Towards Analysis on Fractals: Piecewise C^1-Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff Propinquity

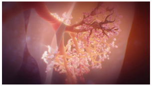
Dr. Therese Basa Landry

March 23, 2022

Overview

Finite Approximations of Fractals

- better understand how fractal structures arise and evolve in nature

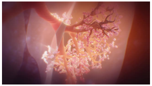
   

- extend methods from mathematical physics classically formulated on smooth manifolds to fractal spaces

Overview

Finite Approximations of Fractals

- better understand how fractal structures arise and evolve in nature




- extend methods from mathematical physics classically formulated on smooth manifolds to fractal spaces

Tools from Noncommutative Geometry

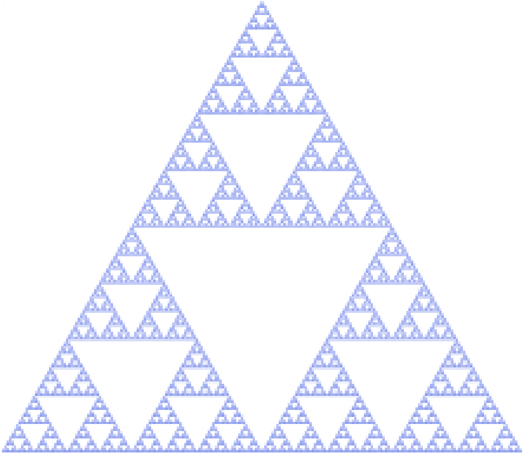
- spectral triples

  -generalize differentiable structure

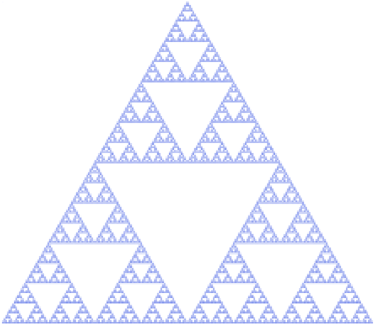
- Gromov-Hausdorff propinquity

  -extends Hausdorff distance to function spaces

Motivating Example: The Sierpinski Gasket



Motivating Example: The Sierpinski Gasket



Let p_i denote the vertices of a regular 3-simplex, and for $i = 1, 2, 3$, let

$$F_i x = \frac{1}{2}(x - p_i) + p_i.$$

The *Sierpinski gasket SG* is the unique nonempty compact subset of $\mathbb{R}^2$ such that $SG = \cup_{i=1}^{3} F_i(SG)$.

Piecewise C^1-fractal Curve (Lapidus, Sarhad)

A *piecewise C^1-fractal curve* is a compact length space $X \subseteq \mathbb{R}^n$ that satisfies the axioms below. Let $L(\gamma)$ denote the length of the continuous curve γ parametrized by its arclength.

Piecewise C^1-fractal Curve (Lapidus, Sarhad)

A *piecewise C^1-fractal curve* is a compact length space $X \subseteq \mathbb{R}^n$ that satisfies the axioms below. Let $L(\gamma)$ denote the length of the continuous curve γ parametrized by its arclength.

- Axiom 1. $X = \overline{R}$ where $R = \bigcup_{j \geq 1} R_j$ and R_j, $j \in \mathbb{N}$, is a rectifiable C^1 curve with $L(R_j) \to 0$ as $j \to \infty$.

Piecewise C^1-fractal Curve (Lapidus, Sarhad)

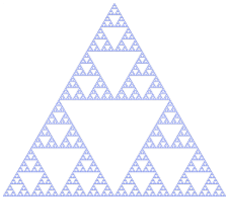
A *piecewise C^1-fractal curve* is a compact length space $X \subseteq \mathbb{R}^n$ that satisfies the axioms below. Let $L(\gamma)$ denote the length of the continuous curve γ parametrized by its arclength.

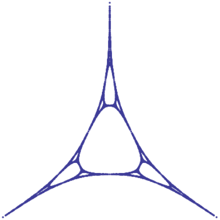
- Axiom 1. $X = \overline{R}$ where $R = \bigcup_{j \geq 1} R_j$ and R_j, $j \in \mathbb{N}$, is a rectifiable C^1 curve with $L(R_j) \to 0$ as $j \to \infty$.
- Axiom 2. There exists a dense subset $\mathcal{B} \subset X$ such that for each $p \in \mathcal{B}$ and $q \in X$, one of the minimizing geodesics from p to q can be given as a countable (or finite) concatenation of the R_j's.

Piecewise C^1-fractal Curve (Lapidus, Sarhad)

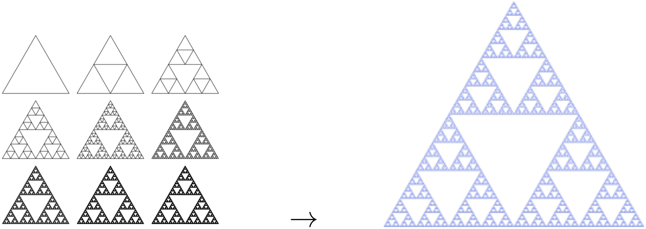
A *piecewise C^1-fractal curve* is a compact length space $X \subseteq \mathbb{R}^n$ that satisfies the axioms below. Let $L(\gamma)$ denote the length of the continuous curve γ parametrized by its arclength.

