leq \eta \left(\frac{d_1(x, y)}{d_1(x, z)} \right) \quad \text{for all } x, y, z \in X_1,\ x \neq z.$$

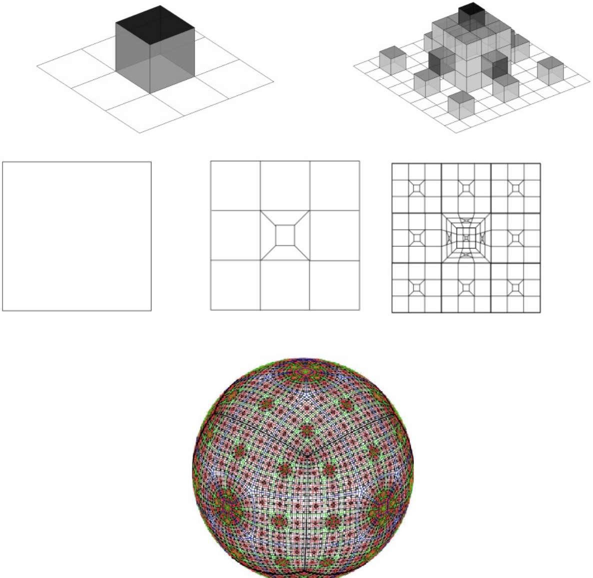
f is a QS (quasisymmetry) it is a quasisymmetry for some η.
Def. Conformal gauge of a metric space (X, d)
$\mathcal{J}(X, d) = \{\theta \text{ is a metric on } X | \mathrm{Id} : (X, d) \to (X, \theta) \text{ is a QS}\}$.



For all $x \in X$, $r > 0$, $A > 0$,
there exists $s > 0$, such that
$B_\theta(x, s) \subset B_d(x, r)$ and
$B_d(x, Ar) \subset B_\theta(x, \eta(A)s)$

Quasisymmetry: example 1

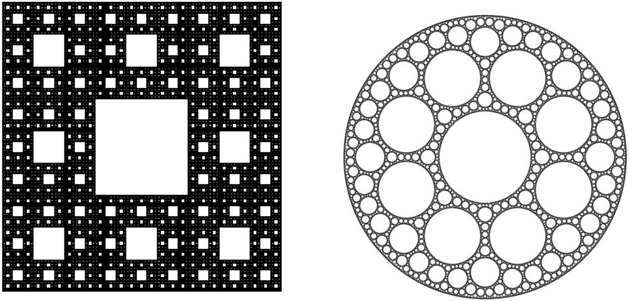
Snowball is quasisymmetric to $\mathbb{S}^2$ (Meyer '02).



Quasisymmetry: example 2

Standard Sierpiński carpet is quasisymmetric to a round Sierpiński carpet (Bonk '11).

Figure: The Sierpiński carpet and a round Sierpiński carpet (image: Kajino)



Ahlfors regular conformal dimension (Pansu '89)

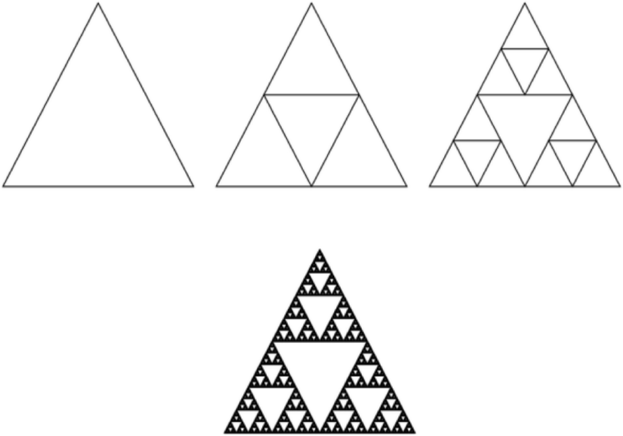
- The Ahlfors regular conformal dimension of a metric space (X, d) is

$$d_{\mathrm{ARC}} = \inf \left\{ Q \;\middle|\; \begin{array}{l} \text{there exists a measure } \mu \text{ and a metric} \\ \theta \in \mathcal{J}(X, d) \text{ such that } \mu(B_\theta(x, r)) \asymp r^Q \\ \text{for all } r < \operatorname{diam}(X, \theta). \end{array} \right\}.$$

