Yoccoz's Inequality and the Parabolic Zoo

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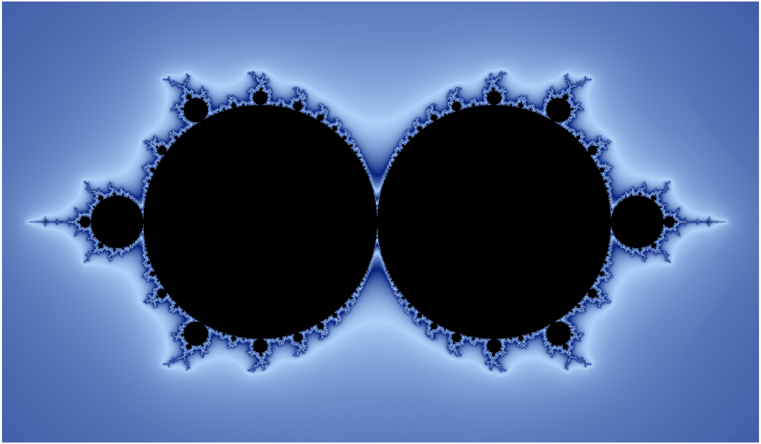
COMD Research Seminar

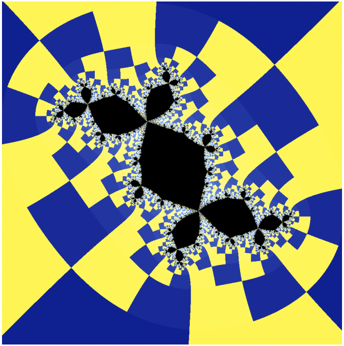
We consider the "Mandelbrot set" $\mathcal{M}$ of all polynomials

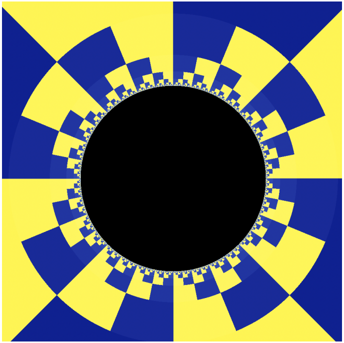
$$f_\lambda(z) = \lambda z + z^2$$

with connected filled Julia set.



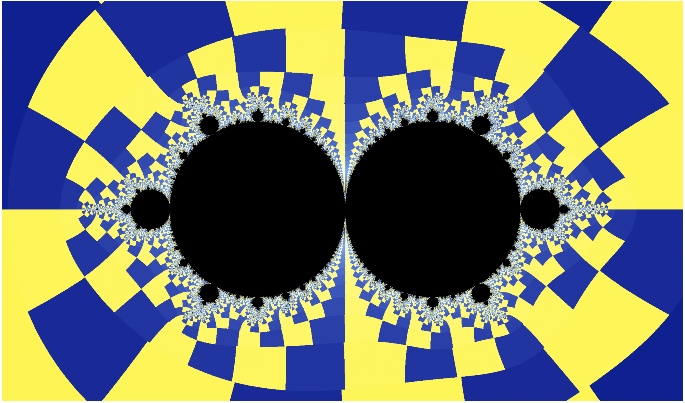
For $\lambda \in \mathcal{M}$, the Böttcher coordinate $\psi_\lambda : \mathbb{C} \setminus K(f_\lambda) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ conjugates f_λ to z^2.



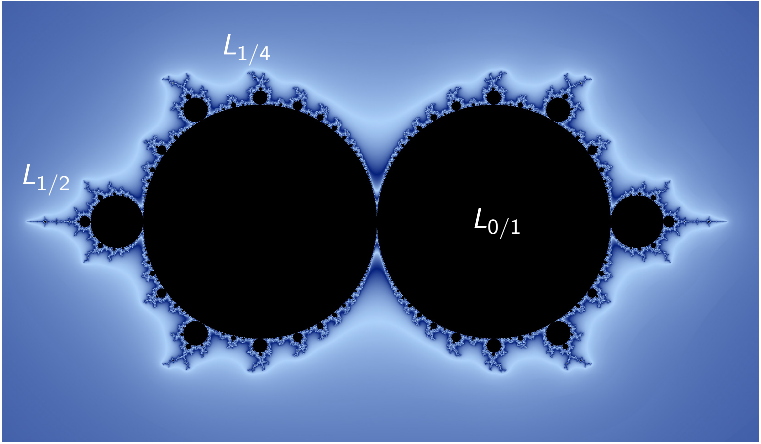


For $z \notin K(f_\lambda)$ we can define the *potential* $G_\lambda(z) := \log |\psi_\lambda(z)|$ and external angle $\arg \psi_\lambda(z)/2\pi$.