- Axiom 1. $X = \overline{R}$ where $R = \bigcup_{j \geq 1} R_j$ and R_j, $j \in \mathbb{N}$, is a rectifiable C^1 curve with $L(R_j) \to 0$ as $j \to \infty$.
- Axiom 2. There exists a dense subset $\mathcal{B} \subset X$ such that for each $p \in \mathcal{B}$ and $q \in X$, one of the minimizing geodesics from p to q can be given as a countable (or finite) concatenation of the R_j's.

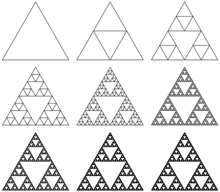




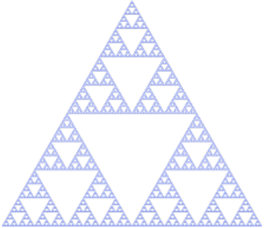
Approximating Piecewise C^1-fractal Curves



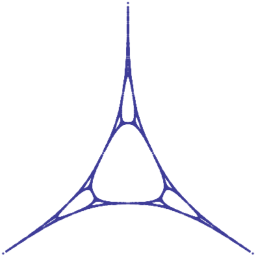
Approximating Piecewise C^1-fractal Curves



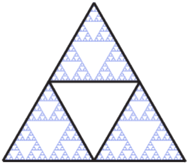
$\rightarrow$



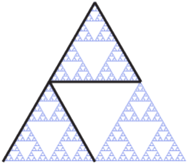
$\rightarrow$



Approximating Piecewise C^1-fractal Curves



$$\mathrm{Haus}_d(SG, SG_1) = inf\{\epsilon > 0 : SG \subseteq B_\epsilon(SG_1), SG_1 \subseteq B_\epsilon(SG)\} = \frac{1}{8}$$



$$\mathrm{Haus}_d(SG, A) = inf\{\epsilon > 0 : SG \subseteq B_\epsilon(A), A \subseteq B_\epsilon(SG)\} = \frac{1}{2}$$

Approximation Sequence for a Piecewise C^1-fractal Curve Compatible with a Given Parameterization (L., Lapidus, Latrémolière)

Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j\in\mathbb{N}}$. An *approximation sequence of X compatible with* $(R_j)_{j\in\mathbb{N}}$ is a strictly increasing function $B:\mathbb{N}\to\mathbb{N}$ such that, for every $\epsilon>0$, there exists $n\in\mathbb{N}$ such that if $n\geq N$, and letting

- $X_n=\bigcup_{j=1}^{B(n)} R_j$,
- V_n denote the set of the endpoints of the curves $R_1,\cdots,R_{B(n)}$,
- d_n be the geodesic distance on X_n,

Approximation Sequence for a Piecewise C^1-fractal Curve Compatible with a Given Parameterization (L., Lapidus, Latrémolière)

Let X be a piecewise C^1-fractal curve with parameterization $(R_j)_{j\in\mathbb{N}}$. An *approximation sequence of X compatible with* $(R_j)_{j\in\mathbb{N}}$ is a strictly increasing function $B:\mathbb{N}\to\mathbb{N}$ such that, for every $\epsilon>0$, there exists $n\in\mathbb{N}$ such that if $n\geq N$, and letting

- $X_n=\bigcup_{j=1}^{B(n)} R_j$,
- V_n denote the set of the endpoints of the curves $R_1,\cdots,R_{B(n)}$,
- d_n be the geodesic distance on X_n,

the following properties hold:

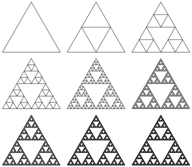
(1) $Haus_{d_n}(V_n,X_n)<\epsilon$,

(2) the restriction of d_∞ to $V_n\times V_n$ is d_n.

Approximation Sequence for the Sierpinski Gasket

Let R_j, denote, for each $j \geq 1$, a continuous, injective functions to edges in SG_n such that

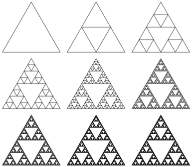
$$R_j : [0,1] \to \text{the edges in } SG_0 \text{ for } j = 1,2,3,$$
$$R_j : [0,2^{-1}] \to \text{the edges in } SG_1 \text{ for } j = 4,5,\cdots,12$$
$$R_j : [0,2^{-2}] \to \text{the edges in } SG_2 \text{ for } j = 13,14,\cdots,39$$
$$\vdots$$
$$R_j : [0,2^{-n}] \to \text{the edges in } SG_n \text{ for } j = 1+\sum_{i=1}^{n} 3^i, 2+\sum_{i=1}^{n} 3^i, \cdots, 3^{n+1}+\sum_{i=1}^{n} 3^i$$



Approximation Sequence for the Sierpinski Gasket

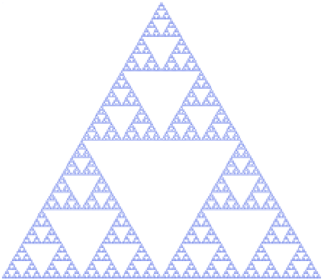
Let R_j, denote, for each $j \geq 1$, a continuous, injective functions to edges in SG_n such that

$$R_j : [0,1] \to \text{the edges in } SG_0 \text{ for } j = 1,2,3,$$
$$R_j : [0,2^{-1}] \to \text{the edges in } SG_1 \text{ for } j = 4,5,\cdots,12$$
$$R_j : [0,2^{-2}] \to \text{the edges in } SG_2 \text{ for } j = 13,14,\cdots,39$$
$$\vdots$$
$$R_j : [0,2^{-n}] \to \text{the edges in } SG_n \text{ for } j = 1+\sum_{i=1}^{n} 3^i, \; 2+\sum_{i=1}^{n} 3^i, \cdots, 3^{n+1}+\sum_{i=1}^{n} 3^i$$

