Jump Diffusions on Metric Measure Spaces

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Collaborators

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In this talk, a jump diffusion means a purely discontinuous (symmetric) Markov process with long range jumps.

Example. Isotropic or rotationally symmetric α-stable process X on $\mathbb{R}^d$ with $0 < \alpha < 2$:

$$\mathbb{E} e^{i\xi\cdot(X_t - X_0)} = e^{-t|\xi|^\alpha}, \qquad \xi \in \mathbb{R}^d.$$

$\alpha = 2$: Brownian motion

Why do we care?

- Jumps model sudden big changes when viewed in a long time scale.
- Anomalous super-diffusions
- Infinitesimal generator of X is $\Delta^{\alpha/2} := -(-\Delta)^{\alpha/2}$:

$$\Delta^{\alpha/2} u(x) = \text{p.v.} \int_{\mathbb{R}^d} (u(y) - u(x)) \frac{c_{d,\alpha}}{|y - x|^{d+\alpha}} dy,$$

where $c_{d,\alpha} \asymp \alpha(2 - \alpha)$. Riesz potential

- Singular obstacle problem; Dirichlet-to-Nuemann map
- Trace of diffusions on subsets.
- Time change of diffusion processes

Challenges

• Non-local operators: not very amenable for calculation

• Typically infinitely many small jumps in any given time intervals.

• Long range jumps

Features:

• Exist in a very general setting

• Very large class and rich behaviors: polynomial, exponential, short range, . . .

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Dirichlet form

A useful tool in the study of symmetric Markov processes is Dirichlet form theory.

$$X \longleftrightarrow \{P_t\}_{t\geq 0} \longleftrightarrow \mathcal{L} \longleftrightarrow (\mathcal{E}, \mathcal{F}),$$

where $\mathcal{F} = \mathrm{Dom}(\sqrt{-\mathcal{L}})$ and

$$\mathcal{E}(f,g) = (\sqrt{-\mathcal{L}}f, \sqrt{-\mathcal{L}}g)_{L^2(\mathcal{X};m)} = (-\mathcal{L}f, g)_{L^2(\mathcal{X};m)}.$$

- Fukushima ('71, '75), Silverstein ('74): regular DF
- Albeverio-Ma ('91), Ma-Röckner ('92): quasi-regular DF
- C.-Ma-Röcknet ('94): quasi-homeomorphism: $\mathrm{QR} \leftrightarrow \mathrm{R}$

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For a regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(\mathcal{X}; m)$,

$$\begin{aligned} \mathcal{E}(u,u) &= \mathcal{E}^{(c)}(u,u) + \frac{1}{2}\int_{\mathcal{X}\times\mathcal{X}} (u(x)-u(y))^2 J(dx,dy) \\ &\quad + \int_{\mathcal{X}} u(x)^2 \kappa(dx), \end{aligned}$$

where $\mathcal{E}^{(c)}$ is strongly local: $\mathcal{E}^{(c)}(u,v) = 0$ for $uv = 0$

$J(dx, dy)$: jumping kernel

$\kappa(dx)$: killing measure.

Stability

In this talk, we are concerned with pure jump processes associated with regular DF $(\mathcal{E}, \mathcal{F})$:

$$\mathcal{E}(u,u) = \frac{1}{2}\int_{\mathcal{X}\times\mathcal{X}} (u(x)-u(y))^2 J(dx,dy)$$

on metric measure space $(\mathcal{X}, d, m)$.

We are interested in the following stability question: Suppose we have another DF $(\mathcal{E}', \mathcal{F})$ with $J' \asymp J$. What properties are transferable from one to the other?

HKE, PHI, EHI

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Example. Isotropic α-stable process X has transition density function

$$p^{(\alpha)}(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{|x-y|^{d+\alpha}}.$$

Its Dirichlet form $(\mathcal{E},\mathcal{F})$ is

$$\mathcal{E}(u,u) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (u(x)-u(y))^2 \frac{c_{d,\alpha}}{|x-y|^{d+\alpha}} dxdy.$$

Q. What is the heat kernel for

$$\mathcal{E}'(u,u) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} (u(x)-u(y))^2 \frac{c(x,y)}{|x-y|^{d+\alpha}} dxdy$$

with $c(x,y) \asymp 1$?

