Analysis on Fractals: Piecewise
Curves, Lapidus-Sarhad Spectra
and the Gromov-Hausdorff
Propinquity
Dr. Therese Basa Landry
March 23, 2022 Towards Analysis on Fractals: Piecewise C 1 -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff **Propinquity**

Dr. Therese Basa Landry

March 23, 2022

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Piecewise C^1 -fractal Curves and [Approximation](#page-1-0) **Sequences**

[Spectral Triples](#page-18-0)

Metric Approximations of Spectral Triples on Piecewise $C¹$ -fractal Curves [via the Spectral](#page-41-0) Propinquity

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Overview

Finite Approximations of Fractals

 \blacktriangleright better understand how fractal structures arise and evolve in nature

 \blacktriangleright extend methods from mathematical physics classically formulated on smooth manifolds to fractal spaces

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Finite Approximations of Fractals

 \blacktriangleright better understand how fractal structures arise and evolve in nature

 \blacktriangleright extend methods from mathematical physics classically formulated on smooth manifolds to fractal spaces

Tools from Noncommutative Geometry

 \blacktriangleright spectral triples

-generalize differentiable structure

▶ Gromov-Hausdorff propinquity -extends Hausdorff distance to function spaces [Towards Analysis](#page-0-0) on Fractals: Piecewise $C^{\mathbf{1}}$ -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff **Propinquity**

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Motivating Example: The Sierpinski Gasket

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Motivating Example: The Sierpinski Gasket

Let p_i denote the vertices of a regular 3-simplex, and for $i = 1, 2, 3$, let

$$
F_i x = \frac{1}{2}(x-p_i)+p_i.
$$

The *Sierpinski gasket SG* is the unique nonempty compact subset of \mathbb{R}^2 such that $SG = \cup_{i=1}^3 F_i(SG)$.

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actal curve is a compact length space
fies the axioms below. Let $L(\gamma)$ denote
inuous curve γ parametrized by its A *piecewise C* 1 *-fractal curve* is a compact length space $X \subseteq \mathbb{R}^n$ that satisfies the axioms below. Let $L(\gamma)$ denote the length of the continuous curve γ parametrized by its arclength.

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► Axiom 1.
$$
X = \overline{R}
$$
 where $R = \bigcup_{j \geq 1} R_j$ and R_j , $j \in \mathbb{N}$, is a rectifiable C^1 curve with $L(R_j) \to 0$ as $j \to \infty$.

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Approximating Piecewise C^1 -fractal Curves

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Approximating Piecewise C^1 -fractal Curves

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Approximating Piecewise C^1 -fractal Curves

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\bigotimes_{\ell=1}^{\ell} S_{\ell} = B_{\epsilon}(SG_{1}),
$$
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$$
SG_{1} \subseteq B_{\epsilon}(SG) \bigg\} =
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$$
\bigotimes_{\ell=1}^{\ell} \bigotimes_{\ell=1}^{\ell} S_{\ell} = B_{\epsilon}(AG_{1}),
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$$

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Haus_d(SG, A) = inf {
$$
\epsilon > 0
$$
 : SG $\subseteq B_{\epsilon}(A)$,
 $A \subseteq B_{\epsilon}(SG)$ } = $\frac{1}{2}$

Approximation Sequence for a Piecewise C 1 -fractal Curve Compatible with a Given Parameterization (L., Lapidus, Latrémolière)

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on (L., Lapidus, Latrémolière)

vise C¹-fractal curve with parameterization

simation sequence of X compatible with

y increasing function $B : \mathbb{N} \to \mathbb{N}$ such t

nere exists $n \in \mathbb{N}$ s Let X be a piecewise C¹-fractal curve with parameterization $(R_i)_{i\in\mathbb{N}}$. An approximation sequence of X compatible with $(R_i)_{i\in\mathbb{N}}$ is a strictly increasing function $B:\mathbb{N}\to\mathbb{N}$ such that, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $n > N$, and letting

$$
\blacktriangleright X_n = \bigcup_{j=1}^{B(n)} R_j,
$$

- \blacktriangleright V_n denote the set of the endpoints of the curves $R_1, \cdots, R_{B(n)}$
- \blacktriangleright d_n be the geodesic distance on X_n ,

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Approximation Sequence for a Piecewise C 1 -fractal Curve Compatible with a Given Parameterization (L., Lapidus, Latrémolière)

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Solomon (L., Lapidus, Latrémolière)

