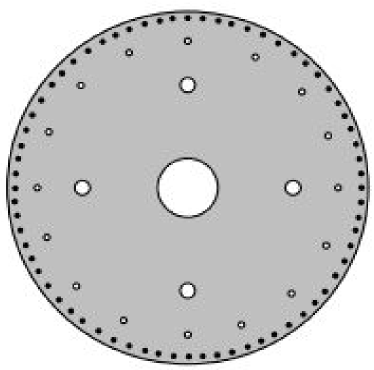
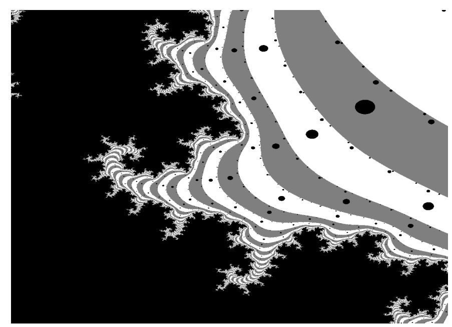
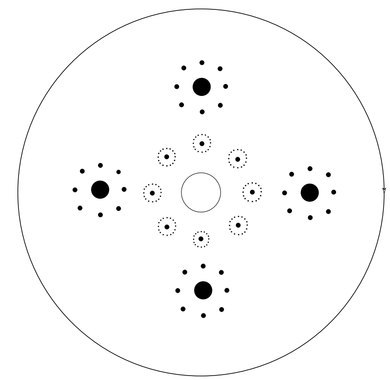
Topology and Geometry of Multiply Connected Wandering Domains

Jack Burkart - University of Wisconsin-Madison







Question: How can we construct examples of non-polynomial entire functions that have Fatou components and Julia set as illustrated?

Joint Work with Kirill Lazebnik

Part 1: Multiply Connected Wandering Domains in Transcendental Dynamics

We outline an argument due to Baker to construct a transcendental entire function (t.e.f) with a wandering domain that is multiply connected.

Choose some large value $C > 1$ and $R_0 > 1$. Define

$$f_0(z) = Cz^2.$$

Define

$$R_1 = \max_{|z|=R_0} |f_0(z)|$$

and

$$f_1(z) = Cz^2 \left(1 + \frac{z}{R_1}\right)$$

Define $R_1 = \max_{|z|=R_0} |f_0(z)|$ and

$$f_1(z) = Cz^2 \left(1 + \frac{z}{R_1}\right)$$

Define $R_2 = \max_{|z|=R_1} |f_1(z)|$ and

$$f_2(z) = Cz^2 \left(1 + \frac{z}{R_1}\right) \left(1 + \frac{z}{R_2}\right)$$

Define $R_{k+1} = \max_{|z|=R_k} |f_k(z)|$ and

$$f_{k+1}(z) = f_k(z) \left(1 + \frac{z}{R_{k+1}}\right) = Cz^2 \prod_{j=1}^{k+1} \left(1 + \frac{z}{R_j}\right).$$

One can calculate directly that $R_n \geq R_{n-1}^2$, and

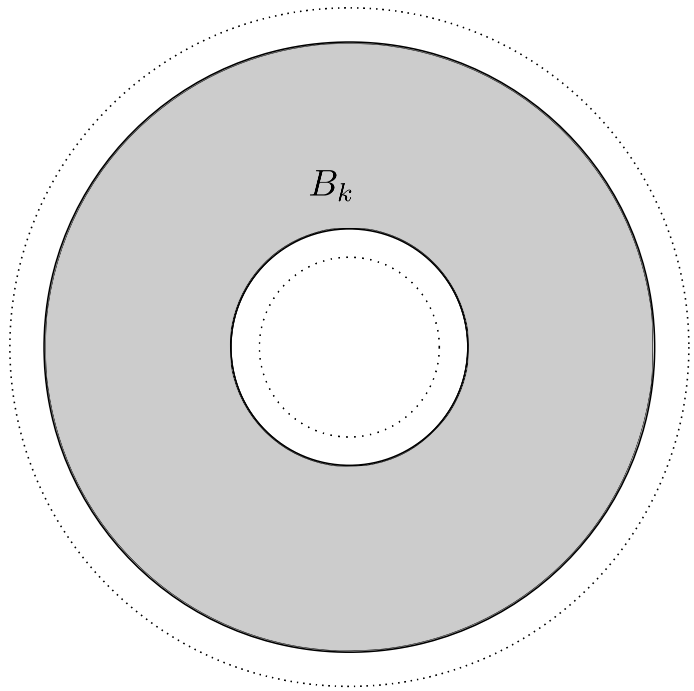
$$f(z) = Cz^2 \prod_{k=1}^{\infty} \left(1 + \frac{z}{R_k}\right).$$

defines an entire function.

Note: 0 is a superattracting fixed point for f, and therefore there is basin of attraction containing 0 in the Fatou set of f.

On $B_k = \{4R_k \le |z| \le \frac{1}{4}R_{k+1}\}$ we have

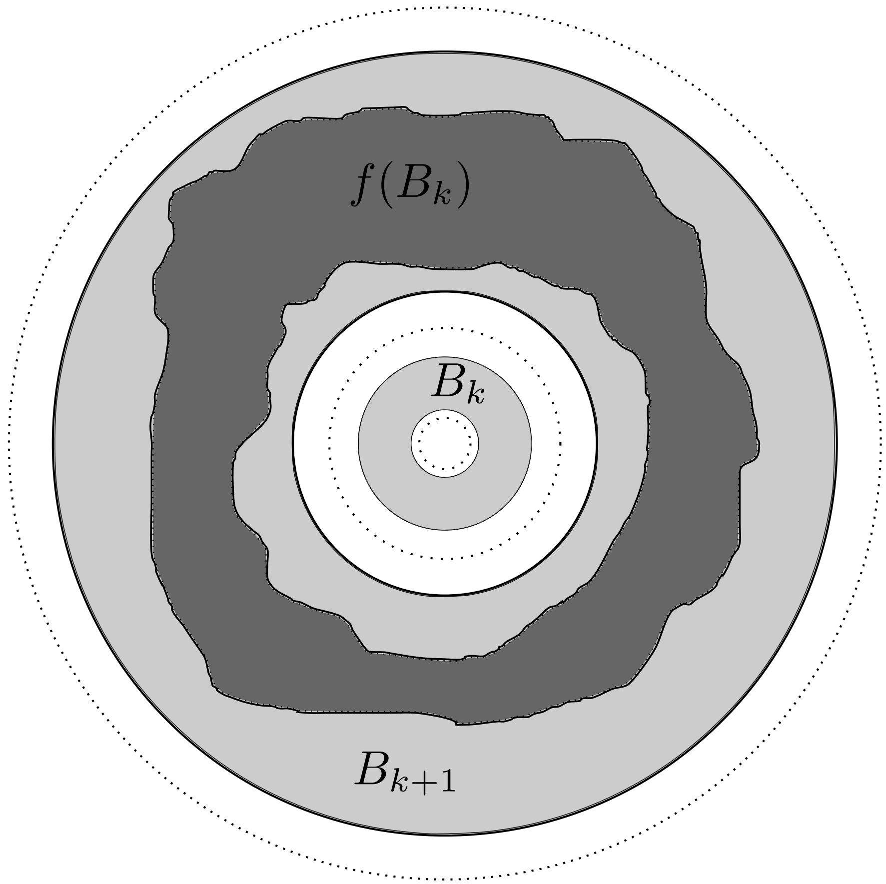
$$f(z) = C_k z^2 \prod_{k=1}^{\infty} \left(1 + \frac{z}{R_k}\right) \sim C_k z^{k+2}(1 + o(1)).$$



Note that

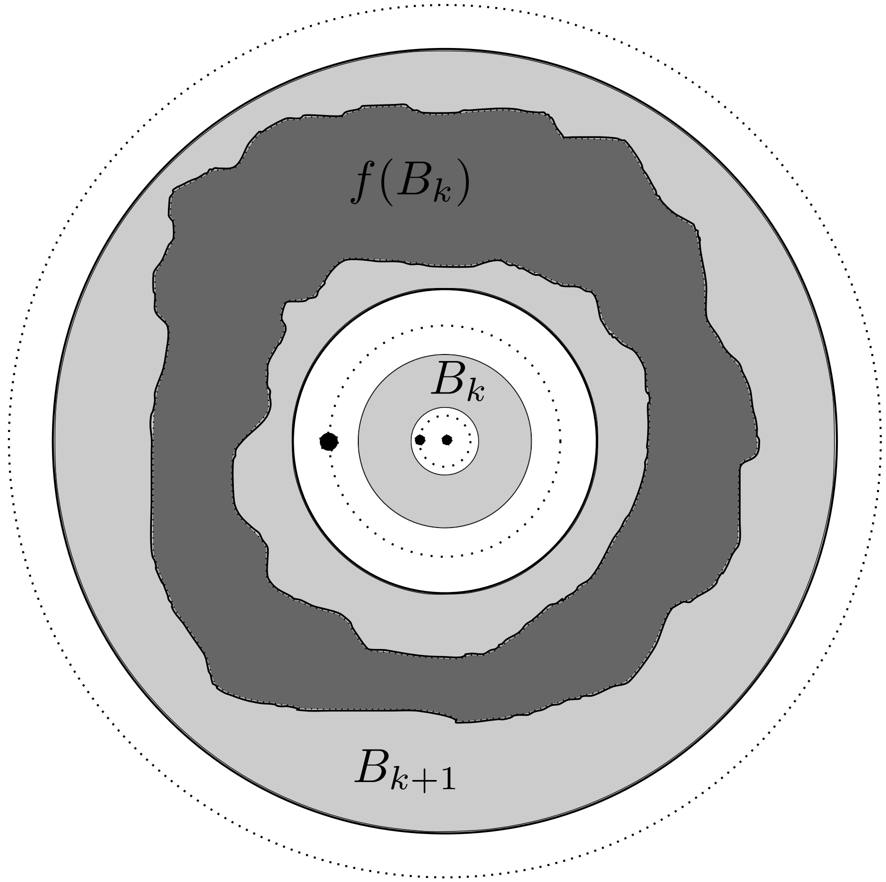
$$\frac{1}{16}\frac{R_{k+1}}{R_k} \to \infty.$$

Lemma: $f(B_k) \subset B_{k+1}$.



Thus each B_k belongs to some Fatou component U_k.

Each B_k surrounds a preimage of basin of attraction containing zero.

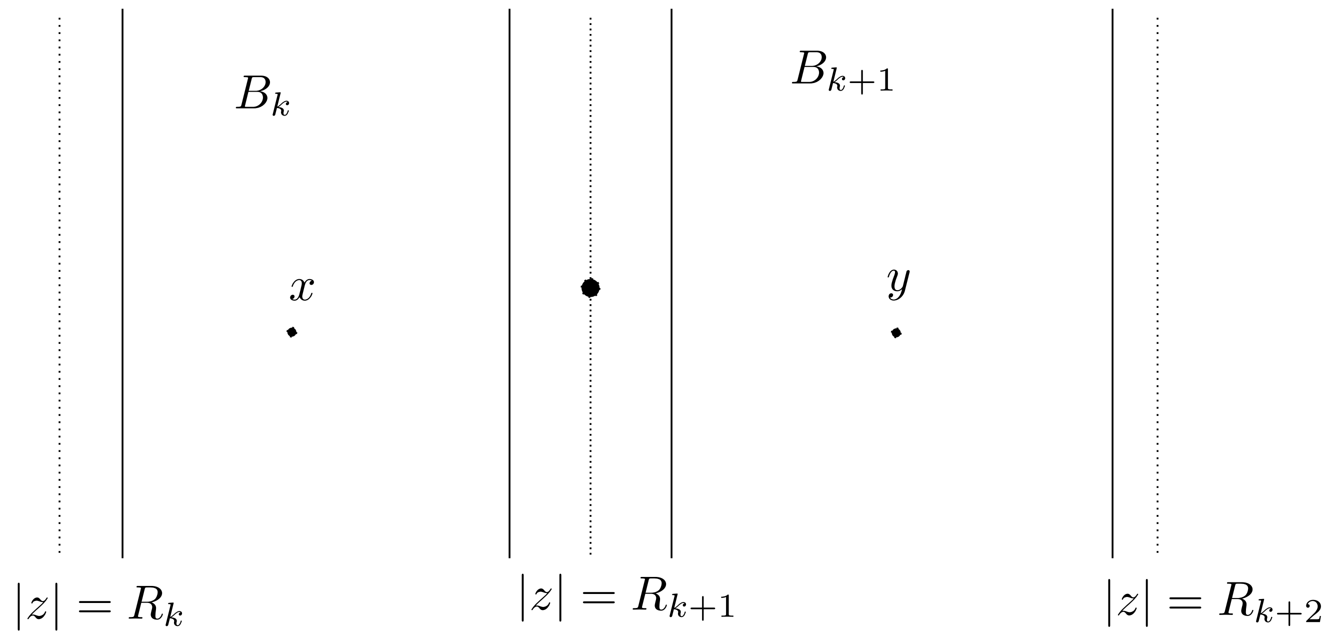


Thus the Fatou component U_k containing B_k is multiply connected

Iterates on B_k tend locally uniformly to ∞.

What's stopping $U_{k+1} = U_k$? Maybe this is an unbounded invariant Fatou component!

Lemma: $U_k \neq U_{k+1}$ - each B_k is in a distinct Fatou component.



Proof: Suppose $U_k = U_{k+1}$ - then $U_k = U_j$ for all $j \geq k$. Take $x \in B_k$ and $y \in B_{k+1}$. We require $d_{\text{hyp}}(f^n(x), f^n(y))$ to be non-increasing. However, because the modulus of B_k is increasing rapidly, we can directly estimate $d_{\text{hyp}}(f^n(x), f^n(y)) \to \infty$.

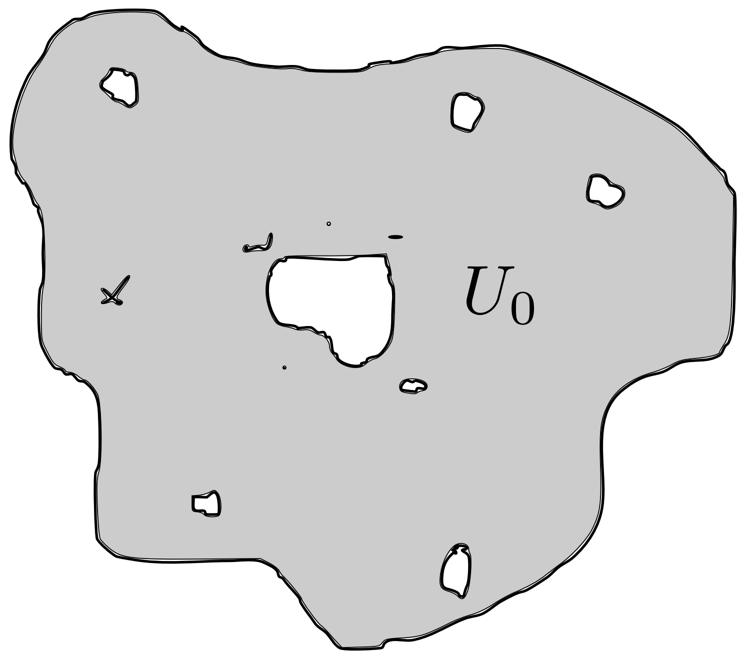
Corollary: Each B_k is in a distinct Fatou component U_k, and U_k is a wandering domain.