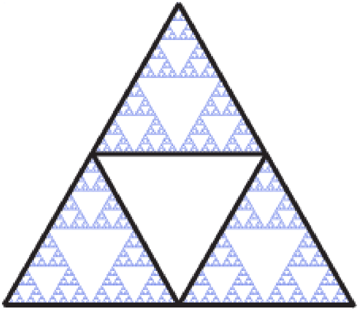


Let $B : \mathbb{N} \to \mathbb{N}$ be given by $B(n) = \Sigma_{i=1}^{n+1} 3^i$. Then $B(n)$ defines an approximation sequence of SG compatible with the parameterization $(R_j)_{j\in\mathbb{N}}$.

Intrinsic Metrics on SG and SG_n

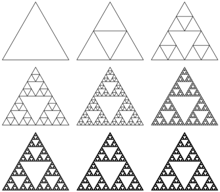


(SG, d_∞)

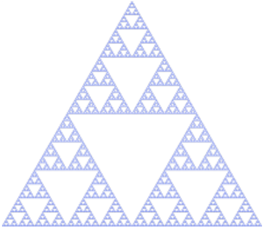


(SG_n, d_n)

Extending Hausdorff Distance to Function Spaces



$\rightarrow$



$$SG_n \xrightarrow{Haus_d} SG$$

$$(SG_n, d_n) \xrightarrow{GH} (SG, d_\infty)$$

$$(C(SG_n), L_{d_n}) \xrightarrow{?} (C(SG), L_{d_\infty})$$

Spectral Triple (Connes)

Let $\mathcal{A}$ be a unital C^*-algebra. An *unbounded Fredholm module* $(\mathcal{H}, D)$ over $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$ together with a unital representation π of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and an unbounded, self-adjoint operator D on $\mathcal{H}$ such that

Spectral Triple (Connes)

Let $\mathcal{A}$ be a unital C^*-algebra. An *unbounded Fredholm module* $(\mathcal{H}, D)$ over $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$ together with a unital representation π of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and an unbounded, self-adjoint operator D on $\mathcal{H}$ such that

- the set

$$\{a \in \mathcal{A} \textit{ for which } [D, \pi(a)] \text{ is densely defined}$$

$$\text{and extends to a bounded operator on } \mathcal{H}\}$$

  is dense in $\mathcal{A}$

Spectral Triple (Connes)

Let $\mathcal{A}$ be a unital C^*-algebra. An *unbounded Fredholm module* $(\mathcal{H}, D)$ over $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$ together with a unital representation π of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and an unbounded, self-adjoint operator D on $\mathcal{H}$ such that

- the set

  $$\{a \in \mathcal{A} \text{ for which } [D, \pi(a)] \text{ is densely defined}$$

  $$\text{and extends to a bounded operator on } \mathcal{H}\}$$

  is dense in $\mathcal{A}$
- the operator $(I + D^2)^{-1}$ is compact.

Spectral Triple (Connes)

Let $\mathcal{A}$ be a unital C^*-algebra. An *unbounded Fredholm module* $(\mathcal{H}, D)$ over $\mathcal{A}$ consists of a Hilbert space $\mathcal{H}$ together with a unital representation π of $\mathcal{A}$ into $\mathcal{B}(\mathcal{H})$ and an unbounded, self-adjoint operator D on $\mathcal{H}$ such that

- the set

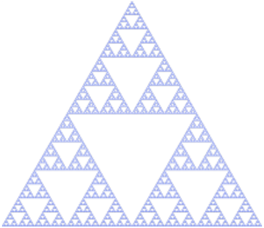
$$\{a \in \mathcal{A} \textit{ for which } [D, \pi(a)] \text{ is densely defined}$$

$$\text{and extends to a bounded operator on } \mathcal{H}\}$$

  is dense in $\mathcal{A}$
- the operator $(I + D^2)^{-1}$ is compact.

If the underlying representation π is faithful, then $(\mathcal{A}, \mathcal{H}, D)$ is called a *spectral triple*, and D a *Dirac operator*.

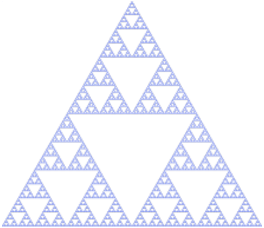
Noncommutative Riemannian Geometry: Recovery of Geodesic Distance (Connes)



$d_{geo}(p,q) = \inf\{ L(\gamma) : \gamma \text{ is a path from } p \text{ to } q \}$ where

$$L(\gamma) = \int_p^q (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}}$$

Noncommutative Riemannian Geometry: Recovery of Geodesic Distance (Connes)



$d_{geo}(p,q) = \inf\{\, L(\gamma) : \gamma \text{ is a path from } p \text{ to } q \,\}$ where

$$L(\gamma) = \int_p^q (g_{\mu\nu} dx^\mu dx^\nu)^{\frac{1}{2}}$$

is ***dual*** to

$$d_{spec}(p,q) = \sup\{|f(p) - f(q)| : f \in C(X), \|[D, \pi(f)]\|_{B(\mathcal{H})} \leq 1\}$$

Building Lapidus-Sarhad Spectral Triples

A Lapidus-Sarhad spectral triple for a piecewise C^1-fractal curve is a direct sum of spectral triples for each curve in a given paramaterization. Each of these spectral triples is built from spectral triples for circles.