Vise C¹-fractal curve with parameterization

Symmation sequence of X compatible with

Vision increasing function $B : \mathbb{N} \to \mathbb{N}$ such there exists $n \$ Let X be a piecewise C¹-fractal curve with parameterization $(R_i)_{i\in\mathbb{N}}$. An approximation sequence of X compatible with $(R_i)_{i\in\mathbb{N}}$ is a strictly increasing function $B:\mathbb{N}\to\mathbb{N}$ such that, for every $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that if $n > N$, and letting

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- \blacktriangleright d_n be the geodesic distance on X_n ,

the following properties hold:

- (1) Haus $_{d_n}(V_n,X_n)<\epsilon$,
- (2) the restriction of d_{∞} to $V_n \times V_n$ is d_n .

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Approximation Sequence for the Sierpinski Gasket

Let R_j , denote, for each $j\geq 1$, a continuous, injective functions to edges in SG_n such that

Draft Rj : [0, 1] → the edges in SG⁰ for j = 1, 2, 3, Rj : [0, 2 −1] → the edges in SG¹ for j = 4, 5, · · · , 12 Rj : [0, 2 −2] → the edges in SG² for j = 13, 14, · · · , 39 . . . Rj −n Pⁿ i ,

$$
R_j: [0, 2^{-n}] \to \text{the edges in } SG_n \text{ for } j = 1 + \sum_{i=1}^n 3^i
$$

$$
2 + \sum_{i=1}^n 3^i, \cdots, 3^{n+1} + \sum_{i=1}^n 3^i
$$

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÷.

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$$

$$
2 + \sum_{i=1}^n 3^i, \cdots, 3^{n+1} + \sum_{i=1}^n 3^i
$$

Let $B : \mathbb{N} \to \mathbb{N}$ be given by $B(n) = \sum_{i=1}^{n+1} 3^i$. Then $B(n)$ defines an approximation sequence of SG compatible with the parameterization $(R_i)_{i\in\mathbb{N}}$. $\mathcal{A} \equiv \mathcal{F} \rightarrow \mathcal{A} \equiv \mathcal{F} \rightarrow \mathcal{A} \equiv \mathcal{F} \rightarrow \mathcal{A}$ 000 [Towards Analysis](#page-0-0) on Fractals: Piecewise $C^{\mathbf{1}}$ -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff **Propinquity**

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Intrinsic Metrics on SG and SG_n

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Extending Hausdorff Distance to Function Spaces

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$$
\begin{aligned}\n &SG_n \xrightarrow{Haus_d} SG \\
 & (SG_n, d_n) \xrightarrow{GH} (SG, d_\infty) \\
 & (C(SG_n), L_{d_n}) \xrightarrow{?} (C(SG), L_{d_\infty})\n \end{aligned}
$$

 \rightarrow

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Spectral Triple (Connes)

 C^* -algebra. An *unbounded Fredholm*
er $\mathcal A$ consists of a Hilbert space $\mathcal H$ toge
ssentation π of $\mathcal A$ into $\mathcal B(\mathcal H)$ and an
djoint operator D on $\mathcal H$ such that Let A be a unital C^* -algebra. An unbounded Fredholm module (H, D) over A consists of a Hilbert space H together with a unital representation π of $\mathcal A$ into $\mathcal B(\mathcal H)$ and an unbounded, self-adjoint operator D on H such that

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▶ the set

 ${a \in \mathcal{A} \text{ for which } [D, \pi(a)] \text{ is densely defined}}$

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and extends to a bounded operator on \mathcal{H} is dense in A

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and extends to a bounded operator on \mathcal{H} is dense in A

• the operator
$$
(I + D^2)^{-1}
$$
 is compact.

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 ${a \in \mathcal{A} \text{ for which } [D, \pi(a)] \text{ is densely defined}}$

and extends to a bounded operator on \mathcal{H} }

is dense in A

ighthroportion $(I + D^2)^{-1}$ is compact.

If the underlying representation π is faithful, then (A, \mathcal{H}, D) is called a spectral triple, and D a Dirac operator.