What can we say about multiply connected wandering domains in general?

It turns out many of the properties highlighted above are **general** for functions with a multiply connected Fatou component.

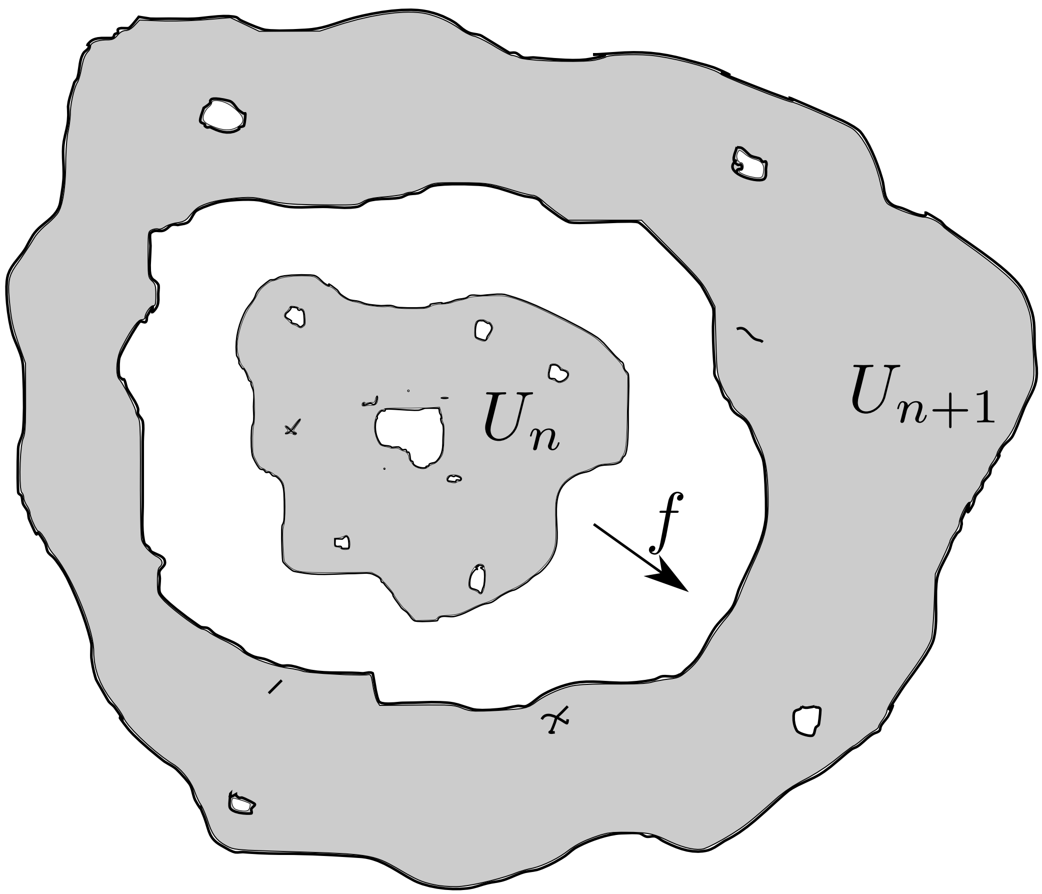
Part 2: General Properties of Multiply Connected Fatou Components.

Let U_0 be a multiply connected Fatou component and let $U_n = f^n(U_0)$.



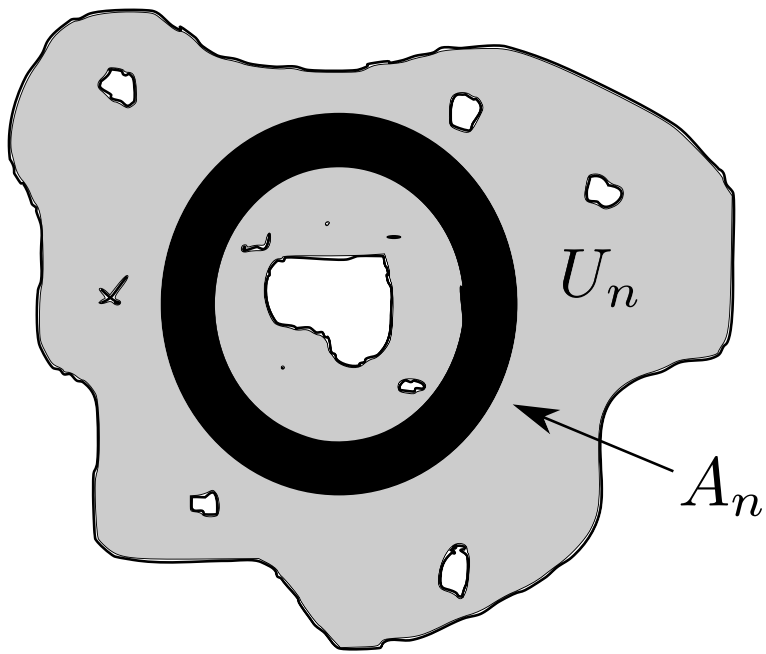
Fact 1 (Baker): For all n, U_n is a bounded wandering domain - $U_n \neq U_m$ for any $n \neq m$.

Let U_0 be a multiply connected Fatou component and let $U_n = f^n(U_0)$.



Fact 2 (Baker): For all sufficiently large n, U_n surrounds the origin, and U_{n+1} surrounds U_n.

Let U_0 be a multiply connected Fatou component and let $U_n = f^n(U_0)$.

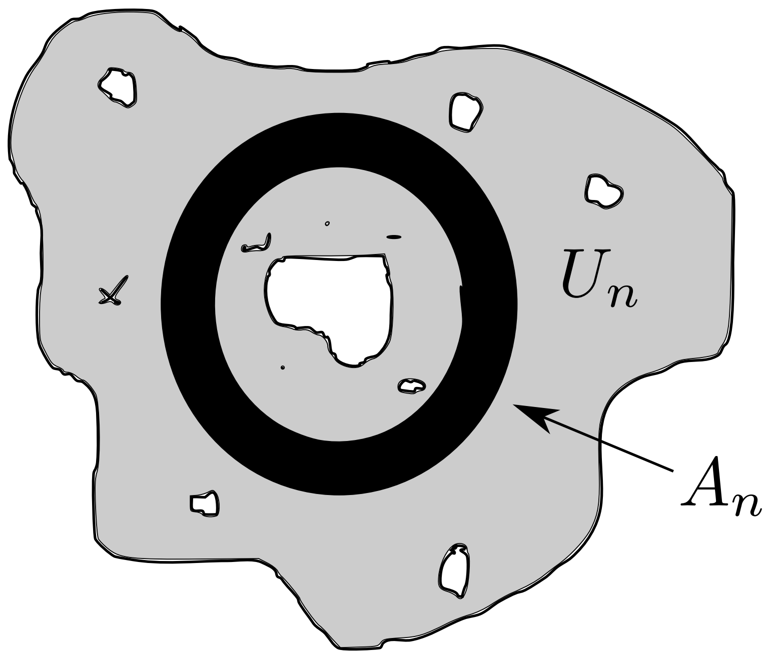


Fact 3 (Zheng): For all sufficiently large n, U_n contains an annulus $A_n = A(r_n, R_n)$ with

$$\lim_{n\to\infty} \frac{R_n}{r_n} = \infty.$$

We call A_n the **Fat Annulus** of U_n.

Let U_0 be a multiply connected Fatou component and let $U_n = f^n(U_0)$.

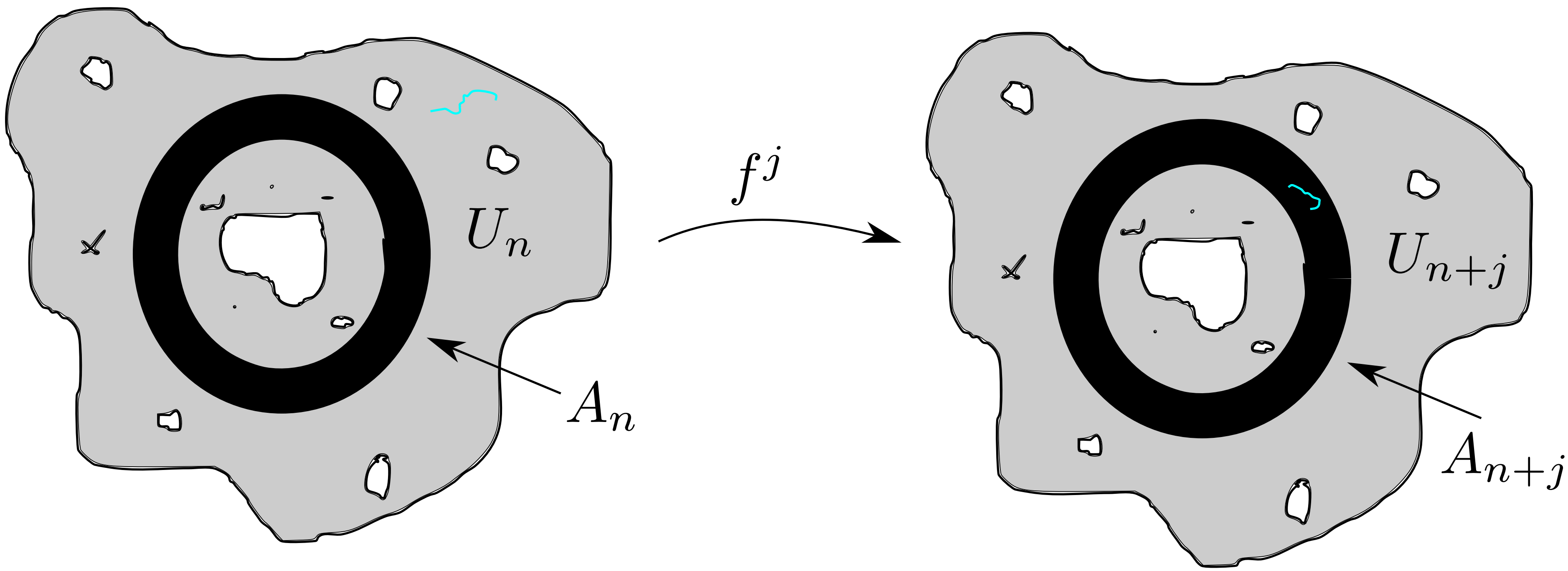


Fact 4 (Bergweiler/Rippon/Stallard): For all sufficiently large n, on the fat annulus A_n we have $f(A_n) \subset A_{n+1}$ and

$$f(z) = C_n \phi(z)^{m_n}$$

where $\phi : A_n \to \mathbb{C}$ is conformal and close to the identity.

Let U_0 be a multiply connected Fatou component and let $U_n = f^n(U_0)$.



Fact 5 (Bergweiler/Rippon/Stallard): In fact, A_n is an **absorbing annulus** - if $K \subset U_n$, then for all j large enough we have $f^j(K) \subset A_{j+n} \subset U_{j+n}$.

In blue, we have illustrated a compact $K \subset U_n$ on the left and its image under f^j to the right.

Let U_0 be a multiply connected Fatou component and let $U_n = f^n(U_0)$.

Fact 6 (Kisaka-Shishikura): The number of complimentary components of U_n is non-increasing and is either ∞ or converges to 2.

Fact 7 (Bergweiler/Rippon/Stallard):

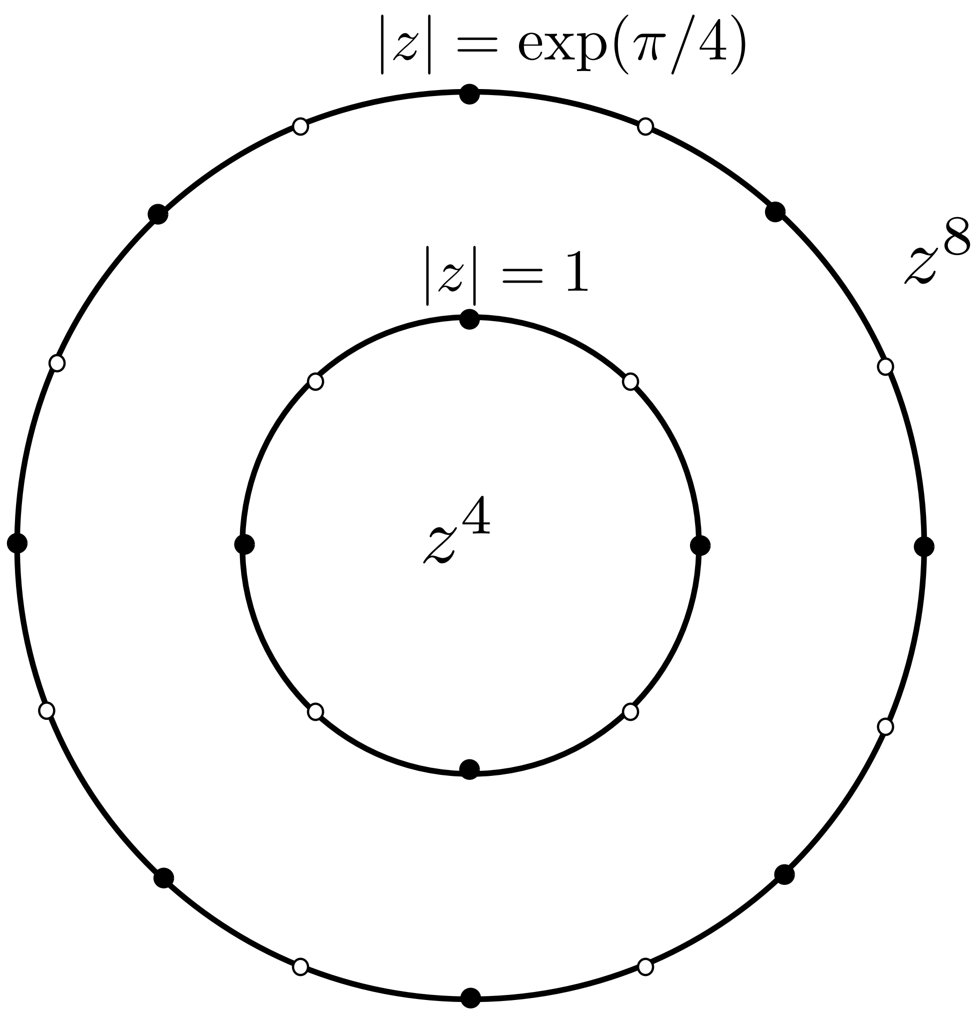
1. The connectivity of U_n is 2 iff $\cup_{m \geq n} U_m$ contains no critical points of f.
2. The connectivity of U_n is ∞ iff $\cup_{m \geq n} U_m$ contains infinitely many critical points of f.
3. The connectivity of U_n is finite iff $\cup_{m \geq n} U_m$ contains finitely many critical points of f.

