Modulus On Orthodiagonal Maps

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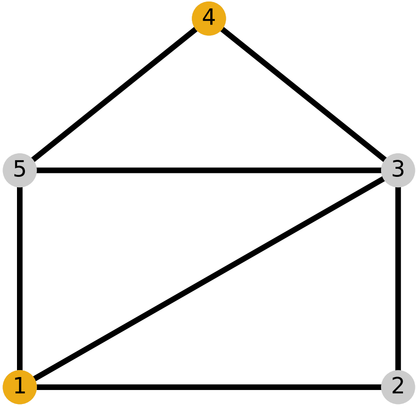
Connecting Families of Paths

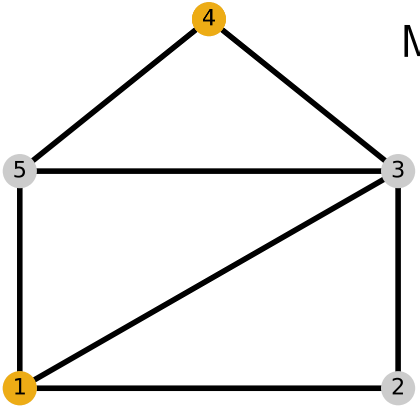


$$\Gamma := \Gamma_{\mathrm{path}}(a, b) \subset \mathbb{R}^{E}_{\geq 0}$$

$$\gamma \in \Gamma \leftrightsquigarrow \mathcal{N}(\gamma, \cdot) \in \{0, 1\}^{E}$$

"Edge Usage"

Modulus



$$\mathrm{Mod}_{2,\sigma}\,\Gamma := \min_{\rho\in \mathrm{Adm}\,\Gamma} \sum_{e\in E} \sigma(e)\rho(e)^2$$

$\mathrm{Adm}\,\Gamma$ is set of $\rho \in \mathbb{R}^E_{\geq 0}$:

$$\sum_{e\in E} \mathcal{N}(\gamma, e)\rho(e) \geq 1, \forall\gamma$$

"Everyone pays 1 dollar"

$$\mathrm{Mod}_{2,\sigma}\,\Gamma = \mathrm{eff}\,\mathcal{C}_\sigma(a, b)$$

Fulkerson Duality For Modulus

The **Fulkerson dual** family is $\hat{\Gamma} := \mathrm{ext}(\mathrm{Adm}(\Gamma)) \subset \mathbb{R}^{E}_{\geq 0}$

Theorem (2017, with Albin et al.)

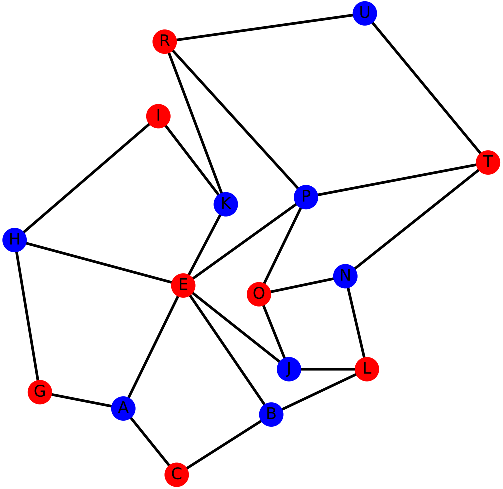
We have

$$\mathrm{Mod}_{2,\sigma}(\Gamma) \cdot \mathrm{Mod}_{2,\sigma^{-1}}(\hat{\Gamma}) = 1$$