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Noncommutative Riemannian Geometry: Recovery of Geodesic Distance (Connes)

 $d_{geo}(p,q) = \inf\{ L(\gamma) : \gamma \text{ is a path from } p \text{ to } q \}$ where $L(\gamma) = \int^q$ p $(g_{\mu\nu}dx^\mu dx^\nu)^{\frac{1}{2}}$

 $\mathcal{A} \equiv \mathcal{A} + \mathcal{A} \equiv \mathcal{A} + \mathcal{A} \equiv \mathcal{A} + \mathcal{A}$

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Noncommutative Riemannian Geometry: Recovery of Geodesic Distance (Connes)

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is dual to

 $d_{spec}(p,q)$

 $= \sup\{|f(p) - f(q)| : f \in C(X), ||[D, \pi(f)]||_{B(H)} \leq 1\}$ 000 [Towards Analysis](#page-0-0) on Fractals: Piecewise $C^{\mathbf{1}}$ -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff **Propinquity**

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Building Lapidus-Sarhad Spectral Triples

spectral triple for a piecewise C^1 -fracta
um of spectral triples for each curve in a
ation. Each of these spectral triples is b
es for circles. A Lapidus-Sarhad spectral triple for a piecewise C^1 -fractal curve is a direct sum of spectral triples for each curve in a given paramaterization. Each of these spectral triples is built from spectral triples for circles.

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spectral triple for a piecewise C^1 -fracta
um of spectral triples for each curve in a
ation. Each of these spectral triples is b
es for circles.
al triple for a circle in the complex plan-
with radius $r > 0$, let
e the A Lapidus-Sarhad spectral triple for a piecewise C^1 -fractal curve is a direct sum of spectral triples for each curve in a given paramaterization. Each of these spectral triples is built from spectral triples for circles.

To define a spectral triple for a circle in the complex plane centered at 0 and with radius $r > 0$, let

- AC_r denote the algebra of complex continuous $2\pi r$ -periodic functions on the real line,
- $\bullet \ \mathcal{H}_r:=L^2([-\pi r,\pi r],(2\pi r)^{-1}\mathfrak{m}),$ where $(2r)^{-1}\mathfrak{m}$ is the normalized Lebesgue measure on $[-\pi r, \pi r]$,
- $D_{\mathsf{C}_r} = -i \frac{d}{d\mathsf{x}}|_{\mathsf{span}(\phi^\mathsf{r}_k)_{k \in \mathbb{Z}}}$ with $\phi^\mathsf{r}_k = \exp(\frac{ik\mathsf{x}}{r})$ $\frac{kx}{r}$), $k \in \mathbb{Z}$,
- \bullet π_{C_r} the representation that sends elements of AC_r to multiplication operators on \mathcal{H}_r

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Theorem (Lapidus, Sarhad 2014)

Let X be a piecewise C^1 -fractal curve. Then $X=\overline{\bigcup_{j\ge 1}R_j}$, where R_j is a rectifiable C^1 curve of length l_j for each $j\in\mathbb{N}.$ Set

\n- \n
$$
\mathcal{H}_{\infty} := \bigoplus_{j \in \mathbb{N}} \mathcal{H}_{l_j}
$$
\n
\n- \n $D_{\infty} := \bigoplus_{j \in \mathbb{N}} D_{l_j}$, where $D_{l_j} = D_{C_{l_j}/\pi} + \frac{1}{2l_j} l$,\n
\n- \n $\pi_{\infty} := \bigoplus_{j \in \mathbb{N}} \pi_{l_j}$, where $\pi_{l_j}(f)h(x) := f(R_j(|t|))h(x)$.\n
\n

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\n• $\mathcal{H}_{\infty} := \bigoplus_{j\in\mathbb{N}} \mathcal{H}_{l_j}$,
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\nThen $ST(X) := (C(X), \mathcal{H}_{\infty}, D_{\infty})$ with representation π_{∞} is
\na spectral triple for X. Furthermore,
\n $d_{\infty}(x, y) = \sup\{|f(x) - f(y)| : f \in C(SG)$,

$$
d_{\infty}(x,y) = \sup\{|f(x) - f(y)| : f \in C(\mathcal{SG}),\|[D_{\infty}, \pi_{\infty}(f)]\|_{\mathcal{B}(\mathcal{H}_{\infty})} \leq 1\}.
$$

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Theorem (Antonescu, Christensen, Lapidus 2008)

Let R_j , denote, for each $j\geq 1$, a continuous, injective functions to edges in SG_n such that