Part 3: Interpolating Between Monomials

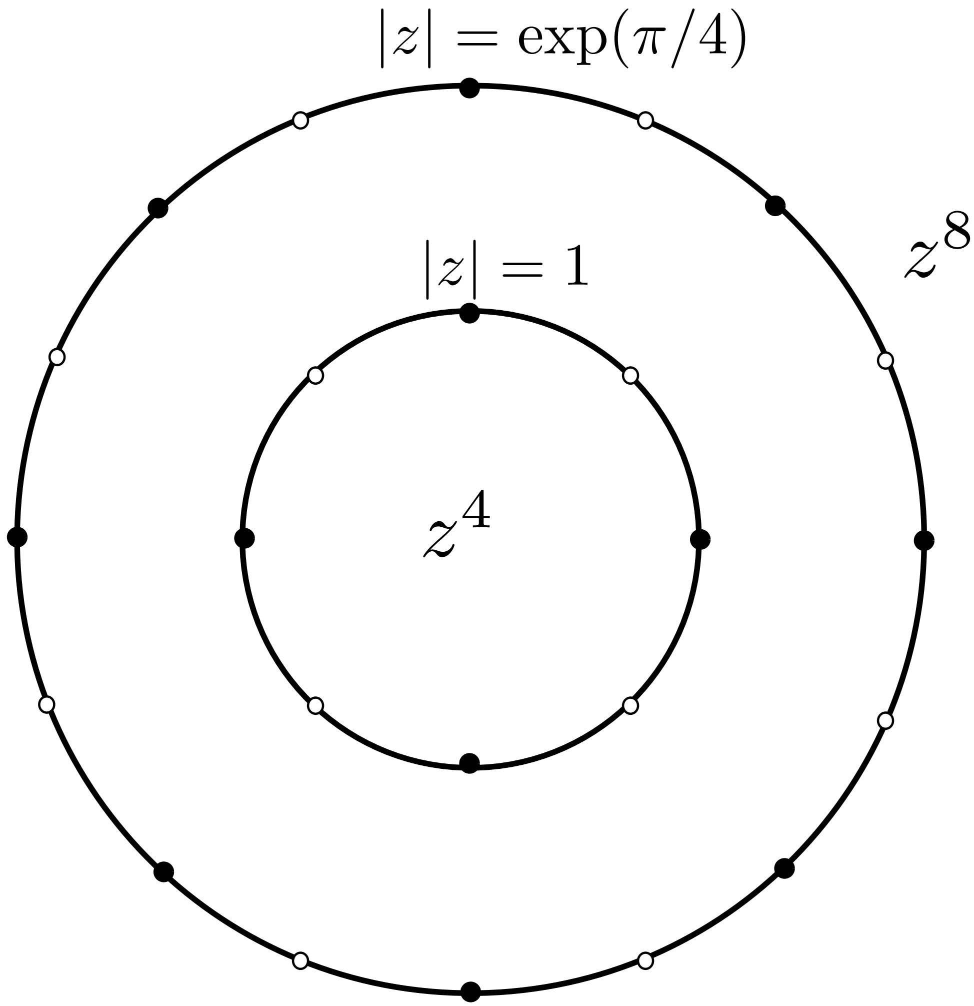
Starting with monomials on ‘fat’ annuli, can we work backwards and find an entire function close to those monomials on these annuli?

In other words, can we reverse engineer multiply connected wandering domains?

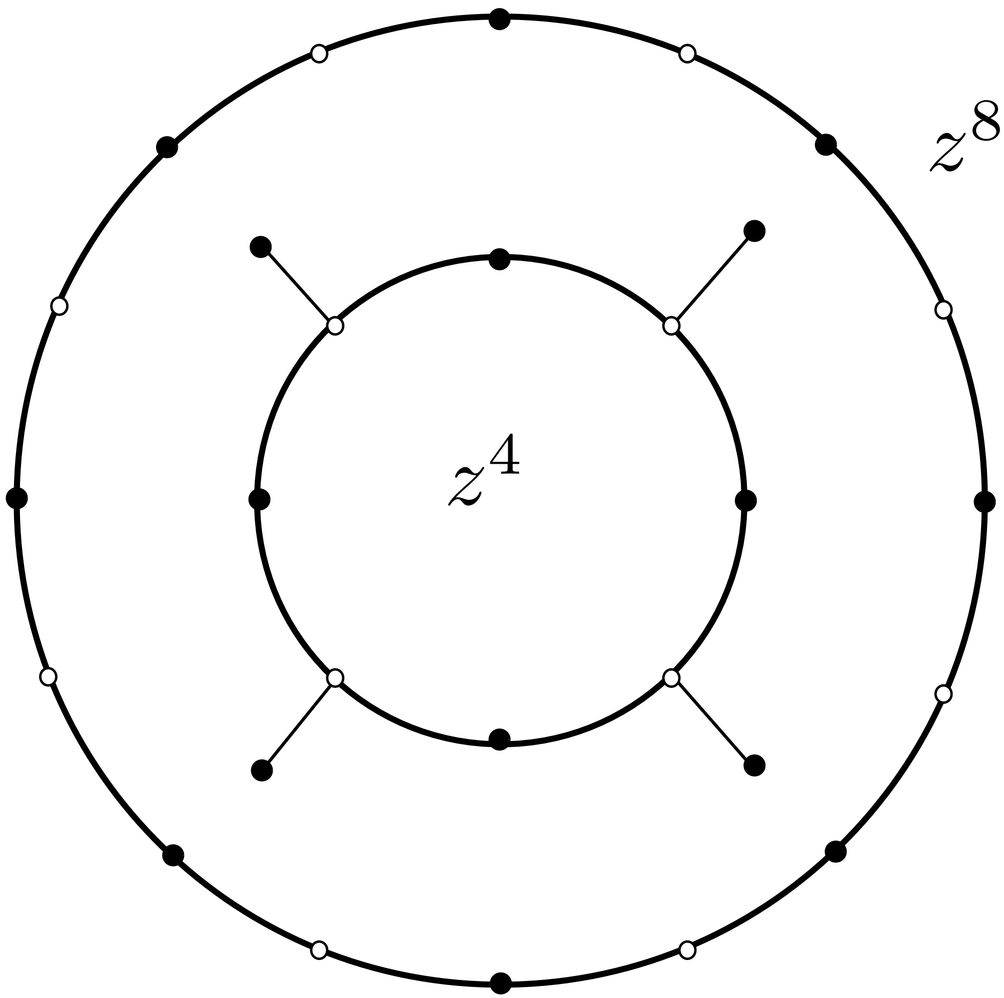
Toy Problem: Find a 'nice' function that is z^4 on $B(0,1)$ and z^8 on $\mathbb{C} \setminus B(0, \exp(\pi/4))$.



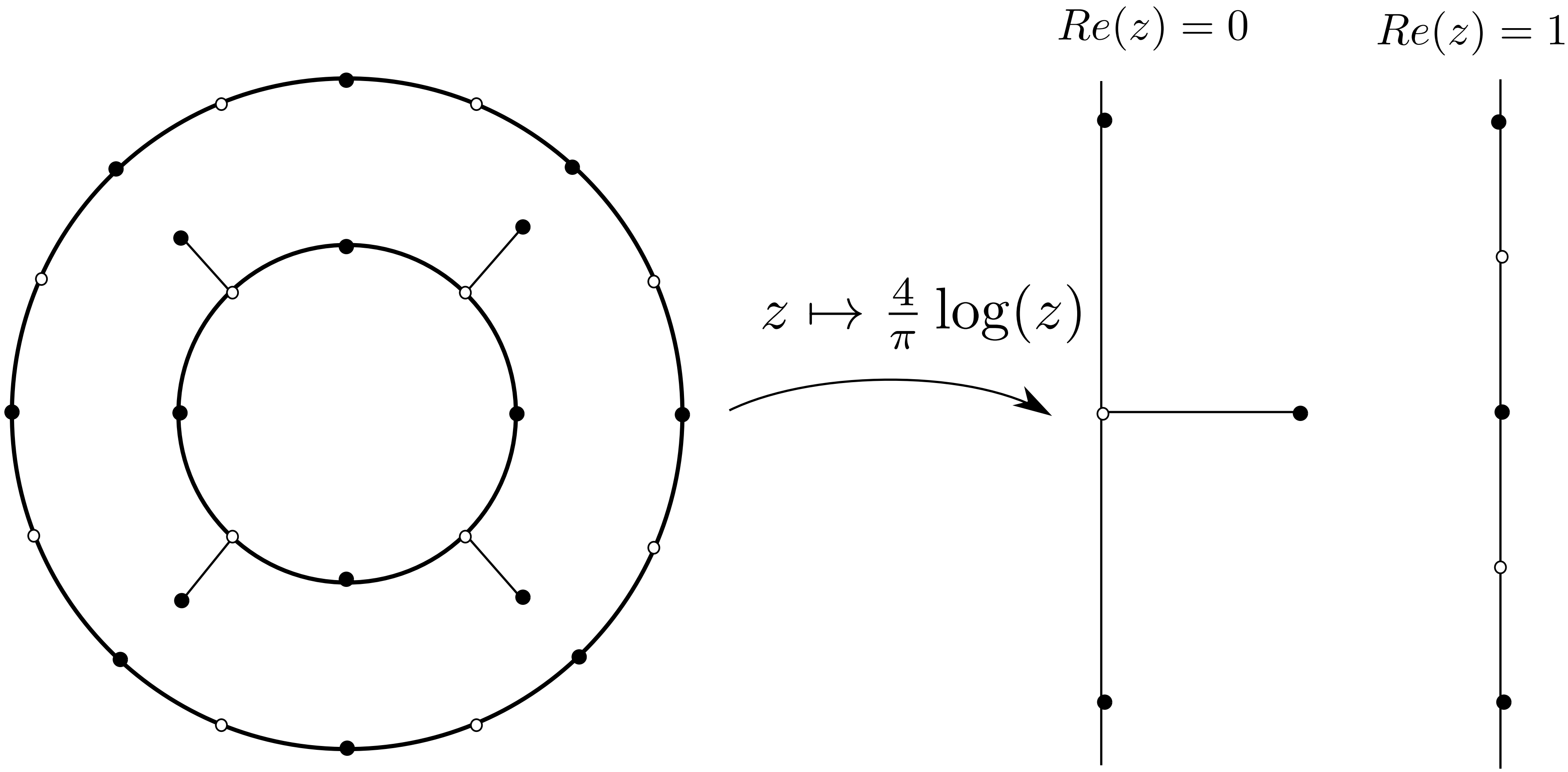
Filled in dots map to positive real axis, and hollow ones to the negative real axis.



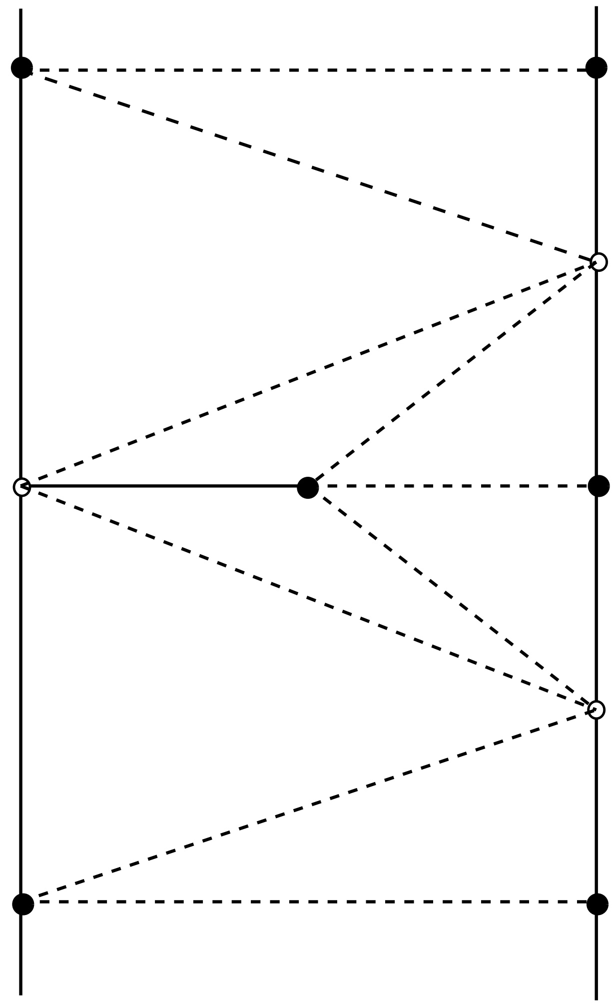
We add **antenna** to the hollow vertices as illustrated.



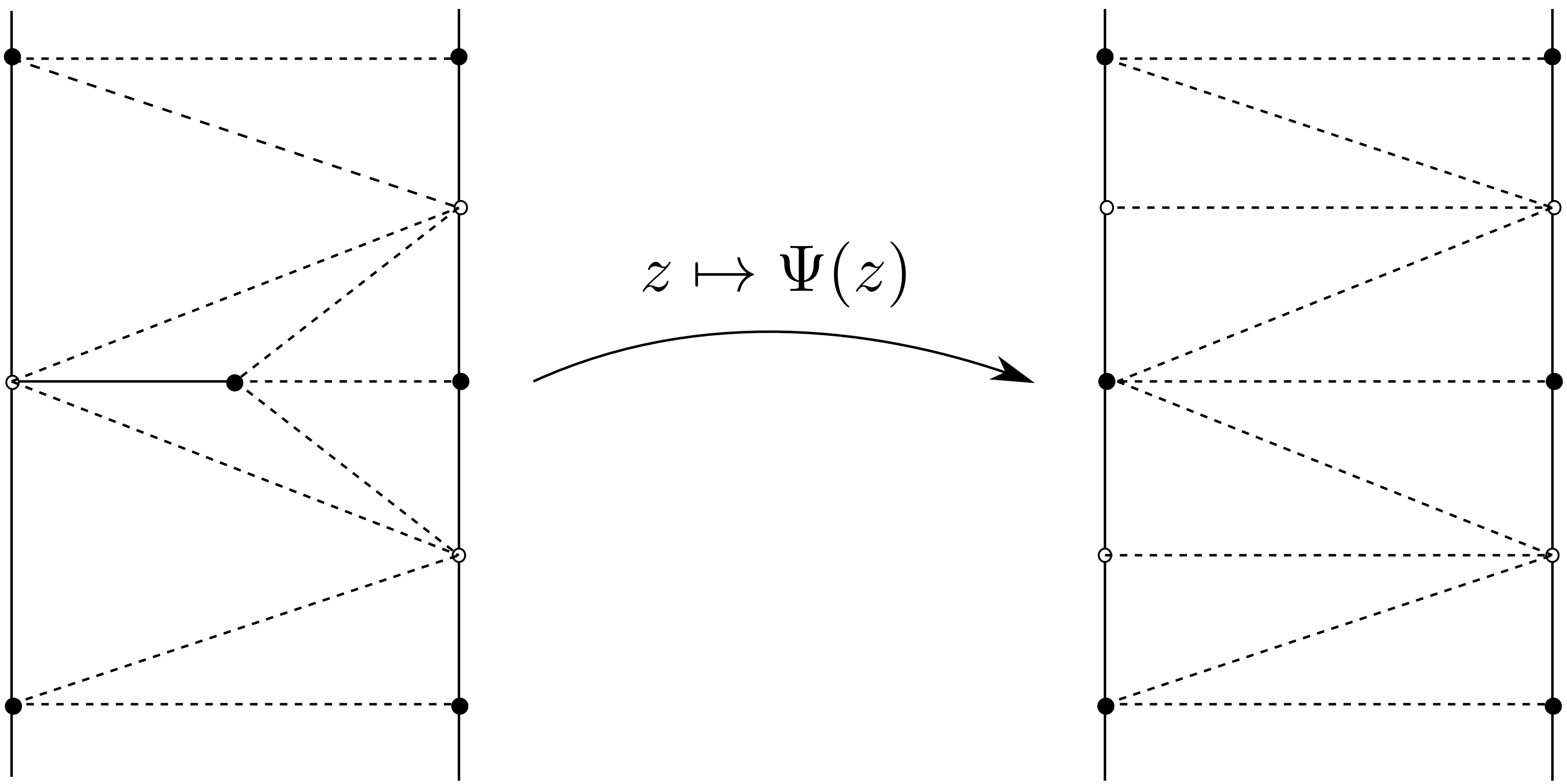
First we map the slit annulus to a slit vertical strip



Triangulate the vertical strip as follows - extend by reflection.