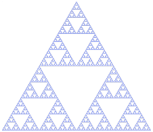
Building Lapidus-Sarhad Spectral Triples

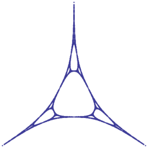
A Lapidus-Sarhad spectral triple for a piecewise C^1-fractal curve is a direct sum of spectral triples for each curve in a given paramaterization. Each of these spectral triples is built from spectral triples for circles.

To define a spectral triple for a circle in the complex plane centered at 0 and with radius $r > 0$, let

- $\mathcal{A}C_r$ denote the algebra of complex continuous $2\pi r$-periodic functions on the real line,
- $\mathcal{H}_r := L^2([-\pi r, \pi r], (2\pi r)^{-1}\mathfrak{m})$, where $(2r)^{-1}\mathfrak{m}$ is the normalized Lebesgue measure on $[-\pi r, \pi r]$,
- $D_{C_r} = -i\frac{d}{dx}|_{\text{span}(\phi_k^r)_{k\in\mathbb{Z}}}$ with $\phi_k^r = \exp(\frac{ikx}{r})$, $k \in \mathbb{Z}$,
- π_{C_r} the representation that sends elements of AC_r to multiplication operators on $\mathcal{H}_r$

Theorem (Lapidus, Sarhad 2014)

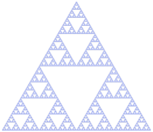


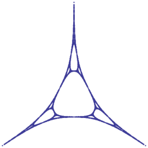


Let X be a piecewise C^1-fractal curve. Then $X = \overline{\bigcup_{j\geq 1} R_j}$, where R_j is a rectifiable C^1 curve of length l_j for each $j \in \mathbb{N}$. Set

- $\mathcal{H}_\infty := \bigoplus_{j\in\mathbb{N}} \mathcal{H}_{l_j}$,
- $D_\infty := \bigoplus_{j\in\mathbb{N}} D_{l_j}$, where $D_{l_j} = D_{C_{l_j/\pi}} + \frac{1}{2l_j} I$,
- $\pi_\infty := \bigoplus_{j\in\mathbb{N}} \pi_{l_j}$, where $\pi_{l_j}(f)h(x) := f(R_j(|t|))h(x)$.

Theorem (Lapidus, Sarhad 2014)





Let X be a piecewise C^1-fractal curve. Then $X = \overline{\bigcup_{j\geq 1} R_j}$, where R_j is a rectifiable C^1 curve of length l_j for each $j \in \mathbb{N}$. Set

- $\mathcal{H}_\infty := \bigoplus_{j\in\mathbb{N}} \mathcal{H}_{l_j}$,
- $D_\infty := \bigoplus_{j\in\mathbb{N}} D_{l_j}$, where $D_{l_j} = D_{C_{l_j/\pi}} + \frac{1}{2l_j} I$,
- $\pi_\infty := \bigoplus_{j\in\mathbb{N}} \pi_{l_j}$, where $\pi_{l_j}(f)h(x) := f(R_j(|t|))h(x)$.

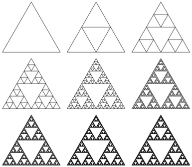
Then $ST(X) := (C(X), \mathcal{H}_\infty, D_\infty)$ with representation π_∞ is a spectral triple for X. Furthermore,

$$d_\infty(x,y) = \sup\{|f(x) - f(y)| \, : \, f \in C(SG), \\ \|[D_\infty, \pi_\infty(f)]\|_{B(\mathcal{H}_\infty)} \leq 1\}.$$

Theorem (Antonescu, Christensen, Lapidus 2008)

Let R_j, denote, for each $j \geq 1$, a continuous, injective functions to edges in SG_n such that

$$R_j : [0,1] \rightarrow \text{the edges in } SG_0 \text{ for } j = 1,2,3,$$
$$R_j : [0,2^{-1}] \rightarrow \text{the edges in } SG_1 \text{ for } j = 4,5,\cdots,12$$
$$R_j : [0,2^{-2}] \rightarrow \text{the edges in } SG_2 \text{ for } j = 13,14,\cdots,39$$
$$\vdots$$
$$R_j : [0,2^{-n}] \rightarrow \text{the edges in } SG_n \text{ for } j = 1+\textstyle\sum_{i=1}^n 3^i, \; 2+\sum_{i=1}^n 3^i, \cdots, 3^{n+1}+\sum_{i=1}^n 3^i$$



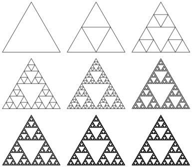
Theorem (Antonescu, Christensen, Lapidus 2008)

Let R_j, denote, for each $j \geq 1$, a continuous, injective functions to edges in SG_n such that