and

$$\hat{\Gamma}_{\mathrm{path}}(a, b) = \Gamma_{\mathrm{cut}}(a, b)$$

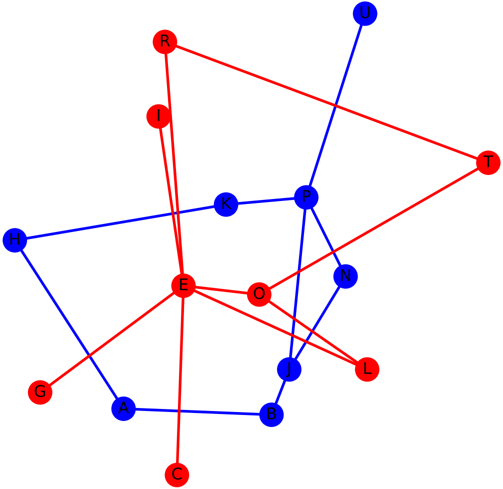
Orthodiagonal maps



Simply connected domain tiled with quadrilaterals with **orthogonal diagonals**

Get a bipartite graph

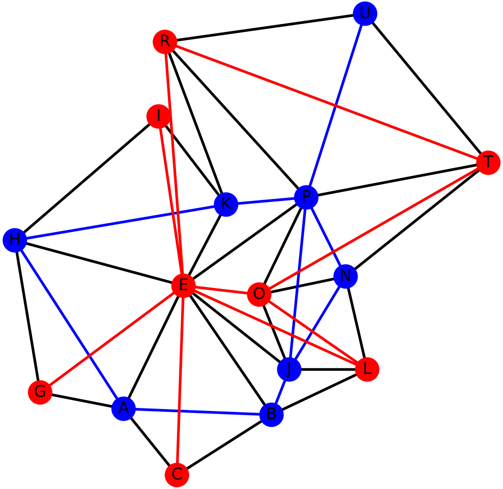
Orthodiagonal maps



Primal graph

Dual graph

Orthodiagonal maps

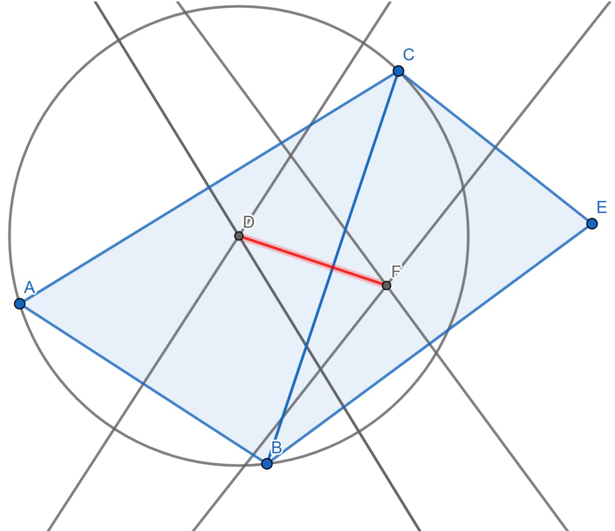