Draft Rj : [0, 1] → the edges in SG⁰ for j = 1, 2, 3, Rj : [0, 2 −1] → the edges in SG¹ for j = 4, 5, · · · , 12 Rj : [0, 2 −2] → the edges in SG² for j = 13, 14, · · · , 39 . . . −n Pⁿ i ,

$$
R_j: [0, 2^{-n}] \to \text{the edges in } SG_n \text{ for } j = 1 + \sum_{i=1}^n 3^i
$$

$$
2 + \sum_{i=1}^n 3^i, \cdots, 3^{n+1} + \sum_{i=1}^n 3^i
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Then $ST(SG)$ with representation π_{∞} is a spectral triple for SG that recovers the Hausdorff dimension, the geodesic metric, and the Hausdorff measure. \equiv 299 [Towards Analysis](#page-0-0) on Fractals: Piecewise $C^{\mathbf{1}}$ -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff **Propinquity**

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Extending Hausdorff Distance to Spectral Triples

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Approximating Lapidus-Sarhad Spectral Triples

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distance that was recently extended to
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Main Result (Informally): If X is a piecewise C^1 -fractal curve X with parameterization $(R_i)_{i\in\mathbb{N}}$ and $B(n)$ is an approximation sequence of X compatible with $\{R_i\}_{i\in\mathbb{N}}$,

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 C^* -algebra. The *state space* $S(A)$ of λ
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isfying some form of Leibniz inequality a Let A be a unital C*-algebra. The state space $\mathcal{S}(A)$ of A is the set of positive linear functionals on A of norm 1. If L is a seminorm defined on a dense subspace of the self-adjoint elements of A satisfying some form of Leibniz inequality and such that

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$$

and the associated Monge-Kantorovich distance, that is, the metric defined for all φ , $\psi \in \mathcal{S}(\mathcal{A})$ by

$$
\mathsf{mk}_L(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathsf{dom}(L), L(a) \leq 1\},\
$$

metrizes the weak* topology of $S(A)$,

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metrizes the weak* topology of $S(A)$, then (A, L) is a quantum compact metric space (A, L) .

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Noncommutative Metric Geometry

 $\mathsf{p}\left(\mathsf{X},d_{\mathsf{X}}\right)$ compact metric space (X, d_X)

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イロン イ押ン イヨン イヨン・ヨー QQQ compact metric space (X, d_X)

compact metric space
$$
(X, d_X)
$$

\n $f \in C(X)$, $L_{d_X}(f) := \sup \left\{ \frac{|f(p) - f(q)|}{d_X(p,q)} : p, q \in X, p \neq q \right\}$
\n(classical) quantum compact metric space $(C(X), L_{d_X})$

(classical) quantum compact metric space $(C(X), L_{d_X})$

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 PQQ

compact metric space (X, d_X)

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f\in C(X), L_{d_x}(f):=\sup\left\{\tfrac{|f(p)-f(q)|}{d_X(p,q)}: p,q\in X, p\neq q\right\}
$$

(classical) quantum compact metric space $(C(X), L_{d_X})$

 $\begin{aligned} &\text{base }(X,d_X) \ &:= \sup \left\{ \frac{|f(p)-f(q)|}{d_X(p,q)}: p,q \in X, p \neq q \right\} \ &\text{mod} \; \text{metric space } (C(X),L_{d_X}) \ &\text{for } d_X \in (\mathcal{S} (C(X)),mk_{L_{d_X}}) \; \text{is an isometry} \end{aligned}$ $\hat{x}: x \in (X,d_X) \mapsto \delta_x \in (\mathcal{S}(\mathcal{C}(X)), {\sf mk}_{L_{d_X}})$ is an isometry onto its image!

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a spectral triple. Given $a \in A$, set
)]| $|B(\mathcal{H})$. If (A, L_D) is a quantum compaint (A, H, D) is a metric spectral triple. Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Given $a \in \mathcal{A}$, set $L_D(a) = ||[D,\pi(a)]||_{\mathcal{B}(\mathcal{H})}.$ If (\mathcal{A},L_D) is a quantum compact metric space, then (A, H, D) is a metric spectral triple.

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Singuity is a metric on the class of metric
 \int Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple. Given $a \in \mathcal{A}$, set $L_D(a) = ||[D,\pi(a)]||_{\mathcal{B}(\mathcal{H})}.$ If (\mathcal{A},L_D) is a quantum compact metric space, then (A, H, D) is a metric spectral triple.

The *spectral propinquity* is a metric on the class of metric spectral triples.