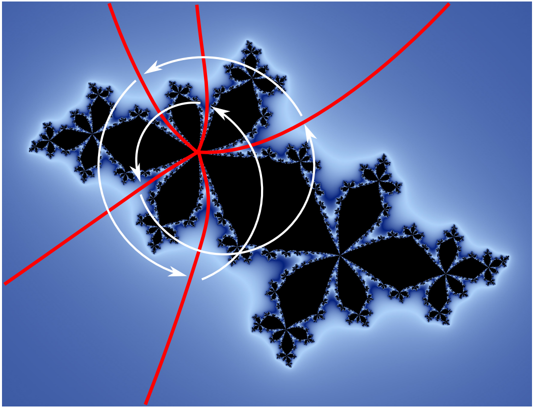
The Böttcher coordinate exists for $\lambda \notin \mathcal{M}$. In this case the potential and external angle of λ is defined as the potential and external angle of the critical value of f_λ.



The p/q-limb $\mathcal{L}_{p/q} \subset \mathcal{M}$ is attached to $\partial\mathbb{D}$ at $e^{2\pi i p/q}$.



For every $\lambda \in \mathcal{L}_{p/q}$, exactly q external rays of f_λ land at 0 and are "p/q-rotated" by f_λ.



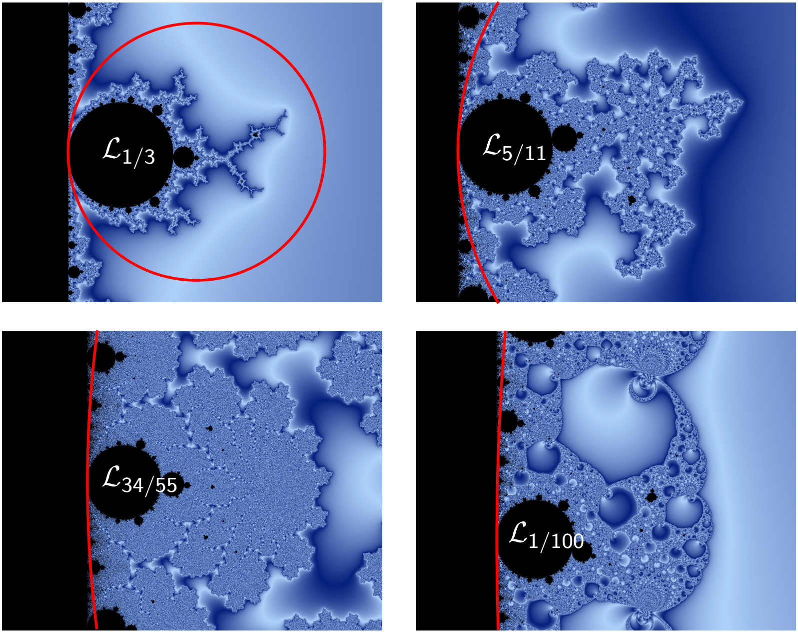
Question: If f_λ is combinatorially a p/q-rotation at 0, is $f'_\lambda(0) = \lambda$ close to $e^{2\pi i p/q}$?

Theorem (The Yoccoz inequality, PLY-inequality)

For every $p/q \in \mathbb{Q}$, if $\lambda \in \mathcal{L}_{p/q}$ then

$$\left| \log \lambda - \frac{2\pi i p}{q} \right| < \frac{2 \log 2}{q}.$$

The Yoccoz inequality relates the combinatorial data of external rays to the analytic data of λ.


