Multiplicative chaos of the Brownian loop soup

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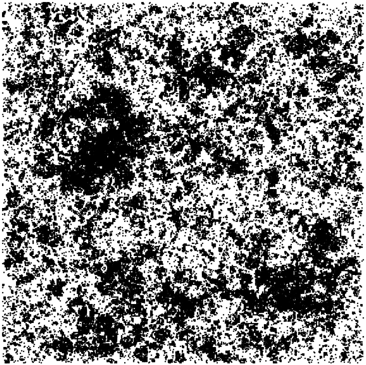
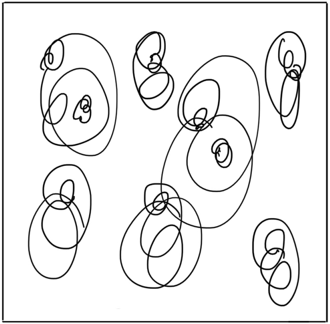
Based on joint work with E. Aïdékon, N. Berestycki and T. Lupu

Brownian loop soup

Countable infinite collection of Brownian-like loops in a domain $D \subset \mathbb{R}^2$





Introduced by Lawler and Werner

Related to many random conformal invariant objects

Definition

Poisson point process $\mathcal{L}_D^\theta \sim \mathsf{PPP}(\theta \mu_D^{\mathsf{loop}})$

- $\theta > 0$: intensity parameter
- μ_D^{loop}: loop measure

$$\mu_D^{\mathrm{loop}}(d\wp) = \int_D \int_0^{+\infty} \mathbb{P}_{D,t}^{z,z}(d\wp) p_D(t,z,z) \frac{\mathrm{d}t}{t} \mathrm{d}z.$$

Brownian bridge heat kernel

Fact

- *Infinitely many small loops*
- *Conformally invariant*
- *Restriction property*

Loop-erased random walk:

(erase chronologically each loop)

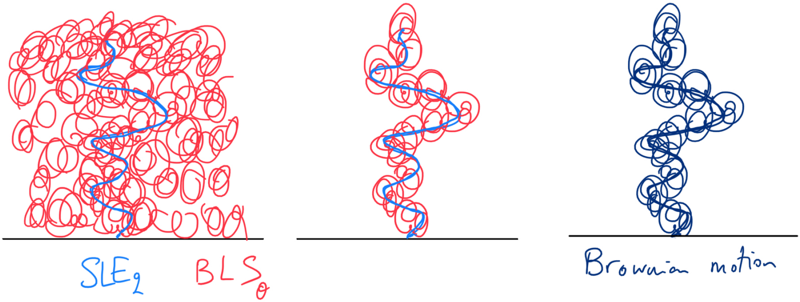
random walk = loop erasure + loops

$\downarrow$

Brownian motion = SLE_2 + loops

$\updownarrow$

Brownian loop soup ($\theta = 1$)



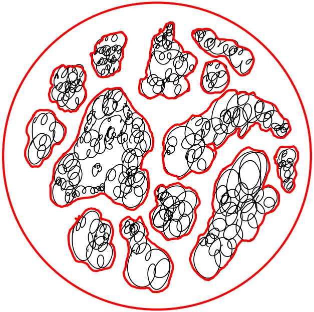
Clusters

Cluster = chain of intersecting loops

Phase transition: (Sheffield–Werner)

• $\theta > 1/2$: one big cluster

• $\theta \leq 1/2$: infinitely many clusters
Outer boundaries of outermost clusters
$= \text{CLE}_\kappa$ where $\kappa = \kappa(\theta) \in (8/3, 4]$
Conformal Loop Ensemble



The Ubiquitous Gaussian free field

$\theta = 1/2$

$(L_x)_{x \in D}$ = occupation field of Brownian loop soup
"L_x = amount of time spent at x"

Le Jan's isomorphism: $(L_x)_{x \in D} \overset{(d)}{=} \frac{1}{2}\mathrm{GFF}^2$

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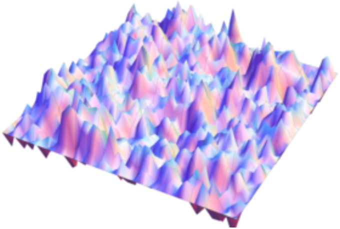