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 (A', H', D') are metric spectral triples w
ty $\Lambda^{\text{spec}}((A, H, D), (A', H', D')) = 0$, then $U : H \to H'$ and a *-isomorphism
that
 $UDU^* = D',$
 A and $\omega \in H',$
 $\theta(a)\omega = (UaU^*)\omega$. If (A, H, D) and (A', H', D') are metric spectral triples with spectral propinquity $\mathsf{\Lambda}^{\rm spec}((\mathcal{A},\mathcal{H},D),(\mathcal{A}',\mathcal{H}',D'))=0$, then there exists a unitary $U:\mathcal{H}\rightarrow \mathcal{H}'$ and a *-isomorphism $\theta : \mathcal{A} \to \mathcal{A}'$ such that

$$
UDU^{\ast }=D^{\prime },
$$

and for every $a \in A$ and $\omega \in \mathcal{H}'$,

$$
\theta(a)\omega=(UaU^*)\omega.
$$

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$$
UDU^{\ast }=D^{\prime },
$$

and for every $a \in A$ and $\omega \in \mathcal{H}'$,

$$
\theta(a)\omega=(UaU^*)\omega.
$$

Note that θ is also a full quantum isometry- that is, $L_{D'} \circ \theta = L_D$.

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) and (C(Let ${R_i}_{i \in \mathbb{N}}$ be a parameterization of SG as a piecewise C^1 -fractal curve and $B(n)$ an approximation sequence of SG compatible with ${R_i}_{i\in\mathbb{N}}$. Denote the Lapidus-Sarhad spectral triple on SG , $(C(SG), \bigoplus_{j \geq 1} \mathcal{H}_{l_j}, \bigoplus_{j \geq 1} D_{l_j})$, by $(\mathit{C}(\mathit{SG}), \mathcal{H}_{\infty}, D_{\infty})$ and $(\mathit{C}(\mathit{SG}_{B(n)}), \bigoplus_{j=1}^{B(n)} \mathcal{H}_{l_j}, \bigoplus_{j=1}^{B(n)} D_{l_j}),$ by $(C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)})$.

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When equipped with $L_{D_\infty}(\mathsf{a}) := ||[D_\infty, \pi_\infty(\mathsf{a})]||_{\mathcal{B}(\mathcal{H}_\infty)}$, $(C(SG), L_{D_{\infty}})$ is a quantum compact metric space. Similarly, $(\mathit{C}(\mathit{SG}_{B(n)}), \mathit{L}_{D_{B(n)}})$ is also a quantum compact metric space.

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Moreover,

$$
\lim_{n\to 0} \Lambda^{\text{spec}}((C(SG_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)}), (C(SG), \mathcal{H}_{\infty}, D_{\infty})) = 0.
$$

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Spectral Propinquity Metric Convergence of Spectral Triples on a Piecewise C^1 -fractal Curve (L., Lapidus, Latrémolière)

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parameterization of *X* as a piecewise

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When equipped with $L_{D_\infty}(\mathsf{a}) := ||[D_\infty, \pi_\infty(\mathsf{a})]||_{\mathcal{B}(\mathcal{H}_\infty)}$, $(C(X), L_{D_{\infty}})$ is a quantum compact metric space. Similarly, $(\mathcal{C}(X_{B(n)}), L_{D_{B(n)}})$ is also a quantum compact metric space.

Moreover,

 $\lim_{n \to \infty} \Lambda^{\text{spec}}((C(X_{B(n)}), \mathcal{H}_{B(n)}, D_{B(n)}), (C(X), \mathcal{H}_{\infty}, D_{\infty})) = 0.$ $n\rightarrow 0$ 000

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Work in Progress: The Stretched Sierpinski Gasket of Parameter α , $0 < \alpha < \frac{1}{3}$

$$
\mathit{SG}_\alpha\xrightarrow{\mathit{Haus}_d}\mathit{SG}
$$

 $(\mathcal{C}(SG_{\alpha}),\mathcal{H}_{SG_{\alpha}},D_{SG_{\alpha}})\xrightarrow{Spectral Propinquity}(\mathcal{C}(SG),\mathcal{H}_{SG},D_{SG})$

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Future Work

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Exploration Fractals
Laplacians on Fractals
Fractality Definition and Study of a "Fractal Manifold"

Classification of C^* -algebras on Fractals

Approximation of Laplacians on Fractals

Noncommutative Fractality

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References

A. Connes

Noncommutative Geometry Academic Press, San Diego, 1994.

晶

T. Landry, M.L. Lapidus, F. Latrémolière,

e Geometry

San Diego, 1994.