$R_j : [0,1] \to$ the edges in SG_0 for $j = 1, 2, 3$,
$R_j : [0, 2^{-1}] \to$ the edges in SG_1 for $j = 4, 5, \cdots, 12$
$R_j : [0, 2^{-2}] \to$ the edges in SG_2 for $j = 13, 14, \cdots, 39$

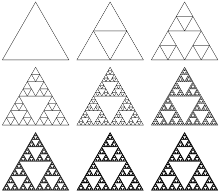
$\vdots$

$R_j : [0, 2^{-n}] \to$ the edges in SG_n for $j = 1 + \sum_{i=1}^{n} 3^i$, $2 + \sum_{i=1}^{n} 3^i, \cdots, 3^{n+1} + \sum_{i=1}^{n} 3^i$

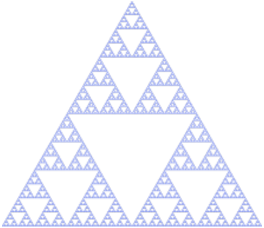


Then $ST(SG)$ with representation π_∞ is a spectral triple for SG that recovers the Hausdorff dimension, the geodesic metric, and the Hausdorff measure.

Extending Hausdorff Distance to Spectral Triples



$\rightarrow$



$$SG_n \xrightarrow{Haus_d} SG$$

$$(SG_n, d_n) \xrightarrow{GH} (SG, d_\infty)$$

$$(C(SG_n), L_{d_n}) \xrightarrow{?} (C(SG), L_{d_\infty})$$

$$(C(SG_n), D_n, \mathcal{H}_n) \xrightarrow{\quad\quad ? \quad\quad} (C(SG), D_\infty, \mathcal{H}_\infty)$$

Approximating Lapidus-Sarhad Spectral Triples

Building on the earlier work of Marc Rieffel, Frédéric Latremoliere introduced a generalization of the Gromov-Hausdorff distance that was recently extended to spectral triples in a form called the *spectral propinquity*.

Approximating Lapidus-Sarhad Spectral Triples

Building on the earlier work of Marc Rieffel, Frédéric Latremoliere introduced a generalization of the Gromov-Hausdorff distance that was recently extended to spectral triples in a form called the *spectral propinquity*.

Main Result (Informally): If X is a piecewise C^1-fractal curve X with parameterization $(R_j)_{j\in\mathbb{N}}$ and $B(n)$ is an approximation sequence of X compatible with $\{R_j\}_{j\in\mathbb{N}}$,

Approximating Lapidus-Sarhad Spectral Triples

Building on the earlier work of Marc Rieffel, Frédéric Latremoliere introduced a generalization of the Gromov-Hausdorff distance that was recently extended to spectral triples in a form called the *spectral propinquity*.

Main Result (Informally): If X is a piecewise C^1-fractal curve X with parameterization $(R_j)_{j\in\mathbb{N}}$ and $B(n)$ is an approximation sequence of X compatible with $\{R_j\}_{j\in\mathbb{N}}$,then the Lapidus-Sarhad spectral triple on X_n converges, in the sense of the spectral propinquity, to the Lapidus-Sarhad spectral triple on X.

Definition (Rieffel)

Let $\mathcal{A}$ be a unital C^*-algebra. The *state space* $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ is the set of positive linear functionals on $\mathcal{A}$ of norm 1. If L is a seminorm defined on a dense subspace of the self-adjoint elements of $\mathcal{A}$ satisfying some form of Leibniz inequality and such that

Definition (Rieffel)

Let $\mathcal{A}$ be a unital C^*-algebra. The *state space* $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ is the set of positive linear functionals on $\mathcal{A}$ of norm 1. If L is a seminorm defined on a dense subspace of the self-adjoint elements of $\mathcal{A}$ satisfying some form of Leibniz inequality and such that

$$\{a \in \mathfrak{sa}(\mathcal{A}) : L(a) = 0\} = \mathbb{R}1_{\mathcal{A}}$$

Definition (Rieffel)

Let $\mathcal{A}$ be a unital C^*-algebra. The *state space* $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ is the set of positive linear functionals on $\mathcal{A}$ of norm 1. If L is a seminorm defined on a dense subspace of the self-adjoint elements of $\mathcal{A}$ satisfying some form of Leibniz inequality and such that

$$\{a \in \mathfrak{sa}(\mathcal{A}) : L(a) = 0\} = \mathbb{R}1_{\mathcal{A}}$$

and the associated *Monge-Kantorovich distance*, that is, the metric defined for all φ, $\psi \in \mathcal{S}(\mathcal{A})$ by

$$\mathrm{mk}_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(L), L(a) \leq 1\},$$

metrizes the weak* topology of $\mathcal{S}(\mathcal{A})$,

Definition (Rieffel)

Let $\mathcal{A}$ be a unital C^*-algebra. The *state space* $\mathcal{S}(\mathcal{A})$ of $\mathcal{A}$ is the set of positive linear functionals on $\mathcal{A}$ of norm 1. If L is a seminorm defined on a dense subspace of the self-adjoint elements of $\mathcal{A}$ satisfying some form of Leibniz inequality and such that

$$\{a \in \mathfrak{sa}(\mathcal{A}) : L(a) = 0\} = \mathbb{R}1_{\mathcal{A}}$$

and the associated *Monge-Kantorovich distance*, that is, the metric defined for all φ, $\psi \in \mathcal{S}(\mathcal{A})$ by

$$\mathrm{mk}_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}(L), L(a) \leq 1\},$$

metrizes the weak* topology of $\mathcal{S}(\mathcal{A})$, then $(\mathcal{A}, L)$ is a *quantum compact metric space* $(\mathcal{A}, L)$.