$\mathcal{L}_{1/3}$
$\mathcal{L}_{5/11}$
$\mathcal{L}_{34/55}$
$\mathcal{L}_{1/100}$

Conjecture

There exists $C > 0$ *such that for every* $p/q \in \mathbb{Q}$ *and* $\lambda \in \mathcal{L}_{p/q}$,

$$\left|\log \lambda - \frac{2\pi i p}{q}\right| < \frac{C}{q^2}.$$

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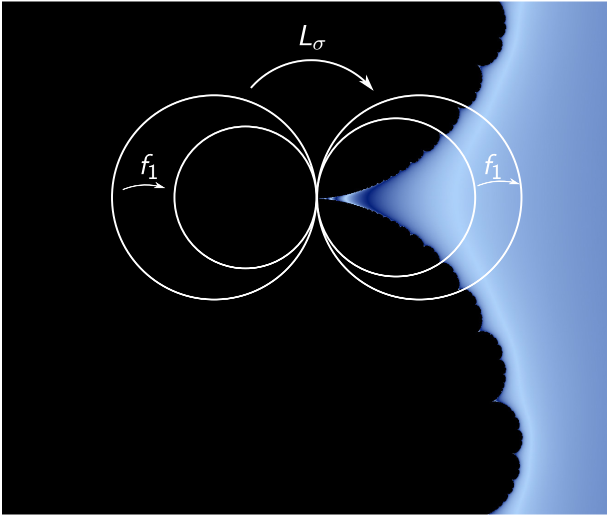
Idea: Maps $f_\lambda \in \mathcal{L}_{1/q}$ converge to $f_1(z) = z + z^2$ when $q \to \infty$, so we will study parabolic implosion.

Recall that a *Lavaurs map* L_σ for f_1, parameterized by $\sigma \in \mathbb{C}$, is an isomorphism from an attracting fundamental domain to a repelling fundamental domain, extended analytically by conjugating with f_1.



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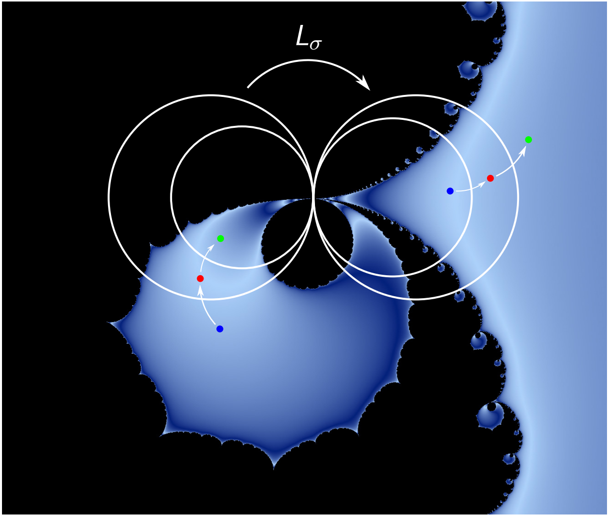

For a Lavaurs map L_σ, we define the *m-escaping set*

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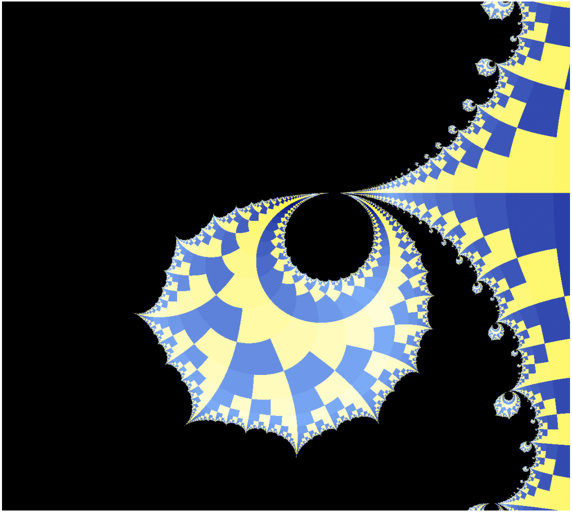
We will assume that the Lavaurs map L_σ is *d-nonescaping*, i.e.