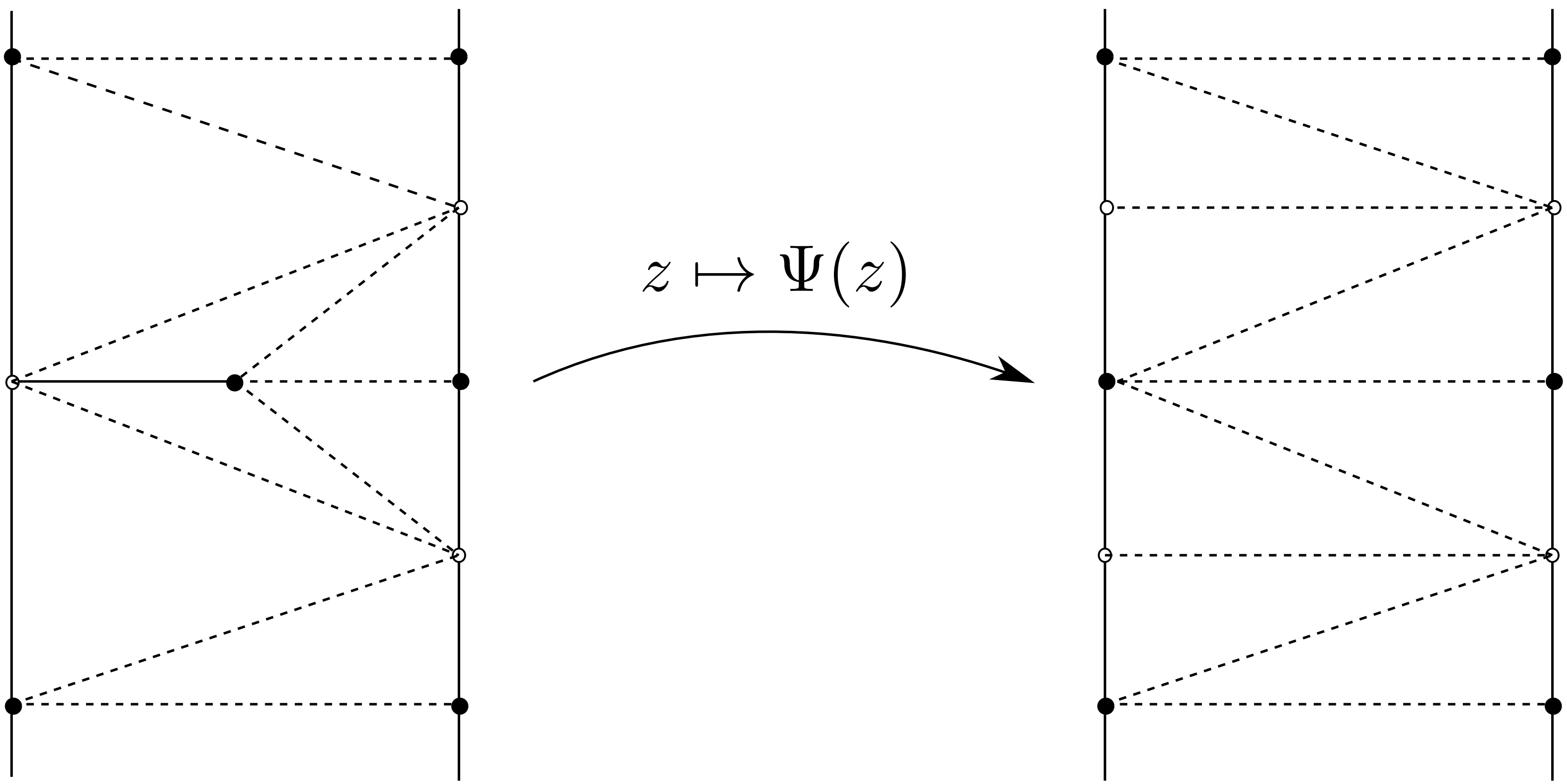


There exists a piecewise linear mapping Ψ compatible with the triangulation to the left with the following triangulation of the vertical strip.



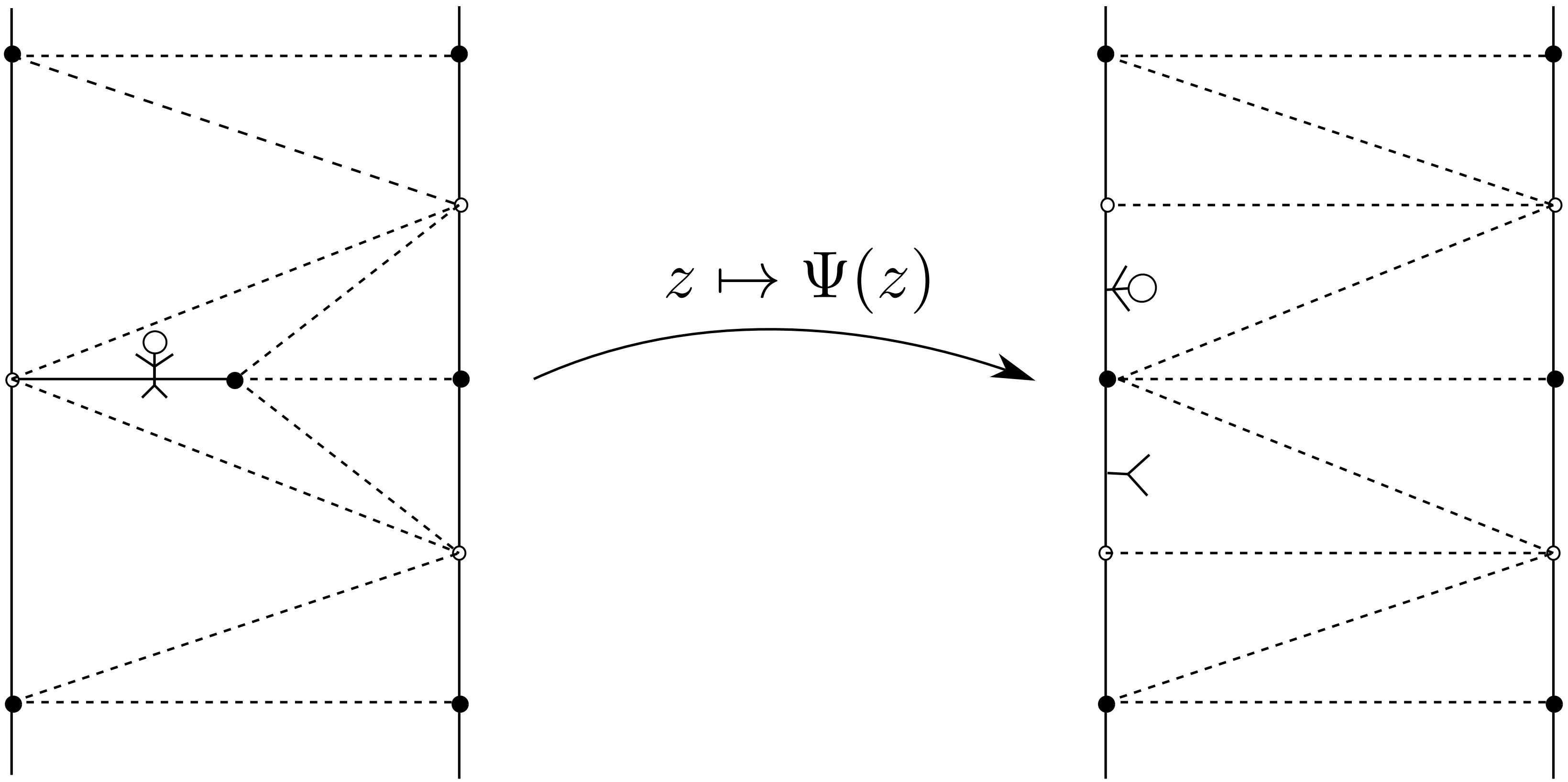
Ψ maps the slit strip to the vertical strip in a piecewise linear way - the "**folding map**".

There exists a piecewise linear mapping Ψ compatible with the triangulation to the left with the following triangulation of the vertical strip.



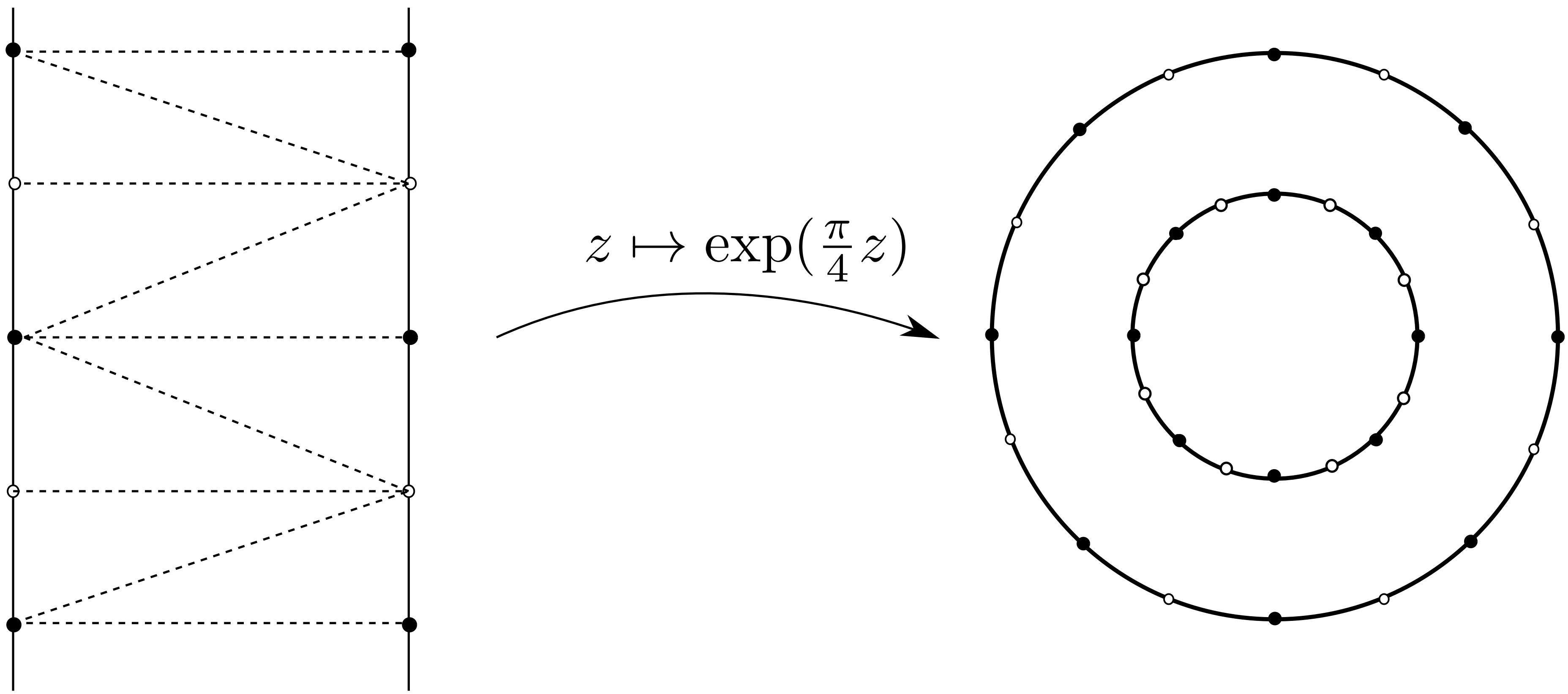
The map Ψ is **quasiconformal** and is to the identity on $Re(z) = 1$.

Find an entire function that is z^4 on $B(0,1)$ and z^8 on $\mathbb{C} \setminus B(0, \exp(\pi/4))$.