Green function
$\downarrow$

h = GFF
Formally, $h = (h_x)_{x \in D} \sim \mathcal{N}(0, G_D)$
Rigorously, random generalised function

Scaling limit of

- Height function in dimers
- Characteristic polynomial of large random matrices

Liouville measure

h : GFF
$\gamma \in (-2, 2)$

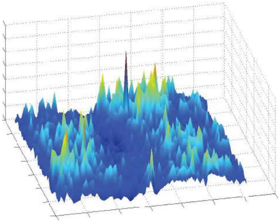
Informally: random measure

$$\mu_\gamma = e^{\gamma h(x)} \mathrm{d}x$$

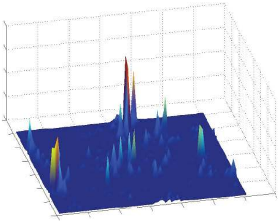
Instance of **Gaussian multiplicative chaos** measure

Rigorously,

$$\mu_\gamma = \lim_{\varepsilon \to 0} \varepsilon^{\gamma^2/2} e^{\gamma h_\varepsilon(x)} \mathrm{d}x$$



(a) $\gamma = 0.2$



(b) $\gamma = 1$

Simulation by Rhodes–Vargas

Thick points

$$\mu_\gamma = e^{\gamma h} \mathrm{d}x$$

x fixed deterministic point $\rightsquigarrow h_\varepsilon(x) = O(\sqrt{|\log \varepsilon|})$

x is μ^γ – typical point $\rightsquigarrow \lim_{\varepsilon \to 0} \frac{h_\varepsilon(x)}{|\log \varepsilon|} = \gamma$

In fact,

Theorem (Folklore?)

$\mathcal{T}_\varepsilon(\gamma) := \{x \in D : h_\varepsilon(x) \geq \gamma |\log \varepsilon|\}$ *(γ-thick points)*

$$\sqrt{|\log \varepsilon|} \varepsilon^{-\gamma^2/2} \mathbf{1}_{\{x \in \mathcal{T}_\varepsilon(\gamma)\}} \mathrm{d}x \xrightarrow[\varepsilon \to 0]{} \mu_\gamma$$

"μ_γ = uniform measure on γ-thick points"

Liouville measure and Brownian loop soup

Loop soup: $\mathcal{L}_D^\theta \sim \mathrm{PPP}(\theta \mu_D^{\mathrm{loop}})$
GFF: h

Couple $(\mathcal{L}_D^\theta, h)$ such that $(L_x)_{x \in D} = \frac{1}{2} h^2$ $\theta = 1/2$

Sample $z \sim e^{\gamma h}$.

Questions:

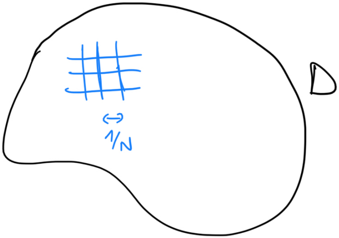
- What does the loop soup look like near z?
- How does the loop soup create a thick local time?
  $\hookrightarrow$ A few very thick loops?
  $\hookrightarrow$ Many loops w/ typical local time?
- What about $\theta \neq 1/2$? What is the associated chaos?

Multiplicative chaos construction

$D_N = \frac{1}{N}\mathbb{Z}^2 \cap D$ discrete approximation of D

$\mathcal{L}^\theta_{D_N}$ random walk loop soup

$$\ell_x = \text{local time at } x = \sum_{\wp \in \mathcal{L}^\theta_{D_N}} \int_0^{\tau_\wp} \mathbf{1}_{\{\wp_t = x\}} \mathrm{d}t$$



typical point: $\mathbb{E}[\ell_x] \sim \frac{\theta}{2\pi} \log N$

Multiplicative chaos construction

$\theta > 0$ intensity

$\mathcal{L}_{D_N}^{\theta}$ random walk loop soup

$a = \frac{\gamma^2}{2}$ thickness parameter

typical point: $\mathbb{E}[\ell_x] \sim \frac{\theta}{2\pi} \log N$

a-thick points: $\mathcal{T}_N(a) := \left\{ x \in D_N : \ell_x \geq \frac{1}{2\pi} a (\log N)^2 \right\}$

Uniform measure on $\mathcal{T}_N(a)$: $\mathcal{M}_a^N := \frac{(\log N)^{1-\theta}}{N^{2-a}} \sum_{x \in \mathcal{T}_N(a)} \delta_x$

Multiplicative chaos construction

$\theta > 0$ intensity

$\mathcal{L}^\theta_{D_N}$ random walk loop soup $\qquad a = \frac{\gamma^2}{2}$ thickness parameter

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Theorem (E. Aïdékon, N. Berestycki, A. J., T. Lupu 21)

$(\mathcal{M}_a^N, \mathcal{L}^\theta_{D_N}) \to (\mathcal{M}_a, \mathcal{L}^\theta_D)$ *as* $N \to \infty$.