Quadrangulation

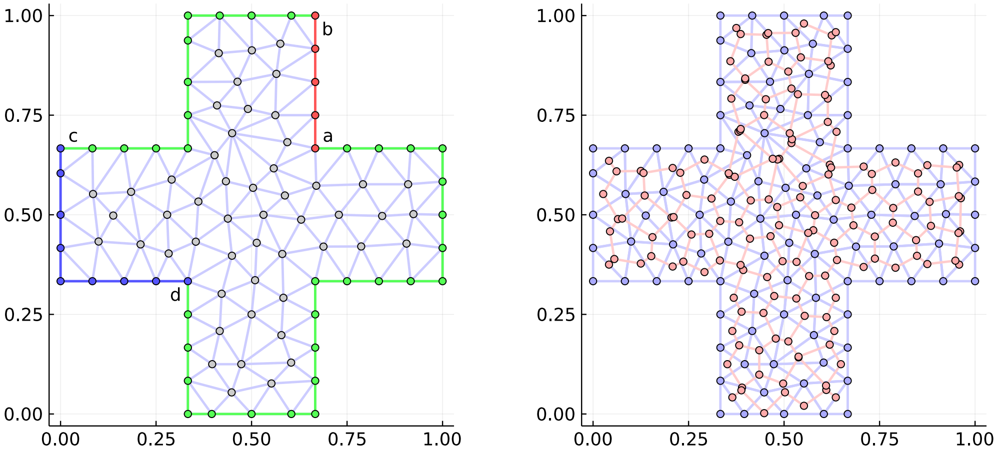
Primal graph

Dual graph

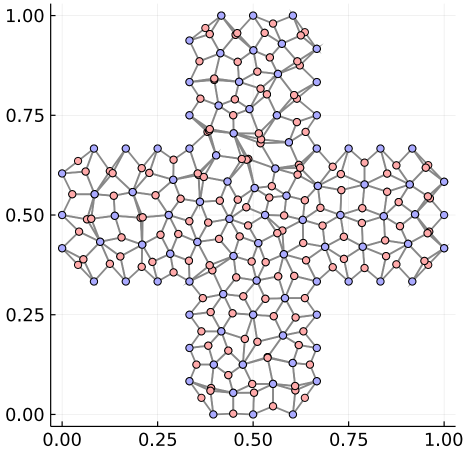
Delaunay triangulations in numerical analysis



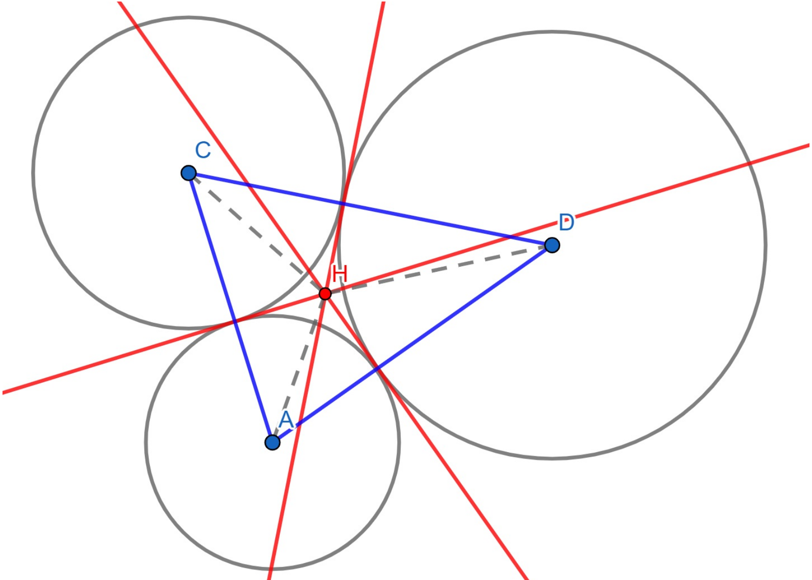
Delaunay triangulation and its dual Voronoi graph



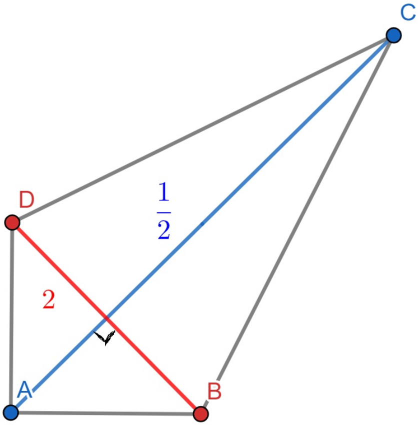
Quadrilaterals with orthogonal diagonals