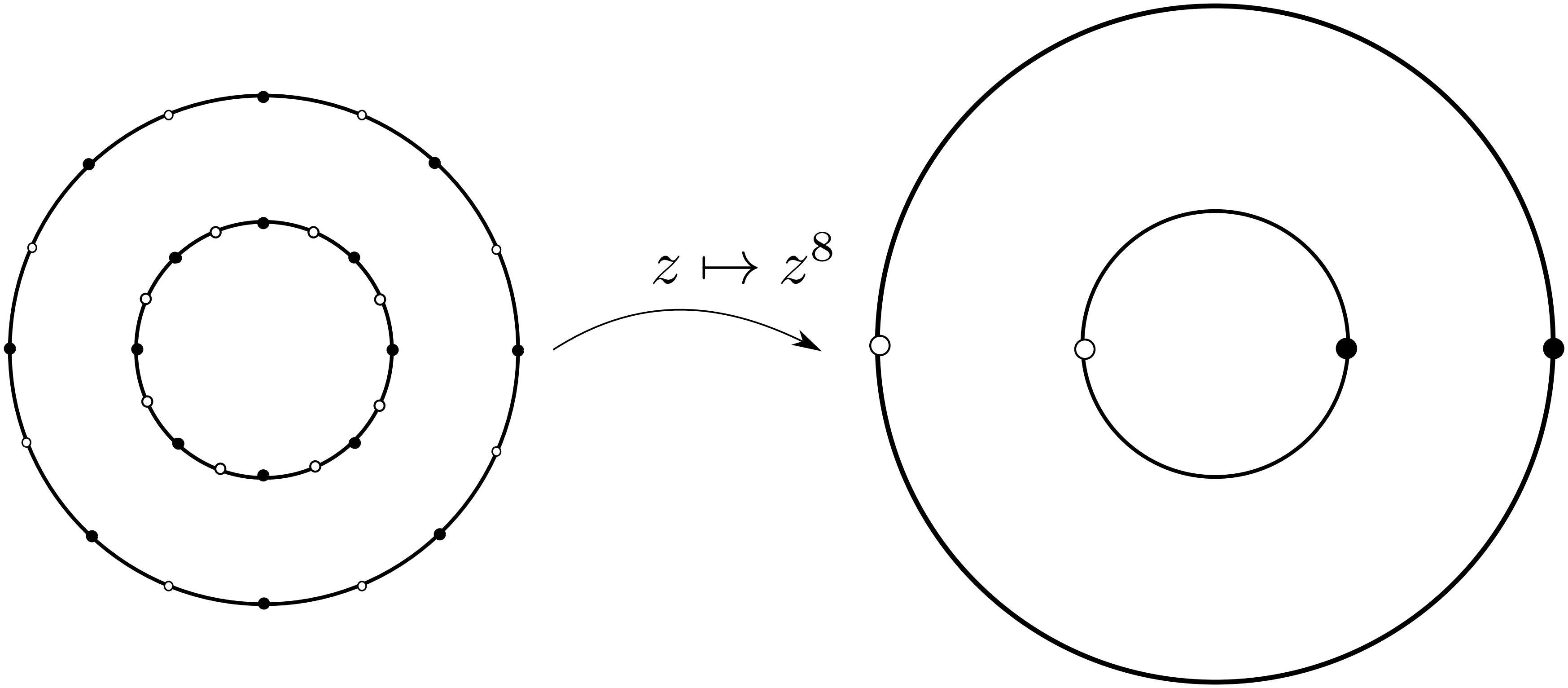
Caution! Ψ is **not continuous over the slit** - an issue we fix later.

Rescale and take the exponential to map back to the annulus $A(1, \exp(\pi/4))$.



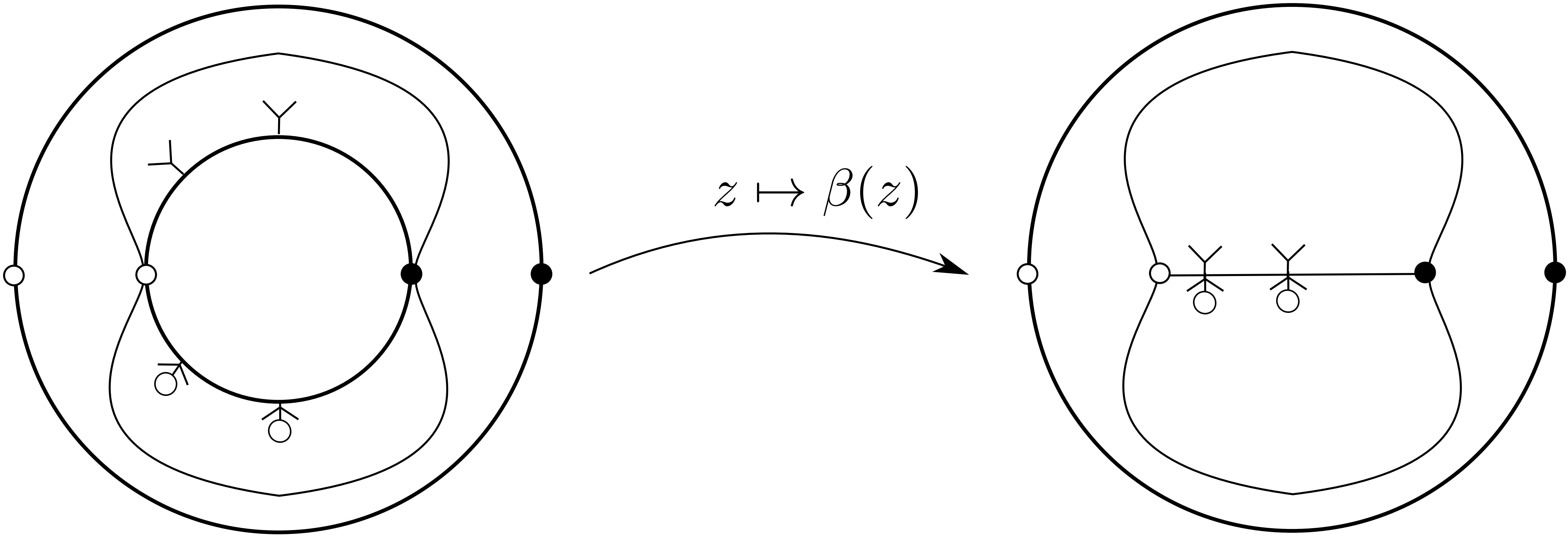
Both pictures now in alignment!

Apply $z \mapsto z^8$

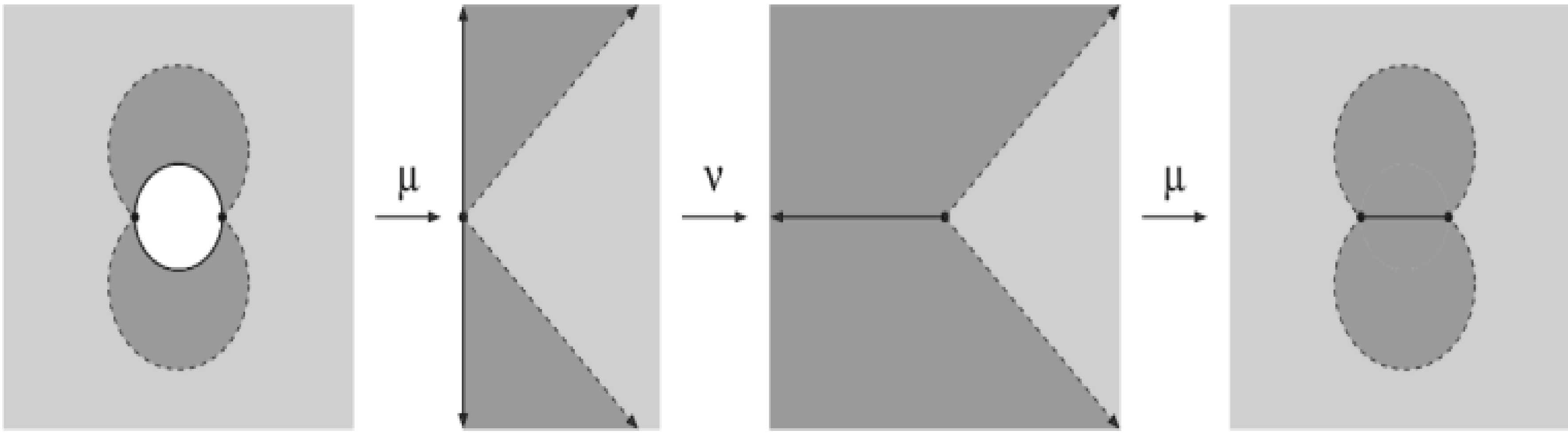


$A(1, \exp(\pi/4))$ gets mapped to $A(1, \exp(2\pi))$.

The composition we have so far defined is continuous everywhere except the antenna where we unfolded.



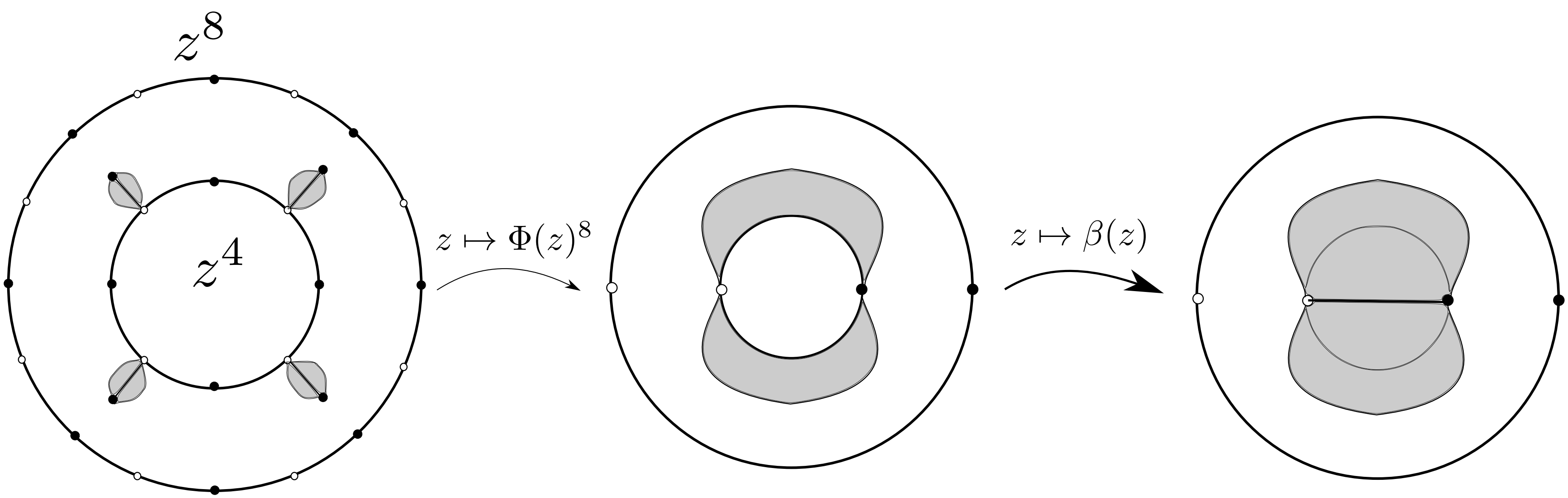
To gain continuity, we want a quasiconformal mapping β that interpolates between $z \mapsto \frac{1}{2}(z + z^{-1})$ on the circle and the identity on the boundary of this crescent region.



Such a mapping can be constructed as a composition of Mobius mappings μ and a quasiconformal mapping ν.

What have we accomplished? We have a **quasiregular** ‘model’ mapping

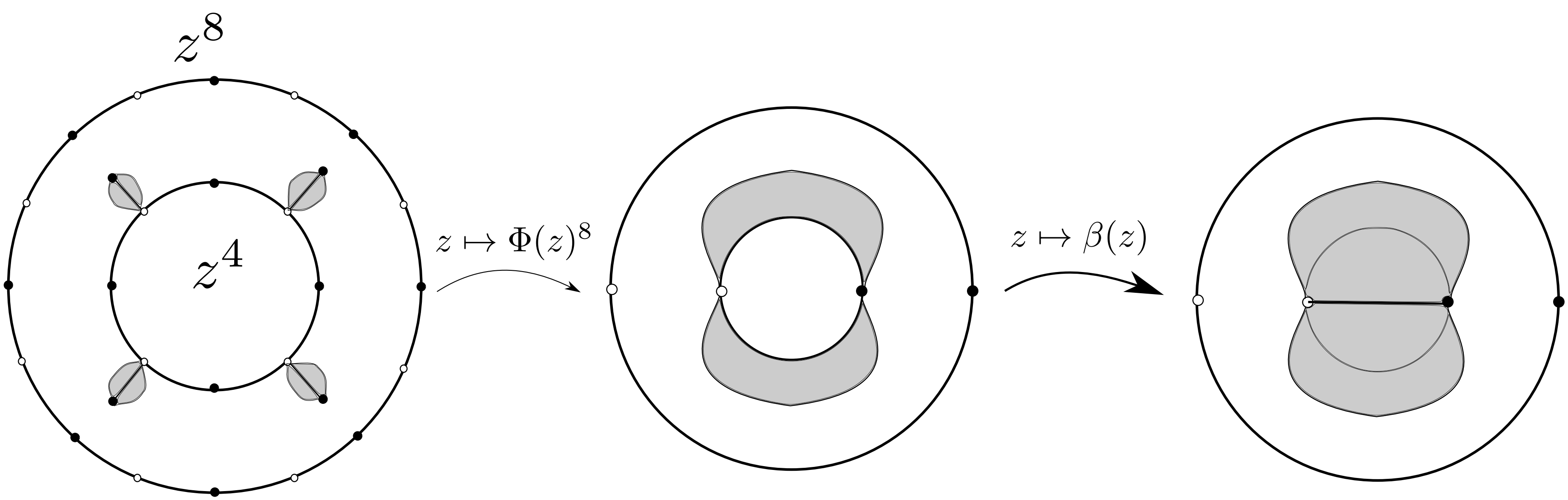
$$h(z) = \begin{cases} z^4 & z \in \overline{B(0,1)} \\ \Phi(z)^8 & z \in \mathbb{C} \setminus B(0,1) \text{ and z far from antenna} \\ \beta \circ \Phi(z)^8 & z \in \mathbb{C} \setminus B(0,1) \text{ and z near the antenna} \end{cases}$$



Φ maps the slit annulus to the round annulus. Note that Φ is the identity on the complement of $B(0, \exp(\pi/4))$. This guarantees continuity on the outer circle.

What have we accomplished? We have a quasiregular 'model' mapping

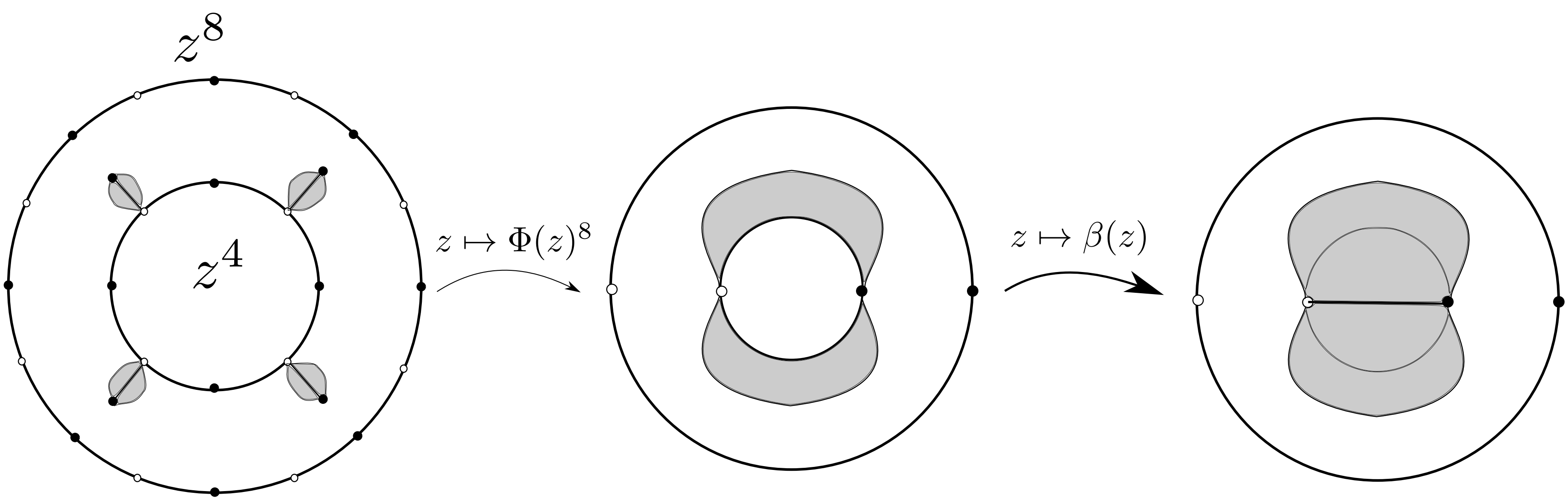
$$h(z) = \begin{cases} z^4 & z \in \overline{B(0,1)} \\ \Phi(z)^8 & z \in \mathbb{C} \setminus B(0,1) \text{ and z far from antenna} \\ \beta \circ \Phi(z)^8 & z \in \mathbb{C} \setminus B(0,1) \text{ and z near the antenna} \end{cases}$$



By the **Measurable Riemann Mapping Theorem**, there is a quasiconformal $\phi : \mathbb{C} \to \mathbb{C}$ and an entire function f such that $f \circ \phi = h$.