Noncommutative Metric Geometry

compact metric space (X, d_X)

Noncommutative Metric Geometry

compact metric space (X, d_X)

$f \in C(X)$, $L_{d_x}(f) := \sup\left\{\frac{|f(p)-f(q)|}{d_X(p,q)} : p, q \in X, p \neq q\right\}$

(classical) quantum compact metric space $(C(X), L_{d_X})$

Noncommutative Metric Geometry

compact metric space (X, d_X)

$f \in C(X)$, $L_{d_x}(f) := \sup\left\{\frac{|f(p)-f(q)|}{d_X(p,q)} : p, q \in X, p \neq q\right\}$

(classical) quantum compact metric space $(C(X), L_{d_X})$

$\hat{x} : x \in (X, d_X) \mapsto \delta_x \in (\mathcal{S}(C(X)), mk_{L_{d_X}})$ *is an isometry onto its image!*

Metric Spectral Triples (Latrémolière)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Given $a \in \mathcal{A}$, set $L_D(a) = |||[D, \pi(a)]|||_{\mathcal{B}(\mathcal{H})}$. If $(\mathcal{A}, L_D)$ is a quantum compact metric space, then $(\mathcal{A}, \mathcal{H}, D)$ is a *metric spectral triple*.

Metric Spectral Triples (Latrémolière)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Given $a \in \mathcal{A}$, set $L_D(a) = |||[D, \pi(a)]|||_{\mathcal{B}(\mathcal{H})}$. If $(\mathcal{A}, L_D)$ is a quantum compact metric space, then $(\mathcal{A}, \mathcal{H}, D)$ is a *metric spectral triple*.

The *spectral propinquity* is a metric on the class of metric spectral triples.

Theorem (Latrémolière 2018)

If $(\mathcal{A}, \mathcal{H}, D)$ and $(\mathcal{A}', \mathcal{H}', D')$ are metric spectral triples with spectral propinquity $\Lambda^{\mathsf{spec}}((\mathcal{A}, \mathcal{H}, D), (\mathcal{A}', \mathcal{H}', D')) = 0$, then there exists a unitary $U : \mathcal{H} \to \mathcal{H}'$ and a *-isomorphism $\theta : \mathcal{A} \to \mathcal{A}'$ such that

$$UDU^* = D',$$

and for every $a \in \mathcal{A}$ and $\omega \in \mathcal{H}'$,

$$\theta(a)\omega = (UaU^*)\omega.$$

Theorem (Latrémolière 2018)

If $(\mathcal{A}, \mathcal{H}, D)$ and $(\mathcal{A}', \mathcal{H}', D')$ are metric spectral triples with spectral propinquity $\Lambda^{\mathrm{spec}}((\mathcal{A}, \mathcal{H}, D), (\mathcal{A}', \mathcal{H}', D')) = 0$, then there exists a unitary $U : \mathcal{H} \to \mathcal{H}'$ and a *-isomorphism $\theta : \mathcal{A} \to \mathcal{A}'$ such that

$$UDU^* = D',$$

and for every $a \in \mathcal{A}$ and $\omega \in \mathcal{H}'$,

$$\theta(a)\omega = (UaU^*)\omega.$$

Note that θ is also a full quantum isometry- that is, $L_{D'} \circ \theta = L_D$.

Spectral Propinquity Metric Convergence of Spectral Triples on SG (L., Lapidus, Latrémolière)

Let $\{R_j\}_{j \in \mathbb{N}}$ be a parameterization of SG as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of SG compatible with $\{R_j\}_{j \in \mathbb{N}}$.

Spectral Propinquity Metric Convergence of Spectral Triples on SG (L., Lapidus, Latrémolière)

Let $\{R_j\}_{j\in\mathbb{N}}$ be a parameterization of SG as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of SG compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the Lapidus-Sarhad spectral triple on SG, $(C(SG), \bigoplus_{j\geq 1} \mathcal{H}_{l_j}, \bigoplus_{j\geq 1} D_{l_j})$, by $(C(SG), \mathcal{H}_\infty, D_\infty)$ and $(C(SG_{B(n)}), \bigoplus_{j=1}^{B(n)} \mathcal{H}_{l_j}, \bigoplus_{j=1}^{B(n)} D_{l_j})$, by $(C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)})$.

Spectral Propinquity Metric Convergence of Spectral Triples on SG (L., Lapidus, Latrémolière)

Let $\{R_j\}_{j\in\mathbb{N}}$ be a parameterization of SG as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of SG compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the Lapidus-Sarhad spectral triple on SG, $(C(SG), \bigoplus_{j\geq 1}\mathcal{H}_{l_j}, \bigoplus_{j\geq 1} D_{l_j})$, by $(C(SG), \mathcal{H}_\infty, D_\infty)$ and $(C(SG_{B(n)}), \bigoplus_{j=1}^{B(n)}\mathcal{H}_{l_j}, \bigoplus_{j=1}^{B(n)} D_{l_j})$, by $(C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)})$.