Circle packings



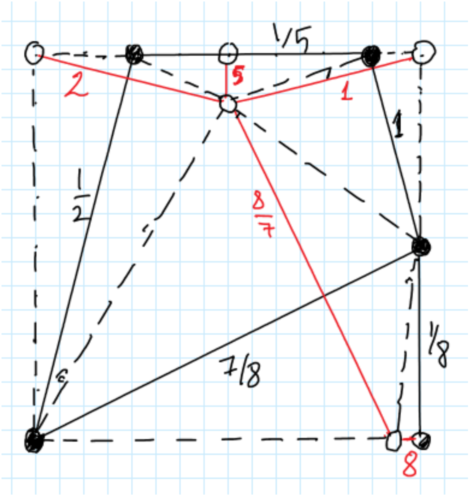
Canonical weights



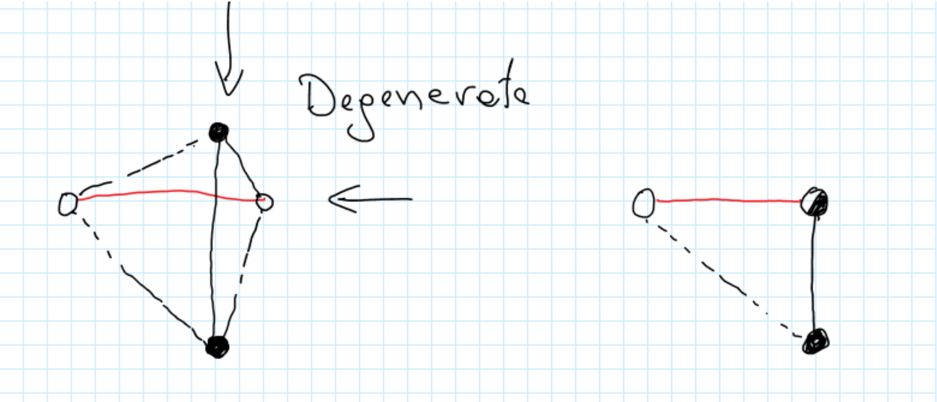
$$\sigma(A, C) = \frac{|BD|}{|AC|}$$

$$\sigma(B, D) = \frac{|AC|}{|BD|}$$

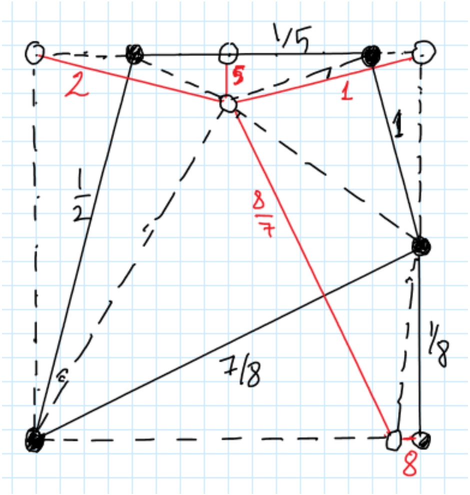
Orthodiagonal Packings of the Square



Degeneration



Orthodiagonal Packings of the Square



$$\mathrm{Mod}\,\Gamma^{\mathrm{b}}_{\mathrm{LR}} = \frac{7}{8} + \frac{1}{2+5+1}$$

$$\mathrm{Mod}\,\Gamma^{\mathrm{r}}_{\mathrm{LR}} = \frac{1}{\frac{1}{2} + \frac{1}{1+\frac{1}{\frac{7}{8}+\frac{1}{8}}}}$$

$$\mathrm{Mod}\,\Gamma^{\mathrm{b}}_{\mathrm{BT}} = \frac{1}{2} + \frac{1}{1+\frac{1}{\frac{7}{8}+\frac{1}{8}}}$$

$$\mathrm{Mod}\,\Gamma^{\mathrm{r}}_{\mathrm{BT}} = \frac{1}{\frac{7}{8} + \frac{1}{1+2+5}}$$

Modulus is Always One

Theorem

Tile a square with an orthodiagonal map (appropriately weighted). Then, all four modulus problems, left-right or bottom-top, either on the primal or dual graph, are equal to one.