On d-sets

- C.-Kumagai '03: When $(\mathcal{X}, \rho, m)$ is an Alfors d-set, and

$$\mathcal{E}(u,u) = \int_{\mathcal{X}\times\mathcal{X}} (u(x)-u(y))^2 \frac{c(x,y)}{\rho(x,y)^{d+\alpha}} m(dx)m(dy),$$

then $p(t,x,y) \asymp t^{-d/\alpha} \wedge \dfrac{t}{\rho(x,y)^{d+\alpha}}$.

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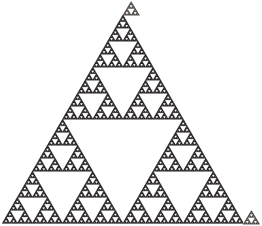

Example: Sierpinski gasket has Hausdorff dimension $d = \log 3/\log 2$ and walk dimension $d_w = \log 5/\log 2$. For any $0 < s < 1$, the s-subordination of BM on $\mathcal{X}$ has

$$p(t,x,y) \asymp t^{-d/\alpha} \wedge \frac{t}{\rho(x,y)^{d+\alpha}},$$

where $\alpha = sd_w \in (0, d_w)$.

The s-subordination of BM on Siepinkski gasket is a symmetric pure jump process with jumping kernel

$$J_0(x, y) \asymp \frac{1}{\rho(x, y)^{d+\alpha}}, \quad \text{where } \alpha = sd_w.$$

Open Question: What can we say about a generic symmetric jump process whose jumping kernel $J(x, y) \asymp \frac{1}{\rho(x,y)^{d+\alpha}}$? Does it have the same HKE?

Framework and results

$(\mathcal{X}, \rho, m)$: Metric measure space (assume $m(\mathcal{X}) = \infty$ for simplicity)

$(\mathcal{E}, \mathcal{F})$: regular pure-jump Dirichlet form on $L^2(\mathcal{X}, m)$

$$\mathcal{E}(f,f) = \int\int_{\mathcal{X}\times\mathcal{X}\setminus d} (f(x) - f(y)^2 J(x,y) m(dx) m(dy).$$

$\{X_t\}_{t\geq 0}$: corresponding Hunt process on $\mathcal{X}$.

We will consider condition

$$J(x,y) \asymp \frac{1}{V(x, \rho(x,y))\phi(\rho(x,y))}. \qquad (J_\phi)$$

$(J_{\phi,\leq})$ (resp. $(J_{\phi,\geq})$): if the upper (resp. lower) bound holds. Here $V(x,r) = m(B(x,r))$.

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Volume and time scale

(VD) and (RVD): There are constants $c_1, c_2 > 1$ s.t.

$$c_1 V(x,r) \leq V(x,2r) \leq c_2 VB(x,r)$$

for all $x \in \mathcal{X}$ and $r > 0$.

Doubling and reverse doubling condition (Polyn) for ϕ:
There are constants $c_3, c_4 > 0$, $\beta_2 \geq \beta_1 > 0$ s.t. for every $0 < r < R < \infty$,

$$c_3 \left(\frac{R}{r}\right)^{\beta_1} \leq \frac{\phi(R)}{\phi(r)} \leq c_4 \left(\frac{R}{r}\right)^{\beta_2}.$$

Examples:

(1) $V(x,r) \asymp r^d$: $J(x,y) \asymp 1/\rho(x,y)^{d+\alpha}$: α-stable-like process (C.-Kumagai '03)

(2) $[\alpha_1, \alpha_2] \subset (0, \infty)$, ν: probability measure on $[\alpha_1, \alpha_2]$

$$\phi(t) := \left(\int_{\alpha_1}^{\alpha_2} t^{-\alpha} \, \nu(d\alpha) \right)^{-1}.$$

Especially, $m(B(x,r)) \asymp r^d$, $0 < \alpha_1 < \cdots < \alpha_n < 2$,

$$J(x,y) = \sum_{k=1}^{n} \frac{c_i(x,y)}{\rho(x,y)^{d+\alpha_i}},$$

where $c^{-1} < c_i(x,y) = c_i(y,x) < c$: mixed-type process (C.-Kumagai '08).