When equipped with $L_{D_\infty}(a) := |||[D_\infty, \pi_\infty(a)]|||_{\mathcal{B}(\mathcal{H}_\infty)}$, $(C(SG), L_{D_\infty})$ is a quantum compact metric space. Similarly, $(C(SG_{B(n)}), L_{D_{B(n)}})$ is also a quantum compact metric space.

Spectral Propinquity Metric Convergence of Spectral Triples on SG (L., Lapidus, Latrémolière)

Let $\{R_j\}_{j\in\mathbb{N}}$ be a parameterization of SG as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of SG compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the Lapidus-Sarhad spectral triple on SG, $(C(SG), \bigoplus_{j\geq 1}\mathcal{H}_{l_j}, \bigoplus_{j\geq 1} D_{l_j})$, by $(C(SG), \mathcal{H}_\infty, D_\infty)$ and $(C(SG_{B(n)}), \bigoplus_{j=1}^{B(n)}\mathcal{H}_{l_j}, \bigoplus_{j=1}^{B(n)} D_{l_j})$, by $(C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)})$.

When equipped with $L_{D_\infty}(a) := \|[D_\infty, \pi_\infty(a)]\|_{\mathcal{B}(\mathcal{H}_\infty)}$, $(C(SG), L_{D_\infty})$ is a quantum compact metric space. Similarly, $(C(SG_{B(n)}), L_{D_{B(n)}})$ is also a quantum compact metric space.

Moreover,

$$\lim_{n\to 0}\Lambda^{\text{spec}}((C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)}), (C(SG), \mathcal{H}_\infty, D_\infty)) = 0.$$

Spectral Propinquity Metric Convergence of Spectral Triples on SG (L., Lapidus, Latrémolière)

Let $\{R_j\}_{j\in\mathbb{N}}$ be a parameterization of SG as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of SG compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the Lapidus-Sarhad spectral triple on SG, $(C(SG), \bigoplus_{j\geq 1} \mathcal{H}_{l_j}, \bigoplus_{j\geq 1} D_{l_j})$, by $(C(SG), \mathcal{H}_\infty, D_\infty)$ and $(C(SG_{B(n)}), \bigoplus_{j=1}^{B(n)} \mathcal{H}_{l_j}, \bigoplus_{j=1}^{B(n)} D_{l_j})$, by $(C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)})$.

When equipped with $L_{D_\infty}(a) := \|[D_\infty, \pi_\infty(a)]\|_{\mathcal{B}(\mathcal{H}_\infty)}$, $(C(SG), L_{D_\infty})$ is a quantum compact metric space. Similarly, $(C(SG_{B(n)}), L_{D_{B(n)}})$ is also a quantum compact metric space.

Moreover,

$$\lim_{n\to 0} \Lambda^{\mathrm{spec}}((C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)}), (C(SG), \mathcal{H}_\infty, D_\infty)) = 0.$$

Spectral Propinquity Metric Convergence of Spectral Triples on a Piecewise C^1-fractal Curve (L., Lapidus, Latrémolière)

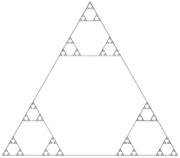
Let $\{R_j\}_{j\in\mathbb{N}}$ be a parameterization of X as a piecewise C^1-fractal curve and $B(n)$ an approximation sequence of X compatible with $\{R_j\}_{j\in\mathbb{N}}$. Denote the Lapidus and Sarhad spectral triple on X, $(C(X), \bigoplus_{j\geq 1}\mathcal{H}_{l_j}, \bigoplus_{j\geq 1} D_{l_j})$, by $(C(SG), \mathcal{H}_\infty, D_\infty)$ and $(C(X_{B(n)}), \bigoplus_{j=1}^{B(n)}\mathcal{H}_{l_j}, \bigoplus_{j=1}^{B(n)} D_{l_j})$, by $(C(X_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)})$.

When equipped with $L_{D_\infty}(a) := |||[D_\infty, \pi_\infty(a)]|||_{\mathcal{B}(\mathcal{H}_\infty)}$, $(C(X), L_{D_\infty})$ is a quantum compact metric space. Similarly, $(C(X_{B(n)}), L_{D_{B(n)}})$ is also a quantum compact metric space.

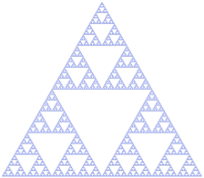
Moreover,

$$\lim_{n\to 0}\Lambda^{\text{spec}}((C(X_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)}), (C(X), \mathcal{H}_\infty, D_\infty)) = 0.$$

Work in Progress: The Stretched Sierpinski Gasket of Parameter α, $0 < \alpha < \frac{1}{3}$