The Dirichlet Problem

Assume $Q = [0,1] \times [0,1] \subset \mathbb{C}$. Recall that $h(z) := \mathrm{Re}(z)$ and $\tilde{h}(z) := \mathrm{Im}(z)$ solve the Dirichlet problems

$$\begin{cases} (\Delta h)(z) = 0 & z \in \mathrm{int}(Q) \\ h(z) = 0 & \mathrm{Re}(z) = 0 \\ h(z) = 1 & \mathrm{Im}(z) = 1 \end{cases} \quad \begin{cases} (\Delta \tilde{h})(z) = 0 & z \in \mathrm{int}(Q) \\ \tilde{h}(z) = 0 & \mathrm{Im}(z) = 0 \\ \tilde{h}(z) = 1 & \mathrm{Re}(z) = 1 \end{cases}$$

Do same on the primal and dual graphs using the weighted discrete Laplacian and get:

$$h^\bullet, h^\circ, \tilde{h}^\bullet, \tilde{h}^\circ$$

The Discrete Laplacian

The weighted discrete Laplacian is

$$(L_\sigma u)(x) := \sum_{y \sim x} \sigma(x, y)\,(u(x) - u(y))$$

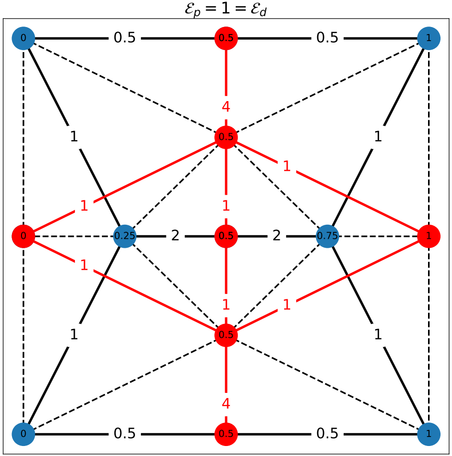
Energy (or quadratic form):

$$\mathcal{E}^\bullet(h^\bullet) := \sum_{\{x,y\} \in E} \sigma(x, y)\,(h^\bullet(x) - h^\bullet(y))^2 = \mathrm{Mod}_{2,\sigma}\,\Gamma^\bullet_{\mathrm{LR}}$$

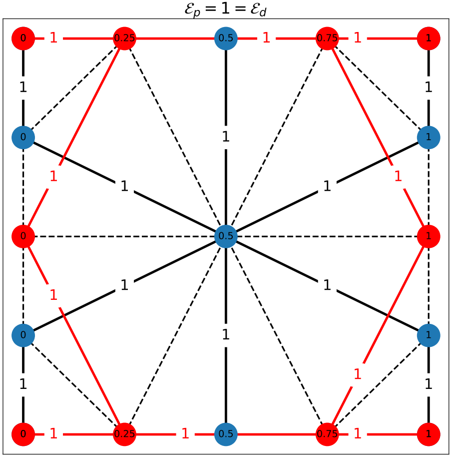
And similar formulas hold for

$\mathrm{Mod}_{2,\sigma}\,\Gamma^\circ_{\mathrm{LR}}$, $\mathrm{Mod}_{2,\sigma}\,\Gamma^\bullet_{\mathrm{BT}}$, $\mathrm{Mod}_{2,\sigma}\,\Gamma^\circ_{\mathrm{BT}}$.

Examples



$\mathcal{E}_p = 1 = \mathcal{E}_d$
0
0.5
0.5
0.5
1
4
1
1
0.5
1
1
1
0
0.25
2
0.5
2
0.75
1
1
1
1
1
1
0.5
4
0
0.5
0.5
0.5
1



$\mathcal{E}_p = 1 = \mathcal{E}_d$
0
1
0.25
0.5
1
0.75
1
1
1
1
0
1
1
1
1
1
1
0
0.5
1
1
1
1
1
0
1
1
1
1
1
1
0
1
0.25
1
0.5
0.75
1
1

Harmonicity of $\operatorname{Re} z$ and $\operatorname{Im} z$

Suppose $z \in V^\bullet \setminus \partial G$. Then, $z \in q^\circ$ face of the dual graph G°. Let $w_0, w_1, \cdots, w_k$ be of q° with $w_0 = w_k$. Consider faces $q_j = (z, w_{j-1}, z_j, w_j)$ with $j = 1, \ldots, k$. Then,

$$w_j - w_{j-1} = i(z_j - z)\sigma(z, z_j).$$

So,

$$0 = \sum_{j=1}^{k}(w_j - w_{j-1}) = i\sum_{j=1}^{k}\sigma(z, z_j)(z_j - z).$$

Now take real and imaginary parts.