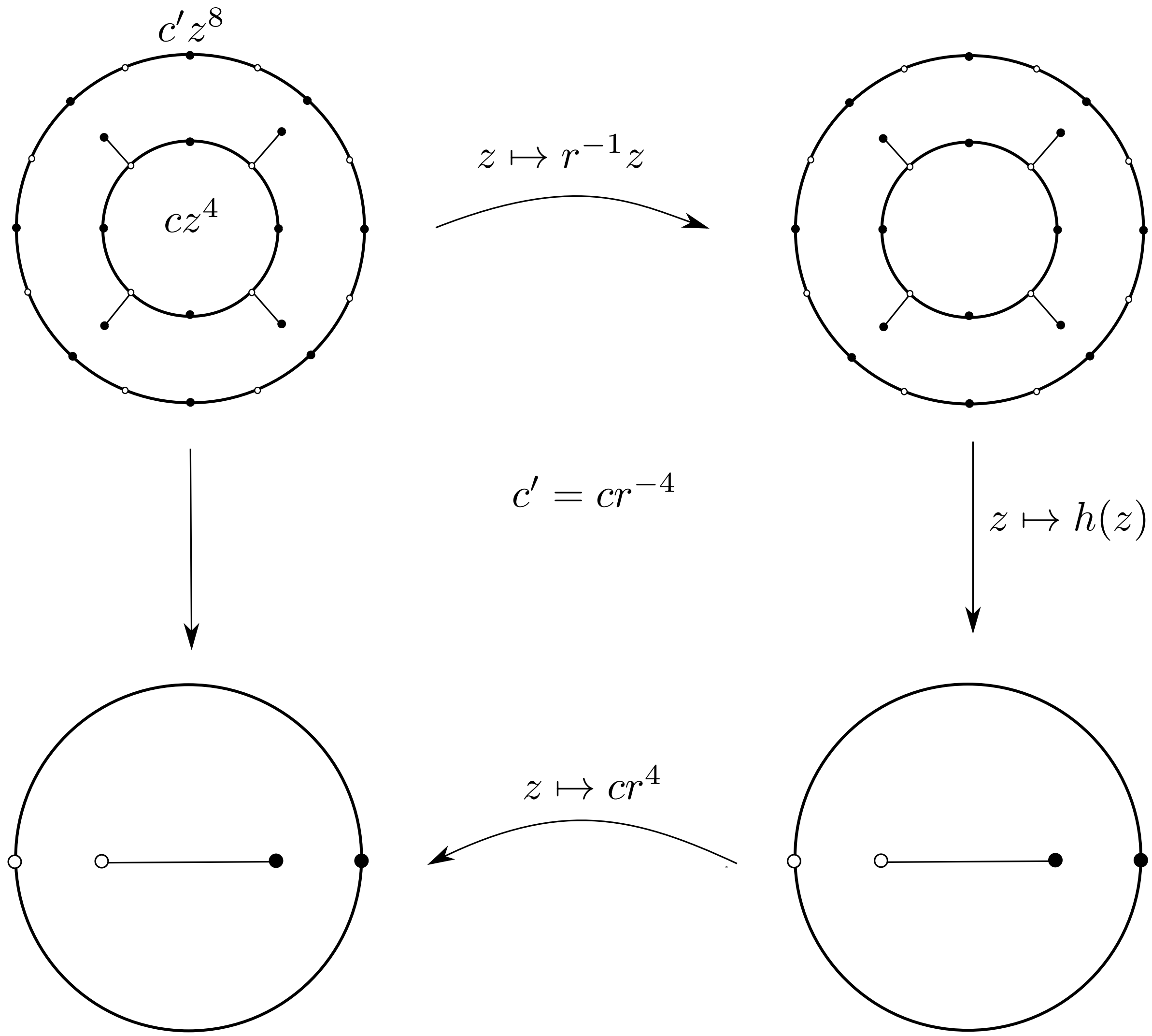
What have we accomplished? We have a quasiregular 'model' mapping

$$h(z) = \begin{cases} z^4 & z \in \overline{B(0,1)} \\ \Phi(z)^8 & z \in \mathbb{C} \setminus B(0,1) \text{ and z far from antenna} \\ \beta \circ \Phi(z)^8 & z \in \mathbb{C} \setminus B(0,1) \text{ and z near the antenna} \end{cases}$$

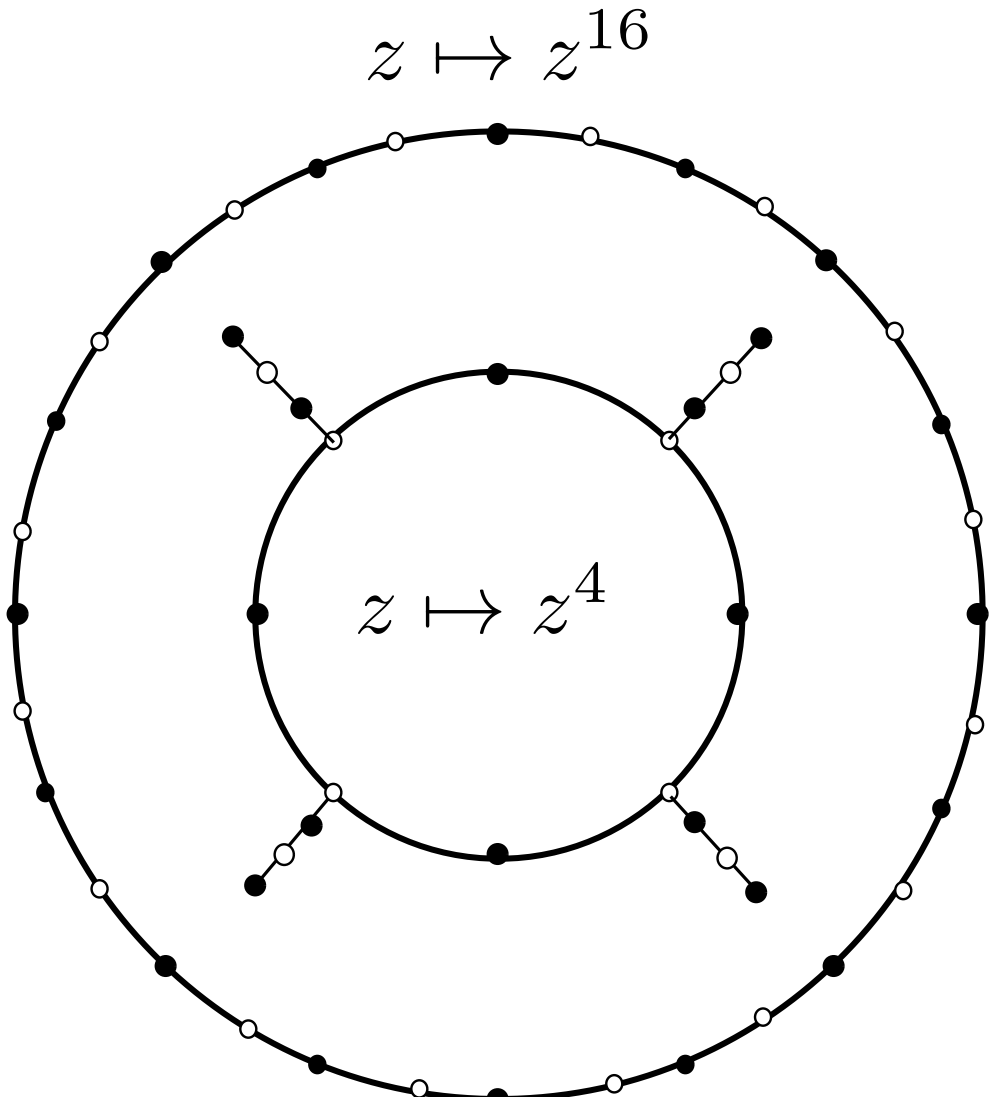


(Imprecise Statement:) Dilatation of the qc correction map ϕ is supported on a 'small' set, so ϕ is close to the identity.

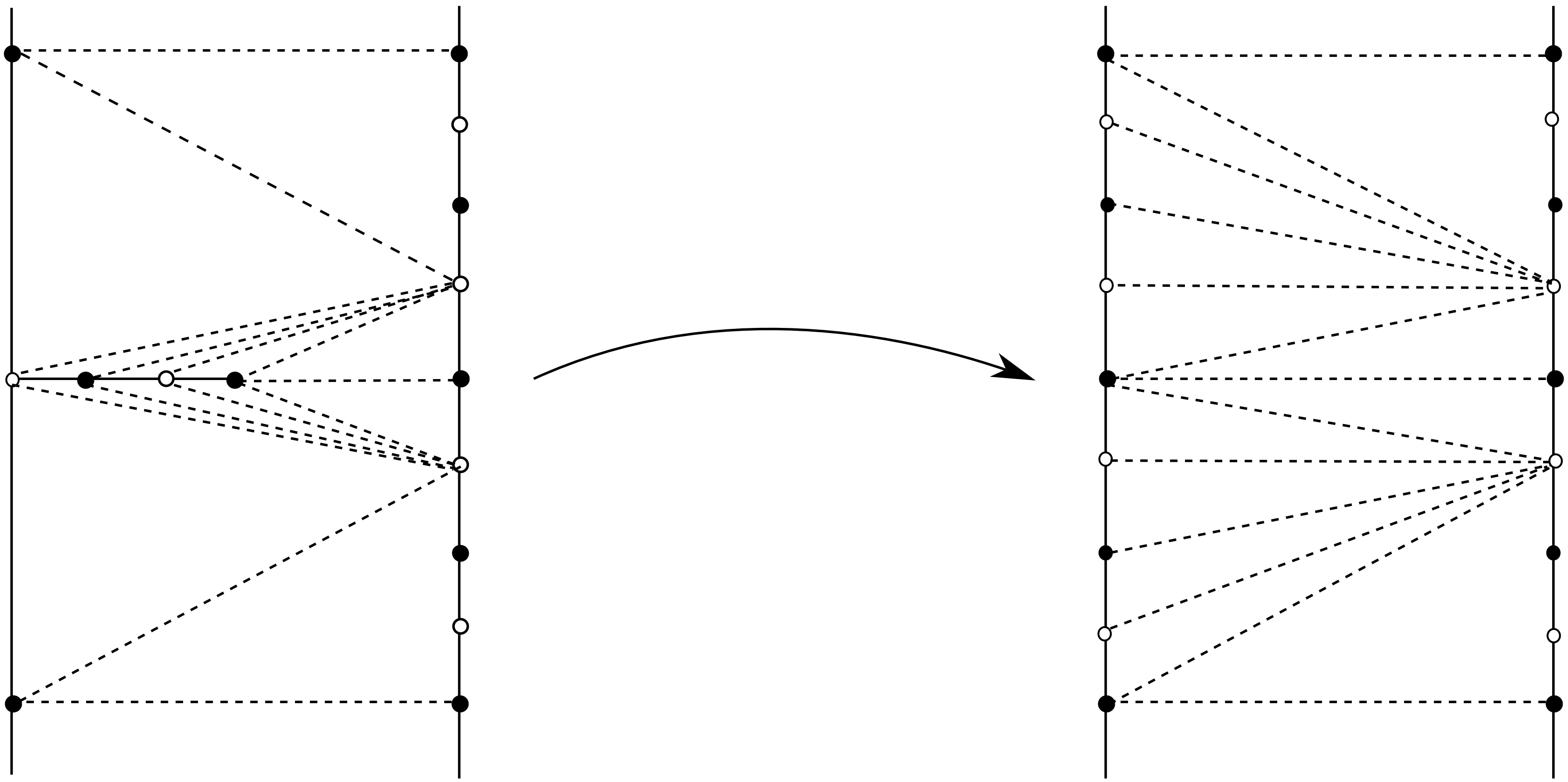
Interpolation at different radii if coefficients are chosen correctly



We can interpolate between larger degrees too.

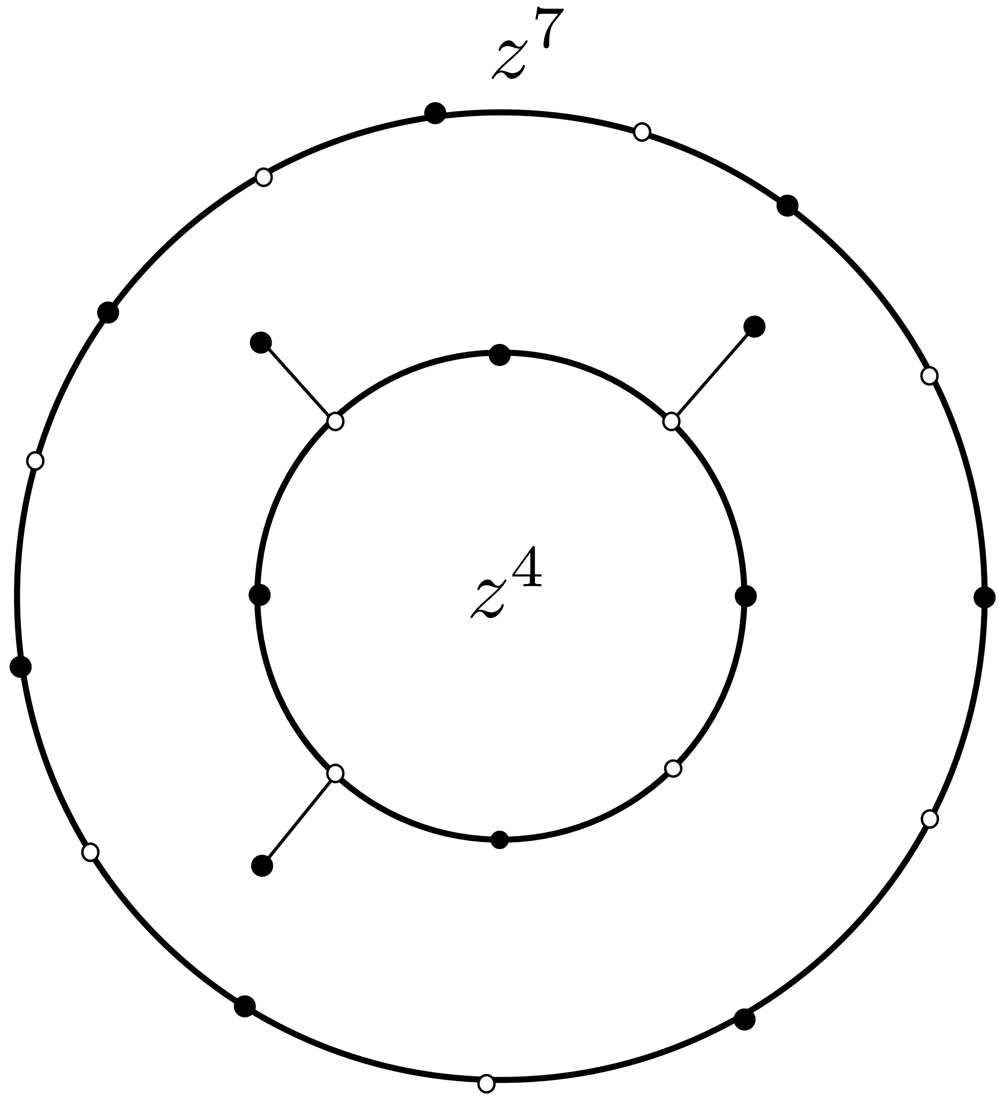


The only thing that changes is the triangulation and unfolding map.