- Ahlfors regular conformal dimension of Julia sets and boundary of hyperbolic groups is used to understand underlying dynamical and group structures respectively.
- Possible values of $d_{\mathrm{ARC}} = \{0\} \cup [1, \infty]$ (Laakso'00, Kovalev'06).
- **Questions**: Given a space, what is the value of d_{ARC}? Is the infimum attained? Both these questions are open for Sierpiński carpet.

Diffusion on fractals

- Diffusions on fractals are often defined as scaling limit of random walks on graph approximations or limit of diffusions on smooth approximations.
- The space-time scaling exponent (walk dimension) is usually strictly larger than two in contrast with smooth settings.

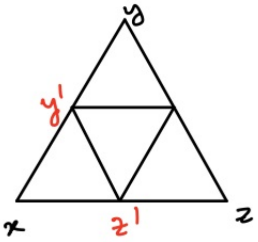


Space time scaling exponent

For the random walk (S_n) on a 'graph' Sierpiński gasket

$$\mathbb{E}d(S_0, S_n) \asymp n^{1/\beta}, \quad \mathbb{E}^x[\tau_{B(x,r)}] \asymp r^{\beta}.$$

where $\beta = \log_2 5$ (Barlow, Perkins '88).



$$\mathbb{E}_x[T_{\{y,z\}}] = 5$$

$$\mathbb{E}_x[T_{\{y',z'\}}] = 1$$

Set up: metric space with a Dirichlet form

Setup: (X, d, m) complete, locally compact, metric measure space, where d is a doubling metric and m is a Radon measure with full support.
$(\mathcal{E}, \mathcal{F}^m)$ regular, strongly local Dirichlet form on $L^2(m)$.
This defines a m-symmetric diffusion process $(Y_t)_{t\geq 0}$ on X.

$$\mathbb{P}_x = \text{law of } Y_t \text{ given the starting point } Y_0 = x.$$

Def.: Let $\Gamma(f, f)$ denote the energy measure (so that $\mathcal{E}(f, f) = \int_X d\Gamma(f, f)$) and energy measure and the intrinsic distance

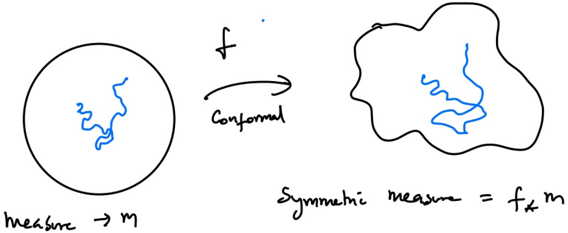
$$d_{\text{int}}(x, y) = \sup\{f(x) - f(y) : f \in C(X) \cap \mathcal{F}^m : \Gamma(f, f) \leq m\}$$

Example.: For the Brownian motion on $\mathbb{R}^n$, m is Lebesgue measure, d is Euclidean metric, $\mathcal{E}(f, f) = \|\nabla f\|_2^2$ and $\mathcal{F}$ is $W^{1,2}$ Sobolev space, $\Gamma(f, f) = |\nabla f|^2 (x)\, dx$ and

$$d_{\text{int}}(x, y) = \sup\{f(x) - f(y) : |\nabla f| \leq 1, f \in C(X) \cap W^{1,2}\} = d(x, y).$$

Revuz correspondence

- Time changes of symmetric Markov process are in one-to-one correspondence with a family of ‘smooth’ measures (Revuz ‘70).
- The diffusion tends to run faster where the new symmetric measure is small and slower if the measure is large.
- Time change is done using a class $\mathcal{A}$ of admissible measures which are defined as Radon measures with full quasi-support such that they do not charge sets of zero capacity.