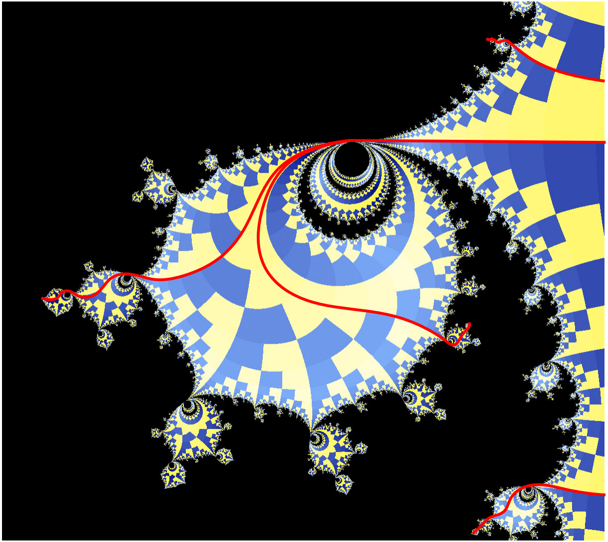
$$-1/4 \notin \bigcup_{m=0}^{d} \overline{E_m}.$$

This assumption simplifies many of the following statements, but the general case isn't too difficult.

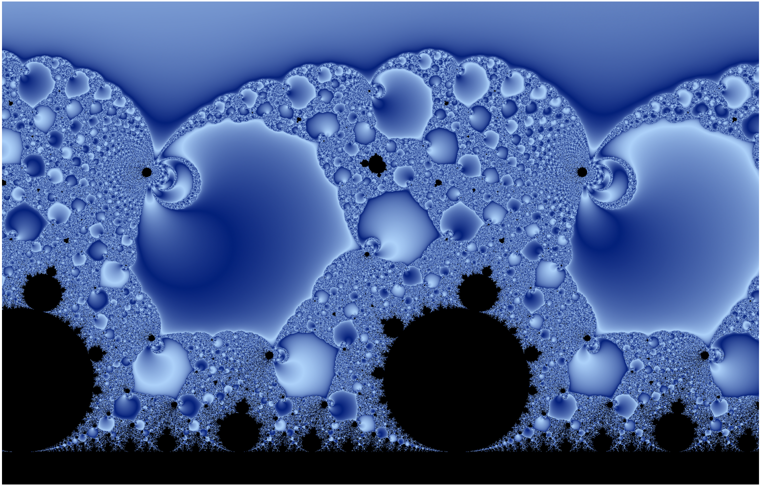
If L_σ is d-nonescaping, then every connected component U of E_d is a croissant and there exists a unique isomorphism $\Psi_U : U \to \mathbb{H}$ which respects the dynamics of f_1 and L_σ.



We can define *enriched rays of depth $d-1$* for L_σ indexed by sequences of angles $(\theta_0, \theta_1, \ldots, \theta_{d-1})$.



We now consider the set $\mathcal{M}_L$ of all $\sigma \subset \mathbb{C}$ with $-1/4 \notin \bigcup_{m\geq 0} E_m$.



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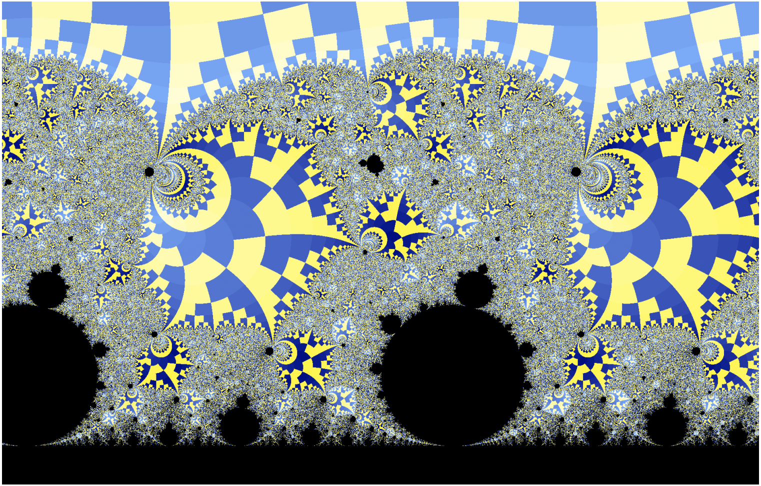