As the ratio of the degrees increases, so does the quasiconformal constant of this piecewise linear map.

No need for powers to be scalar multiples either!



Theorem: (B.-Lazebnik): Let $\{r_j\}_{j=1}^{\infty}$ be a sufficiently quickly increasing sequence and let $\{M_j\}_{j=1}^{\infty}$ be a sequence with $\sup M_{k+1}/M_k < \infty$. Then there exists an entire function f and a quasiconformal mapping ϕ such that

$$f \circ \phi = c_k z^{M_k} \text{ for } z \in A(r_{k-1} \exp(\pi/M_{k-1}), r_k).$$

If $\sum M_j^{-1} < \infty$ we can arrange for ϕ to be close to the identity near ∞:

$$\left|\frac{\phi(z)}{z} - 1\right| \to 0 \text{ as } |z| \to \infty.$$

The only singular values of f are the critical values $(\pm c_j r_j^{M_j})$.

Part 4: Some Applications to Dynamics

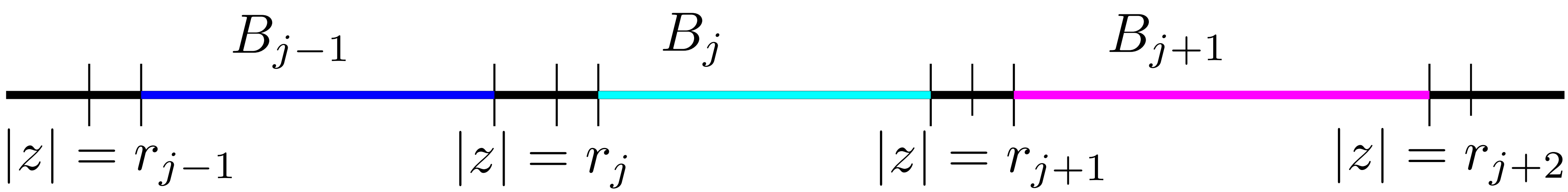
Application 1: A multiply connected wandering domain

Set $M_j = 2^j$, $r_1 = 100$, $c_1 = 1$ and $r_{j+1} = c_j r_j^{M_j}$, and apply the theorem.

We have $f \circ \phi = h$, where h is the model map, ϕ is quasiconformal, and f is entire.

The parameters are defined so that $r_{j+1} = \max_{|z|=r_j} |h(z)|$.

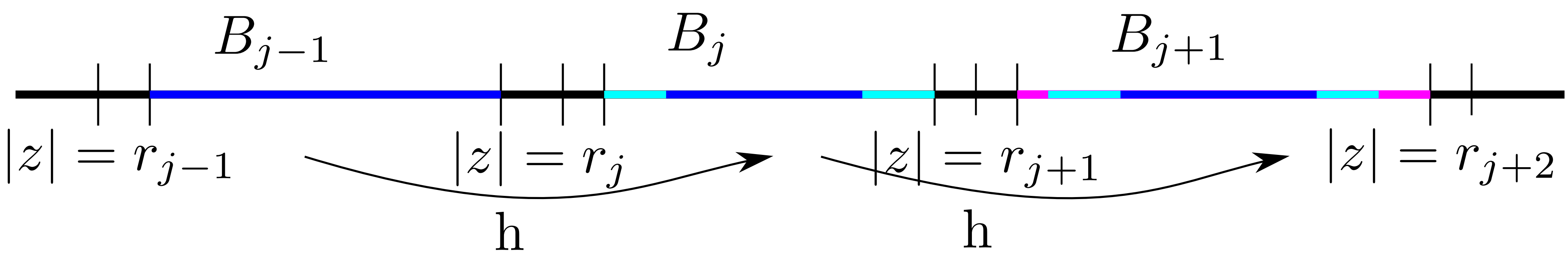
Application 1: A multiply connected wandering domain:



Similar to Baker's example, define

$$B_j = \{4r_j \leq |z| \leq \frac{1}{4}r_{j+1}\}$$

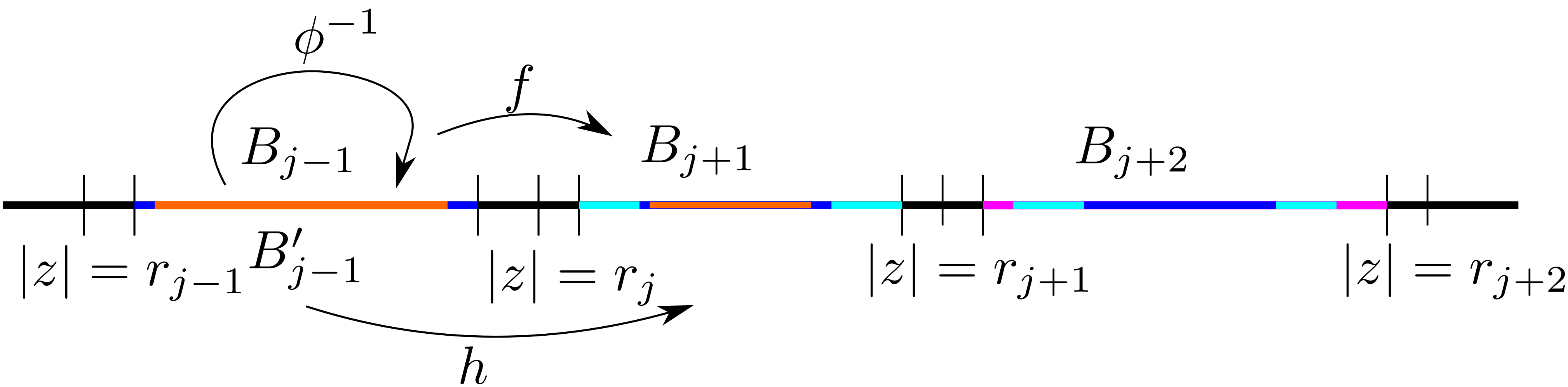
Application 1: A multiply connected wandering domain:



Using the definition of r_j's, we can verify that the map h satisfies

$$h(B_j) \subset B'_{j+1} = \{8r_j \leq |z| \leq \frac{1}{8}r_{j+1}\}$$

Application 1: A multiply connected wandering domain:

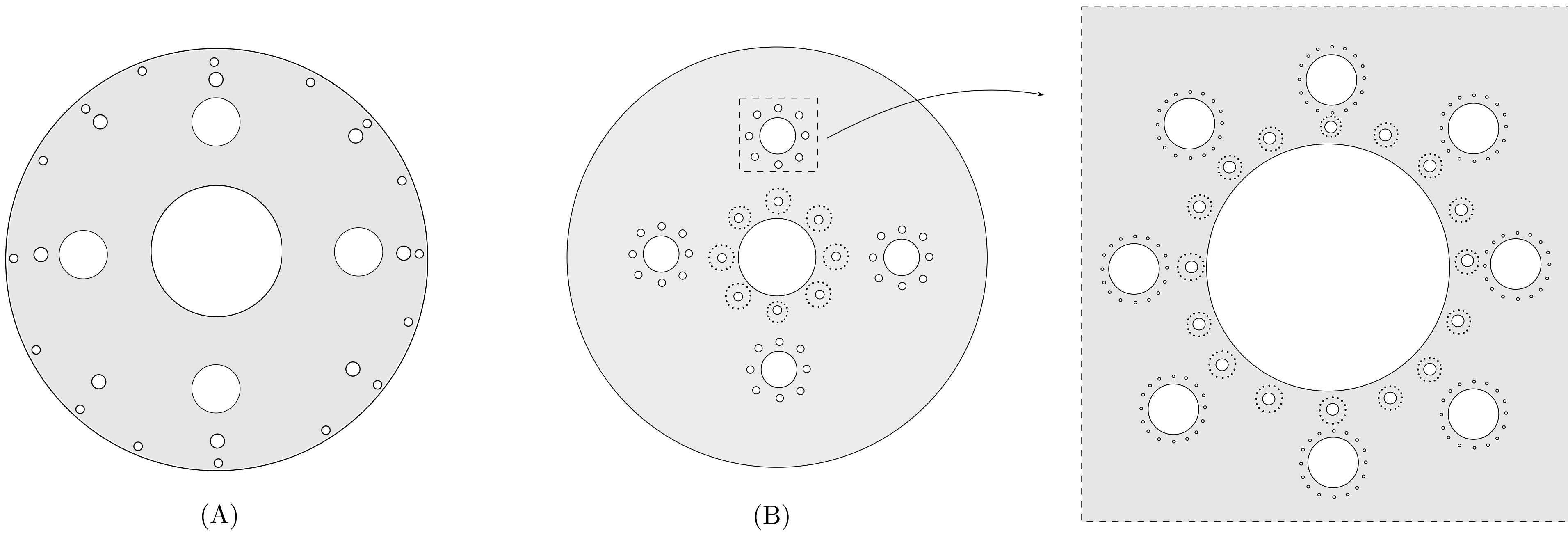


We have $h \circ \phi^{-1} = f$. ϕ^{-1} is close to the identity for large enough j, so for some subannulus we have $\phi^{-1}(B'_j) \subset B_j$

We can show

$$f(B'_j) = h(\phi^{-1}(B'_j)) \subset h(B_j) \subset B'_{j+1}$$

Application 2: Singleton Complementary Components.

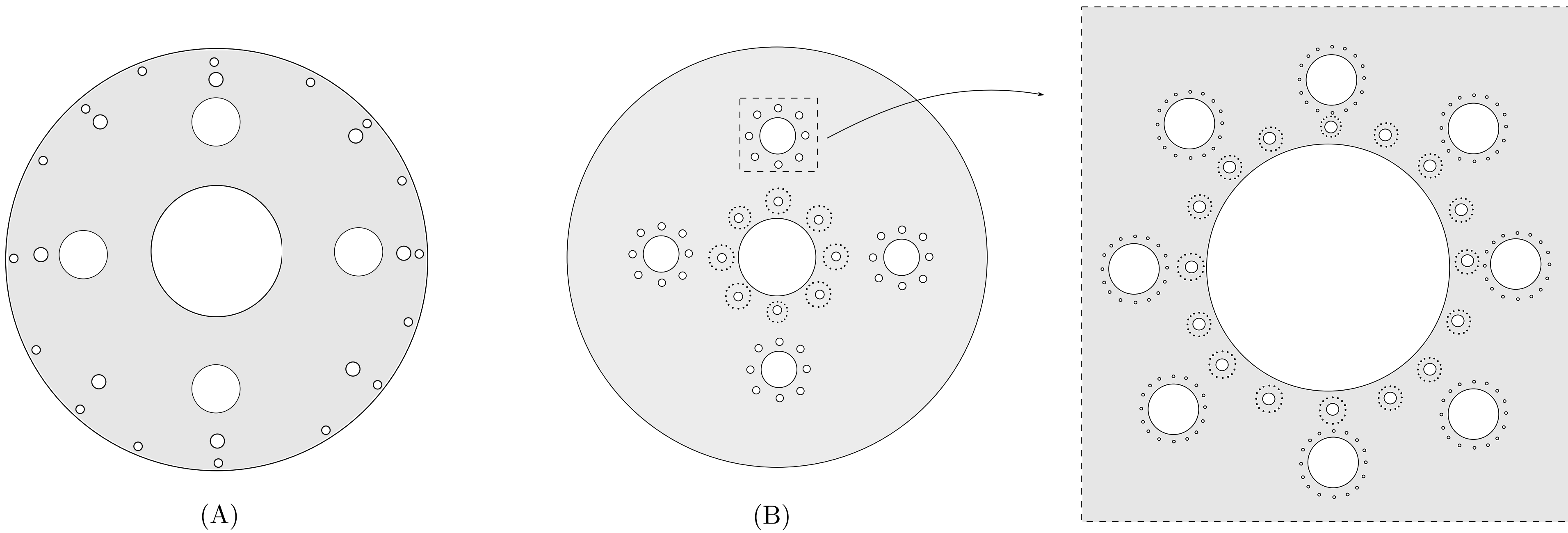


(A) (B)

(A) side corresponds to $M_j = 2^j$ and $r_{j+1} = c_{j+1}(2r_j)^{M_{j+1}}$. Yields examples similar to Bishop's entire functions with $\dim_H(\mathcal{J}(f)) = 1$.

Complementary components converge on the outermost boundary - **infinite outer connectivity**.

Application 2: Singleton Complementary Components.