The Main Identity

Set $Z = z' - z$ and $W = w' - w$, so that $W = i\sigma(z, z')Z$,

$$0 = \sum_q \sigma(W)\,|W - i\sigma(Z)Z|^2$$

$$= \mathcal{E}^\circ(h^\circ) + \mathcal{E}^\circ(\tilde{h}^\circ) + \mathcal{E}^\bullet(h^\bullet) + \mathcal{E}^\bullet(\tilde{h}^\bullet) - 2\sum_q \mathrm{Re}(iZ\overline{W})$$

Note that

$$\mathrm{Re}(iZ\overline{W}) = |Z||W| = 2\,\mathrm{Area}\,q$$

$$\Longrightarrow \mathcal{E}^\circ(h^\circ) + \mathcal{E}^\circ(\tilde{h}^\circ) + \mathcal{E}^\bullet(h^\bullet) + \mathcal{E}^\bullet(\tilde{h}^\bullet) = 4$$

Fulkerson duality

$$\mathcal{E}^\bullet(h^\bullet)\mathcal{E}^\circ(\tilde{h}^\circ) = 1 \qquad \text{and} \qquad \mathcal{E}^\circ(h^\circ)\mathcal{E}^\bullet(\tilde{h}^\bullet) = 1.$$

Therefore,

$$\left(\mathcal{E}^\bullet(h^\bullet) + \frac{1}{\mathcal{E}^\bullet(h^\bullet)}\right) + \left(\mathcal{E}^\circ(h^\circ) + \frac{1}{\mathcal{E}^\circ(h^\circ)}\right) = 4.$$

But this implies that

$$\mathcal{E}^\bullet(h^\bullet) = \mathcal{E}^\circ(h^\circ) = 1,$$

because $x + \frac{1}{x} \geq 2$ for all $x \geq 0$, with equality iff $x = 1$.

Conclusion

We have shown

$$\mathcal{E}^\bullet(h^\bullet) = \mathcal{E}^\circ(h^\circ) = 1,$$

and by Fulkerson duality

$$\mathcal{E}^\bullet(\tilde{h}^\bullet) = \mathcal{E}^\circ(\tilde{h}^\circ) = 1.$$

The claim follows.

Modulus Convergence

Corollary

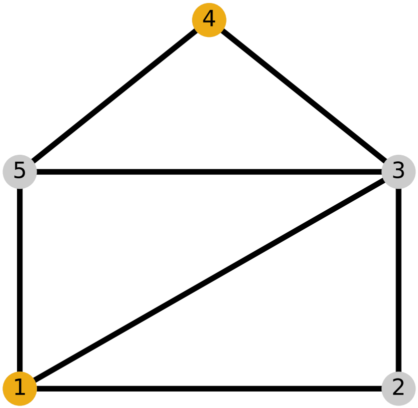
Let G_n be an orthodiagonal tiling of the square Q with mesh size $1/n$. Then $\operatorname{Mod}_2 \Gamma_n \longrightarrow \operatorname{Mod}_2 \Gamma \qquad \text{as } n \to \infty.$

Theorem

Let G_n be an orthodiagonal tiling with mesh size $1/n$ of the domain $\hat{G}_n$ so that $(\hat{G}_n, \alpha_n, \beta_n)$ converges to (Ω, A, B) in Hausdorff distance. Then

$$\operatorname{Mod}_2 \Gamma_n \longrightarrow \operatorname{Mod}_2 \Gamma \qquad \text{as } n \to \infty.$$