HK(ϕ) :

$$p(t,x,y) \asymp \frac{1}{V(x,\phi^{-1}(t))} \wedge \frac{t}{V(x,\rho(x,y))\phi(\rho(x,y))}.$$

UHK(ϕ): if $\le$ holds.

- $\exists c_1 > 1$ s.t. $\forall r > 0$ and a.e. $x \in M$,

$$c_1^{-1}\phi(r) \le \mathbb{E}^x[\tau_{B(x,r)}] \le c_1\phi(r). \qquad (E_\phi)$$

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Main Result I

Theorem (C.-Kumagai-Wang 2021))

Assume that (VD), (RV) and (Polyn) hold. Then the following are equivalent:

(1) $\mathrm{HK}(\phi)$.

(2) J_ϕ *and* (E_ϕ).

(3) J_ϕ *and* $\mathrm{CSJ}(\phi)$.

Any of the above statements implies the process is conservative.

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CSJ(ϕ) condition

- CSJ(ϕ): $\exists C_0 \in (0,1]$, $C_1, C_2 > 0$ s.t. $\forall R \geq r > 0$, $\forall f \in \mathcal{F}$, $\exists$ cutoff function φ for $B(x,R) \subset B(x,R+r)$ (i.e. $\varphi|_{B(x,R)} = 1$, $\varphi|_{B(x,R+r)^c} = 0$) s.t.

$$\begin{aligned} &\int_{B(x,R+(1+C_0)r)} f^2 \, d\Gamma(\varphi,\varphi) \\ \leq \;& C_1 \int_{U\times U^*} (f(x)-f(y))^2 J(dx,dy) \\ &+ \frac{C_2}{\phi(r)} \int_{B(x,R+(1+C_0)r)} f^2 \, dm, \end{aligned}$$

where $U = B(x,R+r) \setminus B(x,R)$,
$U^* = B(x,R+(1+C_0)r) \setminus B(x,R-C_0 r)$.

Remarks

(i) For diffusions, similar inequality was given in Barlow-Andres, Barlow-Bass, Barlow-Bass-Kumagai and Grigor'yan-Hu-Lau.

(ii) $\mathrm{CSJ}(\phi)$ implies
$\mathrm{Cap}(B(x,R), B(x,R+r)) \le c_0 \frac{V(x,R+r)}{\phi(r)}$.
($\mathrm{PI}(\phi)$ gives $\ge$.)

(iii) Under $J_{\phi,\le}$, $\mathrm{CSJ}(\phi)$ always holds if $\beta_2 < 2$.

Remark

Stability of $\mathrm{HK}(r^{\alpha})$ for *d*-set also proved independently and around the same time by

• Murugan and Saloff-Coste ('19): stable-like discrete time Markov chains on *d*-set graphs, by generalizing Davies' method.

• Grigor'yan, E. Hu and J. Hu ('18): stable-like process on *d*-sets.

Main Result II

Theorem (C.-Kumagai-Wang 2021))

Assume that (VD), (RV) and (Polyn) hold. Then the following are equivalent:

(1) $\mathrm{UHK}(\phi)$ *and* $\{X_t\}_{t\geq 0}$ *is conservative.*

(2) $\mathrm{UHKD}(\phi)$, $J_{\phi,\leq}$ *and* (E_ϕ).

(3) $\mathrm{FK}(\phi)$, $J_{\phi,\leq}$ *and* $\mathrm{CSJ}(\phi)$.