Sub-Gaussian heat kernel estimate

A metric measure space (X, d, m) with a Dirichlet form $(\mathcal{E}, \mathcal{F})$ (or equivalently, an m-symmetric diffusion process (Y_t)), satisfies the sub-Gaussian heat kernel bounds HKE(β) if there exists constants $C_{1-4} > 0$ such that the heat kernel (or transition probability density) satisfies

$$p_t(x,y) \le \frac{C_3}{m(B(x,t^{1/\beta}))} \exp\left(-C_4 \left(\frac{d(x,y)^\beta}{t}\right)^{1/(\beta-1)}\right)$$

and

$$\frac{C_1 \mathbf{1}_{\{d(x,y)^\beta \le C_2 t\}}}{m(B(x,t^{1/\beta}))} \le p_t(x,y)$$

for all $t > 0, x, y \in X$.

Possible values of β=$[2,\infty)$ (Barlow '04, Hino'02).
$\beta > 2$ is typically seen on fractals while $\beta = 2$ is seem in smooth settings.

Conformal walk dimension

Definition: Conformal walk dimension d_{cw}

$$d_{\mathrm{cw}} = \inf \left\{ \beta \;\middle|\; \begin{array}{l} \text{there exists an admissible measure } \mu \\ \text{(=time change) and metric } \theta \in \mathcal{J}(X,d) \\ \text{such that } (X,\theta,\mu,\mathcal{E},\mathcal{F}^{\mu}) \text{ satisfies} \\ \mathrm{HKE}(\beta) \end{array} \right\},$$

where admissible measures are Radon measures that do no charge sets of capacity zero and have full quasi support.
We reparametrize time by choosing a symmetric measure, and we reparametrize space by choosing a metric in the conformal gauge.

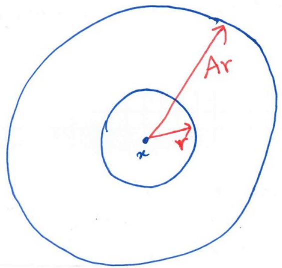
By replacing the metric d with $d^{\alpha} \in \mathcal{J}(X,d)$ where $\alpha \in (0,1)$, we can obtain HKE(β/α) from HKE(β). (reason for taking infimum)

Questions: What is the value of d_{cw}? Is the infimum attained?
Motivation 1: To understand elliptic Harnack inequality.

Elliptic Harnack inequality

Def. Elliptic Harnack inequality (EHI): there exists $C, A > 1$ such that for all $h \geq 0$ harmonic in $B(x, Ar)$

$$\sup_{B(x,r)} h \leq C \inf_{B(x,r)} h.$$



A function $h \in \mathcal{F}$ is harmonic in an open set U if

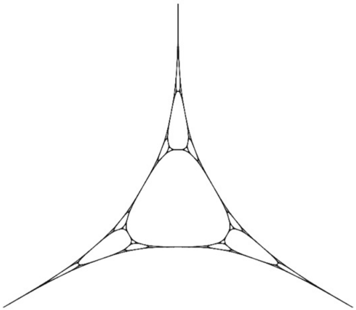
$$\mathcal{E}(h, u) = 0 \text{ for all } f \in C_c(U) \cap \mathcal{F}^m.$$

Motivation 1: elliptic Harnack inequality

- $d_{cw} < \infty$ is equivalent to the elliptic Harnack inequality (Barlow, M. '18, Barlow, Chen, M. '20)
- (Stability of EHI) If M is a Riemannian manifold that satisfies EHI for the Laplace-Beltrami operator. Then M satisfies EHI for any uniformly elliptic divergence form operator (Barlow, M. '18).
- This generalizes Moser's EHI ('61) for the case $M = \mathbb{R}^n$ and follows from stability of HKE(β) shown by Grigor'yan '91, Saloff-Coste '92, Sturm '95, Barlow, Bass, Kumagai '06, Grigor'yan, Hu, Lau '15.