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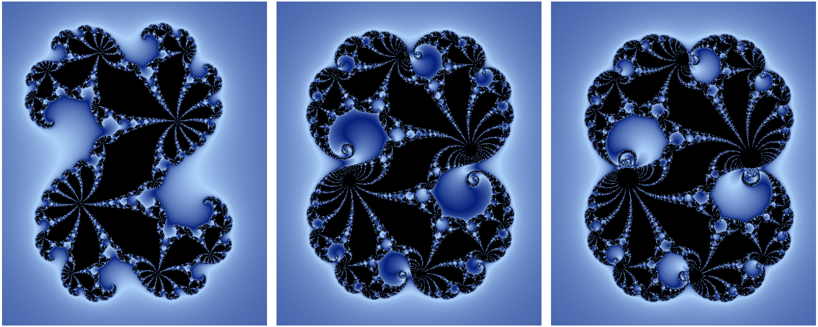
Theorem (Douady, Lavaurs, Shishikura)

For sequences $\lambda_n \to 1$ *and* $k_n \to +\infty$, *if* $k_n - \frac{2\pi i}{\log \lambda_n} \to \sigma \in \mathbb{C}$ *then* $f_{\lambda_n}^{k_n}$ *converges to* L_σ *locally uniformly on* $\mathrm{Int}\ K(f_1)$.

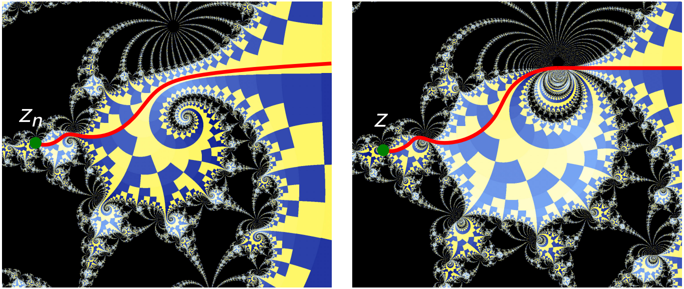


Lemma (Angle lemma)

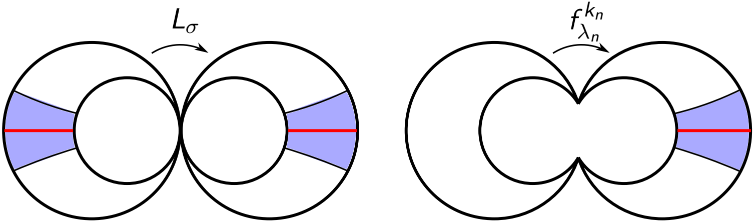
Assume L_σ is d-nonescaping, $f_{\lambda_n}^{k_n} \to L_\sigma$, and $z \in E_d$. If z_n is a sequence converging to z and $G_{\lambda_n}(z_n) > 2^{-mk_n}$ for some $m \geq 0$, then there exist real numbers $\theta_0, \dots, \theta_d$ such that the external angle of z_n under f_{λ_n} is by

$$\theta_0 + \frac{\theta_1}{2^{k_n}} + \cdots + \frac{\theta_{d-1}}{2^{(d-1)k_n}} + \frac{\theta_d + O(1)}{2^{dk_n}}.$$

I.e., external rays of f_{λ_n} converge to enriched rays of L_σ.*



Proof sketch: First we study limits of the fixed ray:



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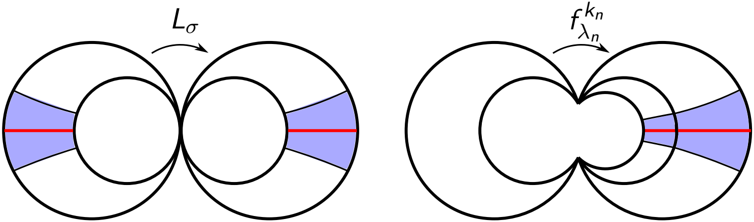

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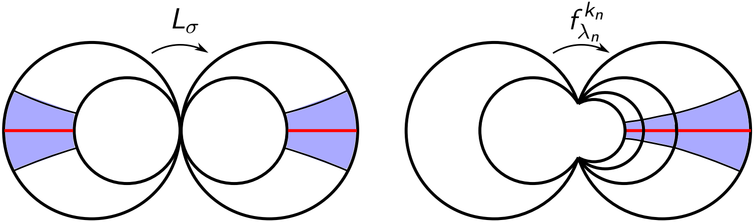