Conditions:

- Faber-Krahn inequality $\mathrm{FK}(\phi)$:

$\exists C, \nu > 0$ s.t. $\forall B(x,r)$, $\forall D \subset B(x,r)$,

$$\lambda_1(D) \geq \frac{C}{\phi(r)} \Big(\frac{V(x,r)}{V(D)} \Big)^{\nu}.$$

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Examples

J_ϕ does not imply $\mathrm{HK}(\phi)$ in general. Let $\mathcal{X} = \mathbb{R}^d$,

$$\phi(r) = r^\alpha + r^\beta, \qquad 0 < \alpha < 2 < \beta.$$

Then for a process satisfying J_ϕ,

$$\mathbb{E}^x[\tau_{B(x,r)}] \asymp r^\alpha \wedge r^2,$$

so (E_ϕ) and hence $\mathrm{HK}(\phi)$ does not hold.

Sharp HKE for this case and their stability has recently been obtained in C.-Kumagai-Wang ('19/'22+), and in Bae-Kang-Kim-Lee ('19).

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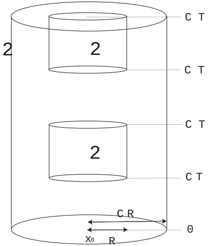
PHI



$u(t,x) : \mathcal{X} \to \mathbb{R}_+$ is *caloric* or *parabolic* on Q if

$$\frac{\partial u}{\partial t}(t,x) = \mathcal{L}u(t,x) \qquad \text{in } Q.$$

• $\mathrm{PHI}(\phi)$ (**parabolic Harnack inequality**): For every $u(t,x) \geq 0$ on $[0,\infty) \times \mathcal{X}$ caloric in Q, $T = \phi(R)$,

$$\sup_{Q_-} u \leq C_6 \inf_{Q_+} u.$$

Main Results III: Parabolic Harnack inequality

Theorem (C.-Kumagai-Wang '20)

Assume that (VD), (RVD) and (Polyn) hold. Then the following are equivalent:

(1) $\mathrm{PHI}(\phi)$.

(2) $\mathrm{NDL}(\phi)$ *and* UJS.

(3) $\mathrm{PI}(\phi)$, $J_{\phi,\le}$, $\mathrm{CSJ}(\phi)$ *and* UJS.

(4) PHR, (E_ϕ) *and* UJS.

(5) EHR, (E_ϕ) *and* UJS.

Corollary (C.-Kumagai-Wang '20)

$\mathrm{HK}(\phi) \Longleftrightarrow \mathrm{PHI}(\phi) + J_{\phi,\ge}$.

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Conditions

- UJS: For $x, y \in \mathcal{X}$ with $\rho(x,y) > 0$,

$$J(x,y) \le \frac{c}{m(B(x,r))} \int_{B(x,r)} J(z,y)m(dz)$$

for any $r \le \rho(x,y)/2$.

- NDL(ϕ): $\exists \varepsilon \in (0,1), c > 0$ s.t. $\forall x_0 \in \mathcal{X}$, $r > 0$, $t \le \phi(\varepsilon r)$ and $B = B(x_0, r)$,

$$p_t^B(x', y') \ge \frac{c}{m(B(x_0, \phi^{-1}(t)))}, \quad x', y' \in B(x_0, \varepsilon\phi^{-1}(t)).$$