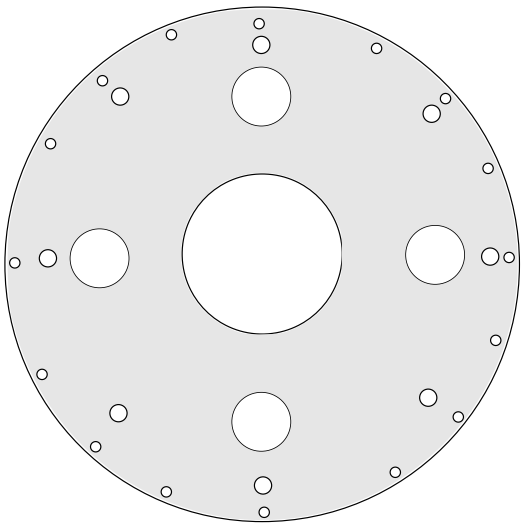


(A) (B)

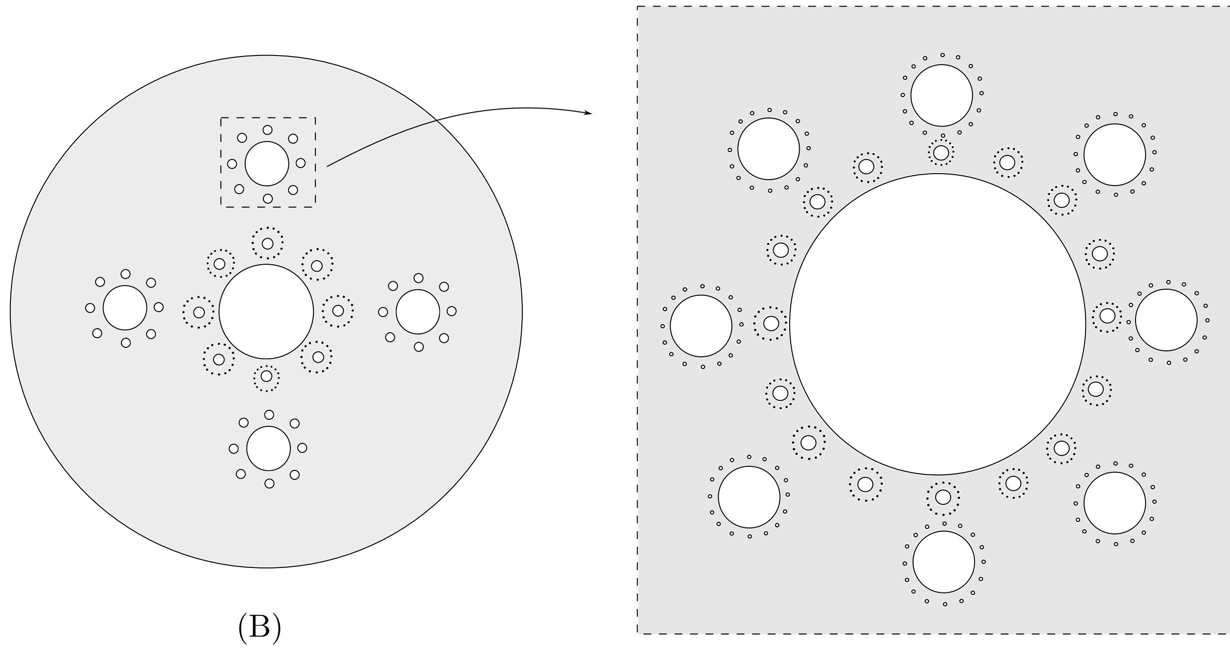
(B) side corresponds to $M_j = 2^j$ and $r_{j+1} = c_j(\frac{1}{2}r_j)^{M_j}$.

Complementary components converge on the inner boundary - **infinite inner connectivity**.

Application 2: Singleton Complementary Components.



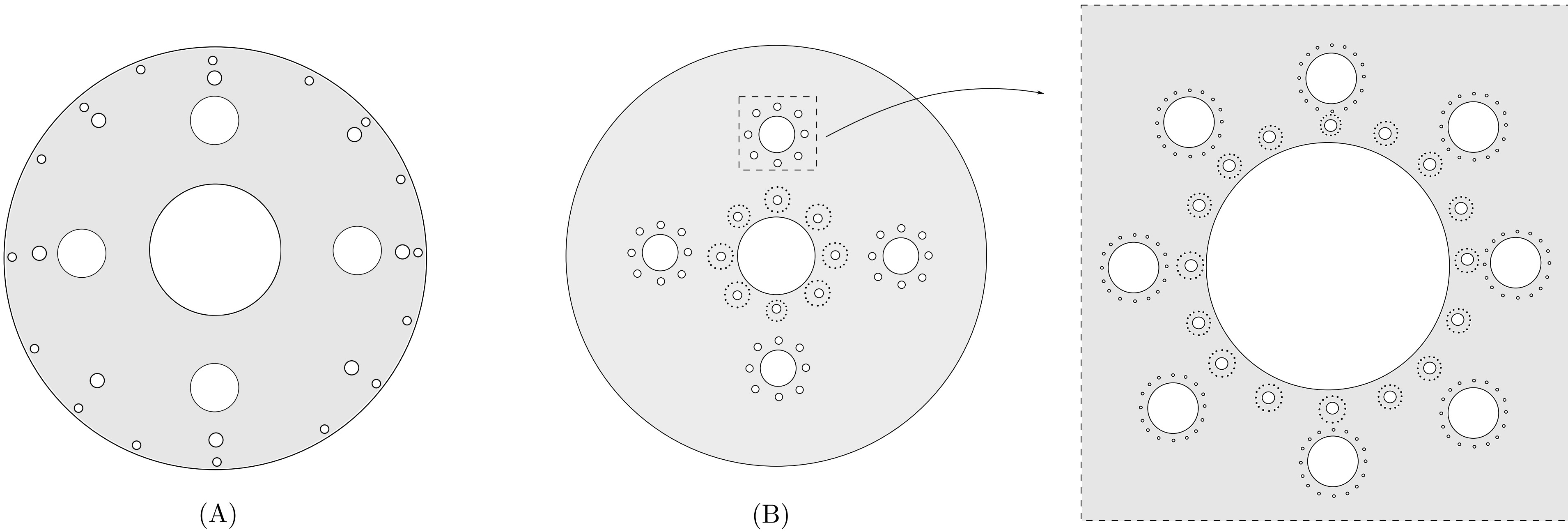
(A)



(B)

Theorem: (B-Lazebnik) The boundary of the wandering domains in the (B) example have uncountably many singleton components. The Julia set still has Hausdorff dimension 1.

Application 2: Singleton Complementary Components.



(A) (B)

Proof ingredients include

- Extending techniques to interpolate between $z^M + \delta z$ and z^N.
- **Conformal mapping estimates** near the antenna
- Various **dimension estimates** of given dynamically defined sets.

Part 5: Some Further Research Directions

Hard Question: Is there a t.e.f with $\mathcal{J}(f)$ being the subset of a finite (spherical) length curve?

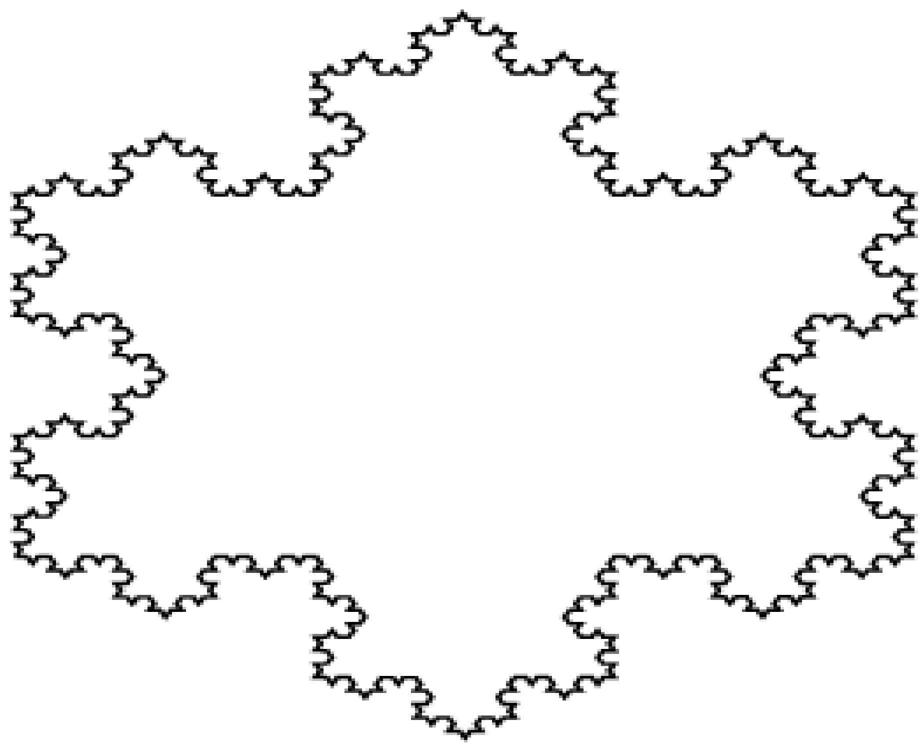
Maybe a function with doubly connected wandering Fatou components will have this property. Have been constructed by Kisaka and Shishikura via quasiconformal surgery.

Question: Is it possible to use this approach to construct annular Fatou components?

Question: If so, what does the Julia set look like? Are the Boundary Curves Jordan curves? Dimension 1? Something else?

Question: Are the connected components of the boundaries in our examples better than C^1?

Question: Can we make the connected components of the boundaries *worse* than C^1?

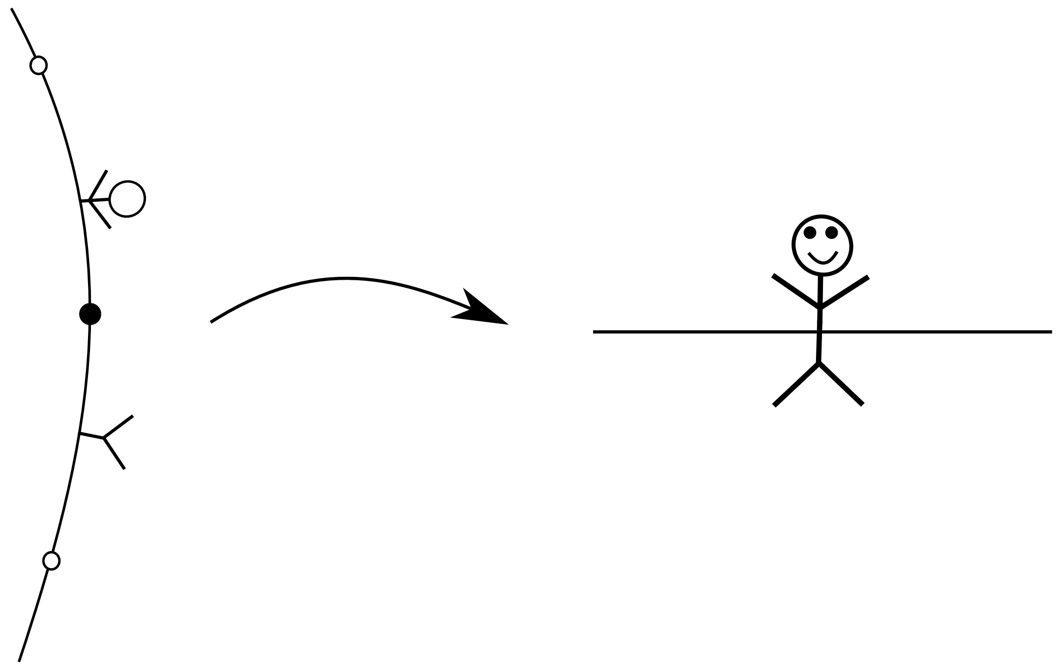


Question: Are there applications to function theory?

Some promising joint work inspired by MSRI interaction going on with Leticia Pardo-Simon and Adi Glucksam currently going on regarding the maximum modulus set of an entire function.

$$M(f) = \{z : |f(z)| = \max_{|z|=r} |f|\}$$

Question: Do these techniques extend to the setting of quasiregular mappings $f : \mathbb{R}^n \to \mathbb{R}^n$?



Thank You!

Theorem (Teichmueller-Wittich-Belinksii) Let $\phi : \mathbb{C} \to \mathbb{C}$ be K-quasiconformal with $\phi(0) = 0$ and

$$I(r) = \frac{1}{2\pi} \int_{|z|<r} \frac{D(z) - 1}{|z|^2} \, dA(z) < \infty \text{ for } r < \infty.$$

Then

$$\left| \frac{\phi(z)}{z} - \phi_z(0) \right| < |\phi_z(0)| \epsilon(|z|) \text{ where } \epsilon(|z|) \to 0 \text{ as } |z| \to 0.$$

The error ϵ depends only on $I(r)$ and K and not otherwise on ϕ.

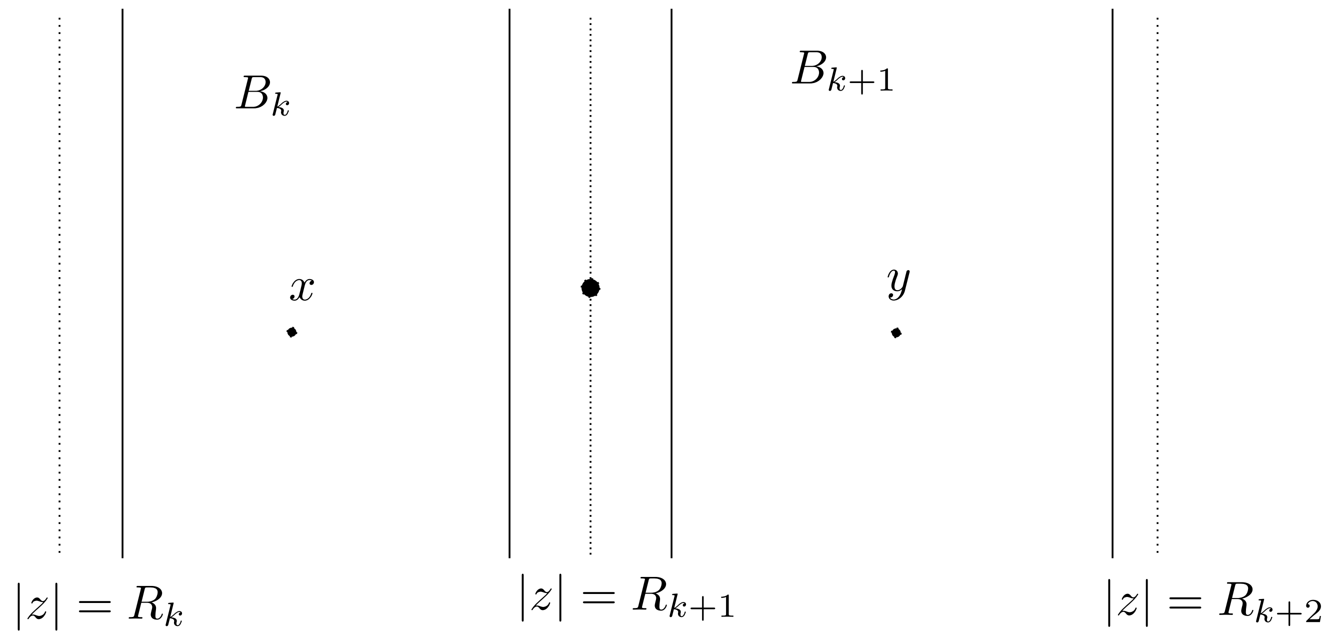
In the interpolation theorem, we change coordinates mapping ∞ to 0, and we estimate for our quasiconformal correction map ϕ that

$$I(r) \lesssim \sum M_j^{-1} < \infty.$$

In some sense, the annuli supporting the dilatation are 'thin' near ∞, so ϕ approximates the identity.

What's stopping $U_{k+1} = U_k$? Maybe this is an unbounded invariant Fatou component!

Lemma: $U_k \neq U_{k+1}$ - each B_k is in a distinct Fatou component.



Proof: Suppose $U_k = U_{k+1}$ - then $U_k = U_j$ for all $j \geq k$. Take $x \in B_k$ and $y \in B_{k+1}$. We require $d_{\text{hyp}}(f^n(x), f^n(y))$ to be non-increasing. However, because the modulus of B_k is increasing rapidly, we can directly estimate $d_{\text{hyp}}(f^n(x), f^n(y)) \to \infty$.