Motivation 2: Sierpiński gasket attains $d_{cw} = 2$.

The 2-dimensional Sierpiński Gasket attains the infimum and $d_{cw} = 2$. (Kigami '08)
μ is the Kusuoka measure and θ is the intrinsic metric corresponding to μ.



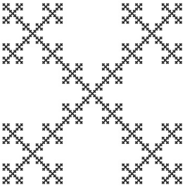
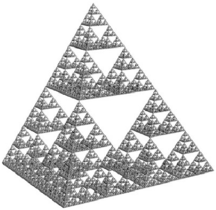
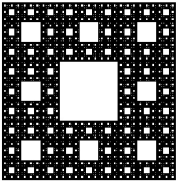
Even fractals can have walk dimension 2! Snowball is another example.

Universal value of d_{cw}

Questions: What is the value of d_{cw}? Is the infimum attained?
Theorem. (Kajino, M. '22+)

$$d_{\mathrm{cw}} < \infty \iff d_{\mathrm{cw}} = 2.$$

That is, we can upgrade from HKE(β) to HKE($2+\epsilon$) for all $\epsilon > 0$.
The following examples do not attain the infimum.

The 2D Sierpiński carpet is ongoing work and the higher dimensional versions of Sierpiński carpet is open (difficult!).

Structure of optimal metric and measure

Theorem (Kajino, M. '20) If $d_{\mathrm{cw}} = 2$ is attained for some metric θ and a measure μ, then θ is bi-Lipschitz equivalent to the intrinsic metric $d_{\mathrm{int}}(\mu)$. (that is, metric is determined by the measure)

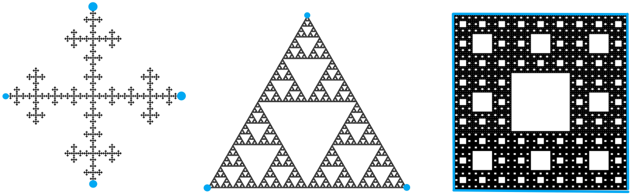
Theorem (Kajino, M. '20) If $d_{\mathrm{cw}} = 2$ is attained for some metric θ and a measure μ, then μ is a minimal energy dominant measure, that is μ satisfies the following

(a) (Energy dominance) $\Gamma(f,f) \ll \mu$ for every $f \in \mathcal{F}$, where $\Gamma(f,f)$ is the energy measure of f.

(b) (Minimality) If $\tilde{\mu}$ satisfies (a), then $\mu \ll \tilde{\mu}$.

(Remark: any two minimal energy dominant measures are mutually absolutely continuous)

A structure theorem for self similar sets



Theorem (Kajino, M. '22+) Let X be a self-similar set and X^{∂} be its 'natural boundary'. If the conformal walk dimension is attained, then the conformal walk dimension is also attained by the energy measure of a harmonic function; that is $\mu = \Gamma(h, h)$ where h is a harmonic function on $X \setminus X^{\partial}$.

In other words, to find 'optimal' metrics and measures it is enough to search for 'optimal' harmonic functions.

Conjecture: An analogue of this result should be true for Ahlfors regular conformal dimension.

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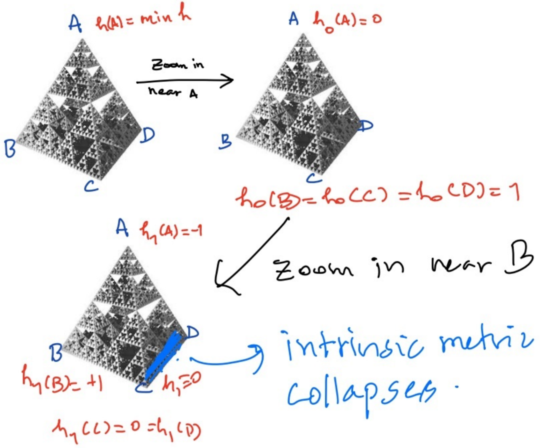
Application: All our examples on non-attainment of d_{cw} relies on this structure theorem.