Probabilistic Interpretation



$$\left(\mathrm{Mod}\,\Gamma\right)^{-1} = \min_{\mu\in\mathcal{P}(\Gamma)} \mathbb{E}_\mu \left|\underline{\gamma} \cap \underline{\gamma'}\right|$$

$\mu \in \mathbb{R}^{\Gamma}_{\geq 0}$ is a pmf

Optimal pmf μ^* not unique

Edge prob $\eta^*(e) := \mathbb{P}_{\mu^*}\left(e \in \underline{\gamma}\right)$ are unique

Problem

Do the optimal pmf's for modulus converge as the mesh goes to zero?

Non-crossing Minimal Subfamily Algorithm

Let G be a finite plane graph, and A and B be boundary arcs. Then $\Gamma_G(A,B)$ has a unique minimal subfamily $\tilde{\Gamma}$ consisting of non-crossing paths that supports an optimal pmf μ^*.

$\tilde{\Gamma}$ is a *minimal subfamily* of Γ if $\operatorname{Mod}\tilde{\Gamma} = \operatorname{Mod}\Gamma$ and

$$\operatorname{Mod}\tilde{\Gamma} \setminus \{\gamma\} < \operatorname{Mod}\Gamma, \qquad \forall \gamma \in \tilde{\Gamma}$$

Main convergence result

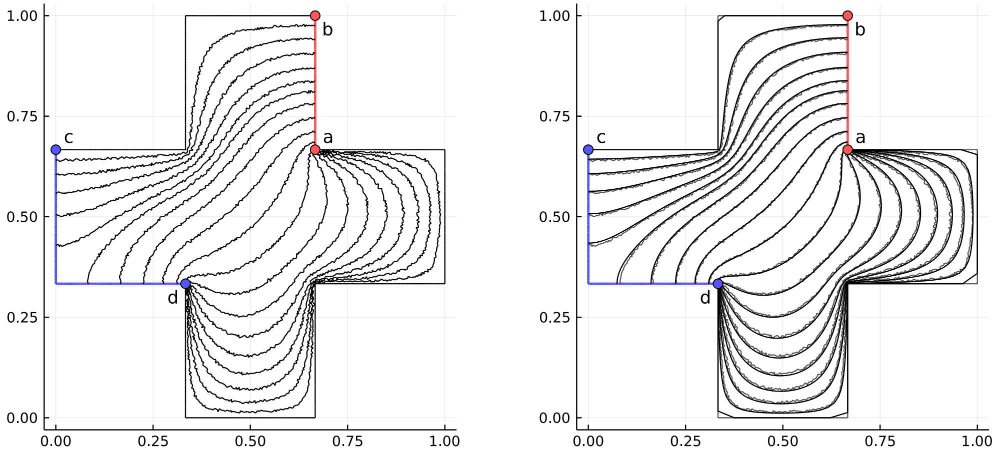
Theorem

Let (Ω, A, B) be a simply connected domain with bdy arcs.

(G_n, A_n, B_n) be approximating orthodiagonal maps

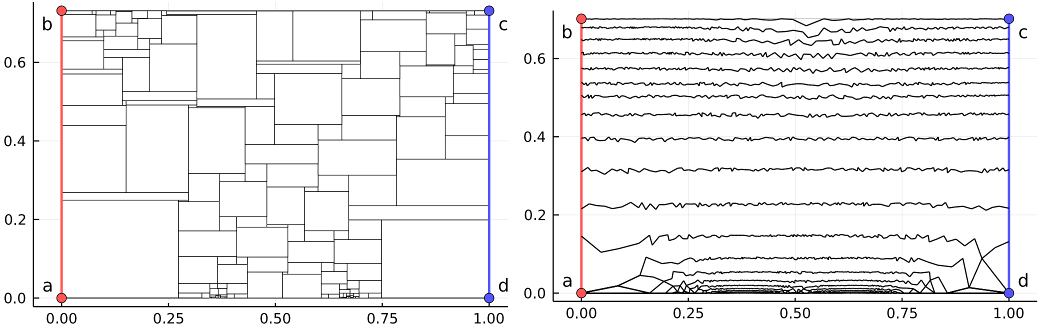
Then the optimal pmfs μ_n output by the algorithm converge to the "transverse" measure on the horizontal trajectories for Ω.