Lapidus, F. Latrémolière,

nations of Spectral Triples on the Sierpiński Ga

al Curves,

21.

nd J. Sarhad,

and Geodesic Metric on the Harmonic Sierpins

r Fractal Sets,

5eo. 8 (2014), 947-9 Metric Approximations of Spectral Triples on the Sierpiński Gasket and other Fractal Curves,

arXiv: 2010.06921.

M. L. Lapidus and J. Sarhad,

Dirac Operators and Geodesic Metric on the Harmonic Sierpinski Gasket and other Fractal Sets,

J. of Noncom. Geo. 8 (2014), 947-985, arXiv: 1212.0878.

F. Latrémolière

The Dual Gromov-Hausdorff Propinquity Journal de Mathematiques Pure et Appliquees 103 (2015) 2, pp.303-351 arXiv: 1311.0104. [Towards Analysis](#page-0-0) on Fractals: Piecewise $C^{\mathbf{1}}$ -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff **Propinquity**

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References

F. Latrémolière

The Gromov-Hausdorff Propinquity for Metric Spectral Triples arXiv:1811.10843v1.

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M. Rieffel

Compact Quantum Metric Spaces

usdorff Propinquity for Metric Spectral Triples
3v1.
um Metric Spaces
as, Quantization, and Noncommutative Geome
.
315-330
8207.
ss from Actions of Compact Groups,
thematica 3 (1998), 215-229,
7084. Operator Algebras, Quantization, and Noncommutative Geometry, Contemp. Math. 365 (2004), pp. 315-330 arXiv:math/0308207.

M.A. Rieffel,

Metrics on States from Actions of Compact Groups,

Documenta Mathematica 3 (1998), 215-229, arXiv: math/9807084.

[Towards Analysis](#page-0-0) on Fractals: Piecewise $C^{\mathbf{1}}$ -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-Hausdorff **Propinquity**

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Spectral Propinquity Metric Convergence

$$
\Lambda^{\text{spec}}((C(SG), H_{SG}, D_{SG}), (C(SG_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}}))
$$

 $\begin{aligned} &\epsilon, D_{SG}), (C(SG_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}}))\ &\epsilon, D_{SG}, H_{SG}, D_{SG}),\\ &\text{mvb}(C(SG_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}}))\ &\epsilon, \\ &\epsilon, D_{SG}, D_{SG}, D_{SG}, D_{SG}),\\ &\text{uqvb}(C(SG_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}}))\} \ &\epsilon, D_{SG}, D_{SG}, C, 0, C(SG), L_{D_{SG}}),\\ &\text{vN}_{SG_{B(n)}}, C, 0, C(SG_{B(n)}), L_{DG_{SG,n}})),\\ &\epsilon, D_{SG$ $=$ max $\{\Lambda^{*met}(m\nu b(C(SG), H_{SG}, D_{SG}),\}$ $mvb(C(SG_{B(n)}), H_{SG_{B(n)}}, D_{SG_{B(n)}})),$ $\wedge^{*{{\mathrm{mod}}},{{\mathrm{cov}}}}(\textit{uqvb}(\textit{C}(\textit{SG}),\textit{H}_{\textit{SG}},\textit{D}_{\textit{SG}}),$ uqvb $(C(\mathsf{SG}_{B(n)}),\mathsf{H}_{\mathsf{SG}_{B(n)}},\mathsf{D}_{\mathsf{SG}_{B(n)}}))\}$ $\epsilon = \max\Big\{ \Lambda^{*}\mathsf{met}((H_{SG},\mathit{DN}_{SG},\mathbb{C},\mathsf{0},\mathcal{C}(SG),L_{D_{SG}}),\Big\}$ $(H_{SG_{B(n)}}, DN_{SG_{B(n)}}, \mathbb{C}, 0, C(SG_{B(n)}), L_{D_{SG_{B(n)}}})),$ $\wedge^{*{{\text{mod}}},{{\text{cov}}}}(\textit{uqvb}(\textit{C}(\textit{SG}),\textit{H}_{\textit{SG}},\textit{D}_{\textit{SG}}),$ uqvb $(C(\mathsf{SG}_{B(n)}),\mathsf{H}_{\mathsf{SG}_{B(n)}},\mathsf{D}_{\mathsf{SG}_{B(n)}}))\}$ 299 [Towards Analysis](#page-0-0) on Fractals: Piecewise $C^{\mathbf{1}}$ -Fractal Curves, Lapidus-Sarhad Spectral Triples, and the Gromov-**Hausdorff Propinquity**

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