$\rightsquigarrow \mathcal{M}_a =$ multiplicative chaos associated to $\mathcal{L}^\theta_D$.

- $\theta = 1/2$: discrete GFF, Biskup–Louidor
- $\theta \to 0$: random walk thick points, Jego

Multiplicative chaos and loop soup

$\theta > 0$ intensity
$a = \frac{\gamma^2}{2}$ thickness

- $\theta = 1/2 \rightsquigarrow \mathcal{M}_a \overset{(\mathrm{d})}{=} e^{\gamma \mathrm{GFF}} + e^{-\gamma \mathrm{GFF}}$
- $\theta \neq 1/2 \rightsquigarrow \mathcal{M}_a =$ multiplicative chaos associated to a permanental field

(new object! not Gaussian!)

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Sample $z \sim \mathcal{M}_a(\cdot)/\mathcal{M}_a(D)$: typical a-thick point.
How many loops go through z? What are the thicknesses associated to each individual loop?

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1) Infinitely many loops.
2) Denote $a_1, a_2, \dots$ thicknesses of loops going through z.
$\{a_1, a_2, \dots\} \sim$ Poisson-Dirichlet(θ) on the interval $[0, a]$.

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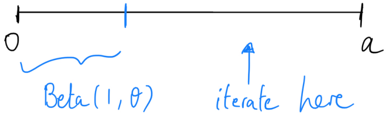
$\{a_1, a_2, \dots\} \sim$ Poisson-Dirichlet(θ) on the interval $[0, a]$.
3) Cond. on $\{a_1, a_2, \dots\}$, the loops that go through z are indep. and distributed like the concatenation of the excursions in PPP($a_i \mu_D^z$).

Poisson-Dirichlet

Random partition of $[0, a]$

Stick breaking construction:



- random permutation: σ random permutation of $\{1, \ldots, n\}$. Decompose σ into disjoint cycles $C_1, \ldots, C_k$:

$$\left\{ \frac{|C_1|}{n}, \ldots, \frac{|C_k|}{n} \right\} \to PD(1)$$

- prime factorisation: sample $n = p_1 \ldots p_k$ uniformly in $[1, N]$, then

$$\left\{ \frac{\log p_1}{\log n}, \ldots, \frac{\log p_k}{\log n} \right\} \to PD(1)$$

Construction from the continuum

Brownian multiplicative chaos: multiplicative chaos associated to **finitely** many trajectories

$$e^{\gamma\sqrt{2L_x}}\mathrm{d}x$$

Bass–Burdzy–Koshnevisan 94
Aïdékon–Hu–Shi 20
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- Kill each loop independently of each other at rate $K > 0$

$\mathbb{P}(\wp \text{ killed}) = 1 - e^{-KT(\wp)}$

- $\mathcal{M}_a^K :=$ multiplicative chaos associated to **killed** loops

Theorem (E.A., N.B., A.J., T.L. 21)

$(\log K)^{-\theta}\mathcal{M}_a^K \xrightarrow[K\to\infty]{\mathbb{P}} \mathcal{M}_a$

Two key ingredients

- Other cutoff? Diameter $> \varepsilon$? Duration $> \varepsilon^2$?

Our cutoff: Exact solvability!

- At $\theta = 1/2$,

$$L_x(\mathcal{L}_K^\theta) + L_x(\mathcal{L}^\theta \setminus \mathcal{L}_K^\theta) = \frac{1}{2}\mathsf{GFF}^2$$

$$\updownarrow$$

$$\frac{1}{2}(\text{massive GFF})^2$$

massive Green function associated to $-\Delta + K$.

$\rightsquigarrow$ Measure-valued martingale (as a function of K)

Thank you for your attention!