Conjecture: An analogue of this result should be true for Ahlfors regular conformal dimension where harmonic is replaced by d_{ARC}-harmonic.

Recent preprints of Kigami '21 and Shimizu '21 construct non-linear analogues of Dirichlet forms and energy measures on some fractals.

Non-attainment on higher dimensional Sierpiński gasket



Challenge

The Sierpinski carpet case is much more involved since there is an infinite dimensional space of harmonic functions. The proof involves more advanced techniques: Boundary Harnack principle, Doob h-transform, Jordan curve theorem.

We would like to know if the energy measure of a non-constant harmonic function has full support. By a Poincaré inequality, this is equivalent to unique continuation principle.

Unique continuation principle is not known for harmonic functions on the Sierpiński carpet.

Challenge: Find a probabilistic proof of the unique continuation principle for harmonic function on $\mathbb{R}^n$.

Another description of d_{cw}

$$d_{\mathrm{cw}} = \inf\left\{ \beta \;\middle|\; \begin{array}{l} \text{there exists an admissible measure } \mu \\ \text{(=time change) and metric } \theta \in \mathcal{J}(X,d) \\ \text{such that } (X,\theta,\mu,\mathcal{E},\mathcal{F}^{\mu}) \text{ satisfies} \\ \mathrm{HKE}(\beta) \end{array} \right\}$$

Using known characterizations of heat kernel bounds, we can rewrite the definition as

$$d_{\mathrm{cw}} = \inf\left\{ \beta \;\middle|\; \begin{array}{l} \text{there exists an admissible measure } \mu \text{ (=time change) and metric } \theta \in \mathcal{J}(X,d) \text{ such that} \\ (X,\theta,\mu,\mathcal{E},\mathcal{F}^{\mu}) \text{ satisfies } \mathrm{EHI} \text{ and} \\ \mu(B_\theta(x,r)) \asymp r^{\beta}\,\mathrm{Cap}_{B_\theta(x,Ar)}(B_\theta(x,r)) \end{array} \right\},$$

Here $\mathrm{Cap}_D(A) = \inf\{\mathcal{E}(f,f) : f \in C_c(D)\cap\mathcal{F}, f|_A \geq 1\}$.

Ingredients of the proof of $d_{cw} < \infty \implies d_{cw} = 2$.

- Heuristic idea: The construction of the metric is such that the 'new' diameter of a ball is proportional to the gradient of equilibrium potential at all locations and scales.
- A given metric space can be viewed as the boundary of a Gromov hyperbolic space (Elek '97, Bourdon-Pajot '03). This Gromov hyperbolic space is a graph and called the hyperbolic filling of a metric space.
- (Bonk, Schramm '01) A bi-Lipschitz change of the metric of the hyperbolic filling results in a quasisymmetric change of the metric on its boundary.
- (Bonk, Schramm '01) All metrics on the conformal gauge can be obtained as a bi-Lipschitz change of metric on its hyperbolic filling.
- (Carrasco '13) A combinatorial description of the conformal gauge $\mathcal{J}(X, d)$ by providing conditions on edge weights that can be used to perturb the distances on the hyperbolic filling.

Thank you for your attention

M. T. Barlow, M. Murugan, Stability of the elliptic Harnack inequality, *Annals of Math.*, **187** (2018), 777–823.

M. T. Barlow, Z.-Q. Chen, M. Murugan, Stability of EHI and regularity of MMD spaces, revision requested by *Advances in Math.* (arXiv:2008.05152)

N. Kajino, M. Murugan, On the conformal walk dimension: Quasisymmetric uniformization for symmetric diffusions, revision requested by *Inventiones Math.* (arXiv:2008.12836)