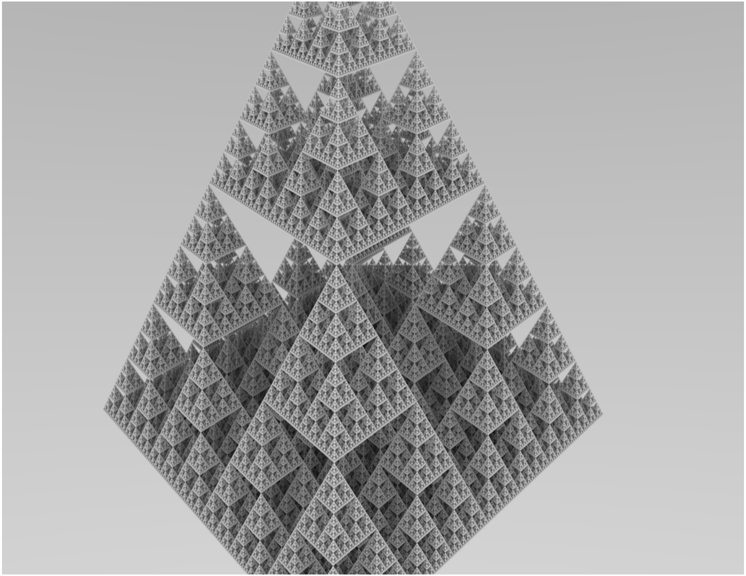
$\xrightarrow{\alpha \downarrow 0}$



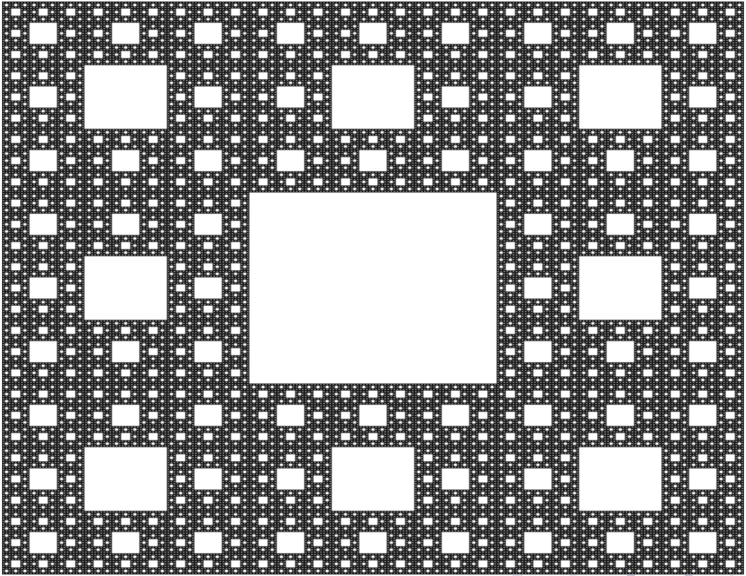
$$SG_\alpha \xrightarrow{Haus_d} SG$$

$$(C(SG_\alpha), \mathcal{H}_{SG_\alpha}, D_{SG_\alpha}) \xrightarrow{\text{Spectral Propinquity}} (C(SG), \mathcal{H}_{SG}, D_{SG})$$

Future Work



Future Work



Future Work

Definition and Study of a "Fractal Manifold"

Classification of C^*-algebras on Fractals

Approximation of Laplacians on Fractals

Noncommutative Fractality

References

A. Connes
Noncommutative Geometry
Academic Press, San Diego, 1994.

T. Landry, M.L. Lapidus, F. Latrémolière,
Metric Approximations of Spectral Triples on the Sierpiński Gasket and other Fractal Curves,
arXiv: 2010.06921.

M. L. Lapidus and J. Sarhad,
Dirac Operators and Geodesic Metric on the Harmonic Sierpinski Gasket and other Fractal Sets,
J. of Noncom. Geo. **8** (2014), 947-985,
arXiv: 1212.0878.

F. Latrémolière
The Dual Gromov-Hausdorff Propinquity
Journal de Mathematiques Pure et Appliquees
103 (2015) 2, pp.303-351
arXiv: 1311.0104.

References

F. Latrémolière
The Gromov-Hausdorff Propinquity for Metric Spectral Triples
arXiv:1811.10843v1.

M. Rieffel
Compact Quantum Metric Spaces
Operator Algebras, Quantization, and Noncommutative Geometry, Contemp. Math.
365 (2004), pp. 315-330
arXiv:math/0308207.

M.A. Rieffel,
Metrics on States from Actions of Compact Groups,
Documenta Mathematica **3** (1998), 215-229,
arXiv: math/9807084.

Spectral Propinquity Metric Convergence

$$\Lambda^{\text{spec}}((C(SG), H_{SG}, D_{SG}), (C(SG_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}}))$$

$$= \max\{\Lambda^{*\text{met}}(mvb(C(SG), H_{SG}, D_{SG}),$$
$$mvb(\mathrm{C}(\mathrm{SG}_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}})),$$
$$\Lambda^{*\text{mod,cov}}(uqvb(C(SG), H_{SG}, D_{SG}),$$
$$uqvb(\mathrm{C}(\mathrm{SG}_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}}))\}$$

$$= \max\left\{\Lambda^{*\text{met}}((H_{SG}, DN_{SG}, \mathbb{C}, 0, C(SG), L_{D_{SG}}),\right.$$
$$(H_{SG_{B(n)}}, DN_{SG_{B(n)}}, \mathbb{C}, 0, C(SG_{B(n)}), L_{D_{SG_{B(n)}}})),$$
$$\Lambda^{*\text{mod,cov}}(uqvb(C(SG), H_{SG}, D_{SG}),$$
$$uqvb(\mathrm{C}(\mathrm{SG}_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}}))\}$$