- PI(ϕ) (Poincaré inequality): $\forall B = B(x,r)$ and $f \in \mathcal{F}$,

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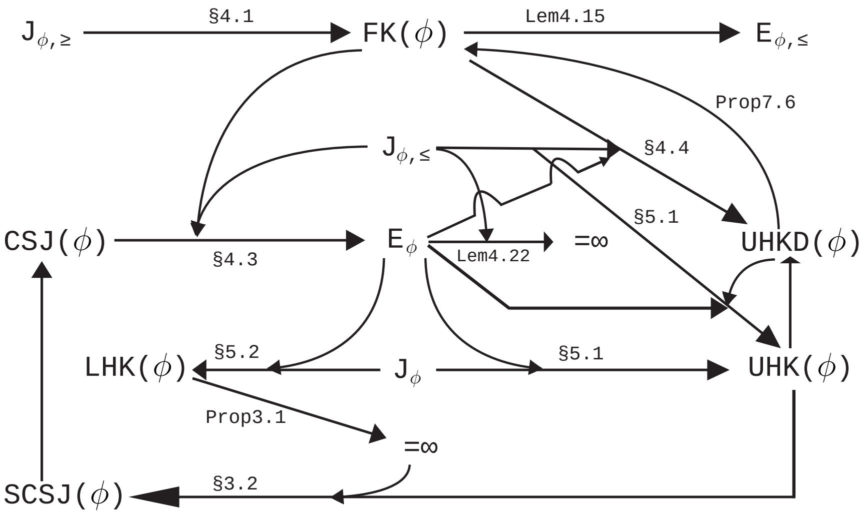
Idea of the proof for HKE (very sketchy)



Cacciopoli inequality

It is relatively easier to show that $\mathrm{HK}(\phi) \Leftrightarrow J_\phi + E_\phi$.

For their equivalence to $J_\phi + \mathrm{CSJ}(\phi)$, a key part of our approach uses the Caccioppoli inequality.

We show $\mathrm{CSJ}(\phi) + J_{\phi,\leq} \Rightarrow$ Cacciopoli inequality.

We then show this with $\mathrm{FK}(\phi)$ and through a comparison of L^2-norm over balls yields the L^2- as well as L^1-mean value inequalities for $\mathcal{E}$-subharmonic functions for truncation Dirichlet forms.

The latter can be used to derive $E_{\phi,\geq}$.

On the other hand, it is quite easy to deduce that

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Caccioppoli inequality: $x_0 \in \mathcal{X}$, $s > 0$, $B_s = B(x_0, s)$.
Assume VD, $\mathrm{CSJ}(\phi)$, $J_{\phi,\le}$.
For $0 < r \le r'$, let u: sub–harmonic on $B_{r'+r}$, and $v = (u - \theta)_+$ for $\theta \ge 0$. φ: cutoff function for $B_{r'-r} \subset B_{r'}$ in $\mathrm{CSJ}(\phi)$. Then

$$\int_{B_{r'+r}} d\Gamma(v\varphi, v\varphi)$$

$$\le \frac{c}{\phi(r)} \left[1 + \frac{1}{\theta} \left(1 + \frac{r'}{r} \right)^{d_2+\beta_2-\beta_1} \mathrm{Tail}\,(u; x_0, r' + r) \right] \int_{B_{r'+r}} u^2 \, dm.$$

where $$\mathrm{Tail}\,(u; x_0, r) := \phi(r) \int_{B(x_0,r)^c} \frac{|u(x)| m(dx)}{V(x_0, \rho(x_0, x))\phi(\rho(x_0, x))}.$$

Elliptic Harnack inequality

For non-local Dirichlet forms, we have some stability results under some strong "bounded goemetry" assumption. For strongly local Dirichlet forms,

Theorem (Barlow-C.-Murugan '22+)

Let $(\mathcal{X}, d)$ be a complete, locally compact, separable metric doubling space, and m a Radon measure on $\mathcal{X}$ with full support. Let $(\mathcal{E}, \mathcal{F})$ and $(\mathcal{E}', \mathcal{F})$ be strongly local symmetric regular Dirichlet forms on $L^2(\mathcal{X}; m)$ satisfying

$$C^{-1}\mathcal{E}(f, f) \leq \mathcal{E}'(f, f) \leq C\mathcal{E}(f, f) \quad \text{for all } f \in \mathcal{F}.$$

If $(\mathcal{E}, \mathcal{F})$ satisfies EHI, then so does $(\mathcal{E}', \mathcal{F})$.

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Thank you!