Non-crossing curves



Non-crossing curves vs. level lines of conformal map

Convergence of the measures



rectangle packing and its induced coordinates

Theorem (Gurel-Gurevich, Jerison, Nachmias)

Ω is a bounded simply connected domain and $g \in C^2(\mathbb{R}^2)$. Given $0 < \epsilon, \delta <$ Diam(Ω), let G be orthodiagonal map with max edge length ϵ and Hausdorff dist at most δ between $\partial\Omega$ and ∂G. Let $h_c \in C(\overline{\Omega})$ be the solution to the continuous Dirichlet problem with data g. Let $h_d \in \mathbb{R}^{V^\bullet}$ be the solution to the discrete Dirichlet problem with data $g\,|_{\partial V^\bullet}$. Then

$$|h_d(x) - h_c(x)| \leq \frac{C \operatorname{Diam}(\Omega)(C_1 + C_2\epsilon)}{\log^{1/2}(\operatorname{Diam}(\Omega)/(\delta \wedge \epsilon))}$$

Two flows on the primal graph from left to right

Idea from Gurel-Gurevich, Jerison, Nachmias *The Dirichlet problem for orthodiagonal maps*. Let $q = [z, w, z', w']$

$$\begin{aligned} f(z, z') &:= (\sigma dh)(z, z') = \sigma(z, z')\,(h(z') - h(z)) \\ \tilde{f}(z, z') &:= \tilde{h}(w') - \tilde{h}(w) \end{aligned}$$

Then if $f^{\bullet} := \sigma dh^{\bullet}$ is current flow, we have

$$f - f^{\bullet} \perp f^{\bullet} - \tilde{f}$$

in the inner product $\langle f, g \rangle_r = \sum_e r(e) f(e) g(e)$, $(r = \sigma^{-1})$

Orthogonality

Lemma

If $Div_g(x) = 0$ for $x \in V \setminus (A \cup B)$, and $h = \sigma d\phi$ with ϕ constant on A and B, then

$$\langle g, h \rangle_r = \text{Val}(g)(\phi(B) - \phi(A))$$

Let $h = f - f^{\bullet} = \sigma d(h - h^{\bullet}) = \sigma d\phi$. Then $\phi(A) = \phi(B) = 0$.
Need to check that if $g = f^{\bullet} - \tilde{f}$, is divergence-free for $x \in V \setminus (A \cup B)$. Enough To Show for $\tilde{f}$.

Pythagorean Theorem

$$\mathcal{E}^\bullet(f - f^\bullet) + \mathcal{E}^\bullet(f^\bullet - \tilde{f}) = \mathcal{E}^\bullet(f - \tilde{f})$$

For instance in the case of the square.

$$\begin{aligned} f(z, z') - \tilde{f}(w, w') &= \sigma(z, z')\,(h(z') - h(z)) - (\tilde{h}(w') - \tilde{h}(w)) \\ &= \mathrm{Re}[\sigma(z, z')(z' - z)] - \mathrm{Im}(w' - w) = 0 \end{aligned}$$

Since $w' - w = i\sigma(z', z)(z' - z)$ and $\mathrm{Im}\, iZ = \mathrm{Re}\, Z$.

References

Albin, N., Lind, J., Poggi-Corradini, P.
Convergence of the Probabilistic Interpretation of Modulus.
https://arxiv.org/pdf/2106.11418.pdf.