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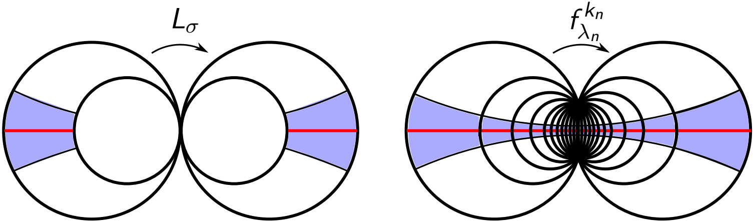
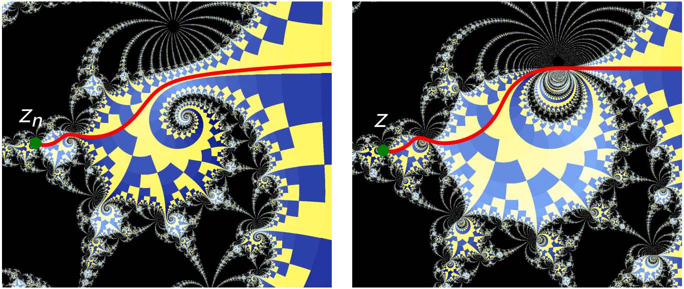


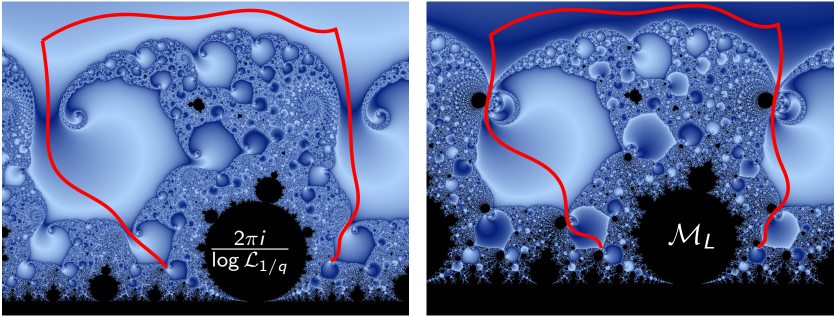
We then pull back by f_1 and L_σ.



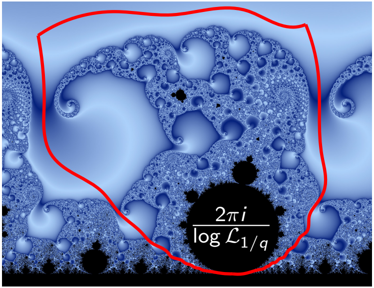
When $f_{\lambda_n}^{k_n} \to L_\sigma$, Lavaurs' theorem and the angle lemma put analytic and combinatorial constraints respectively on λ_n and σ.

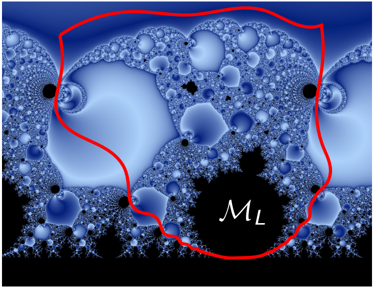
This relationship between analytic and combinatorial data is the heart of the improved Yoccoz inequality.

It follows from the angle lemma that parameter rays in the $-\frac{2\pi i}{\log \lambda}$ plane converge* to enriched parameter rays.



If we could extend to infinite depth enriched rays, then we could build a cage around $\mathcal{L}_{1/q}$ out of external rays and an equipotential.





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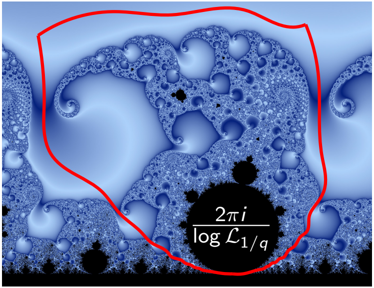

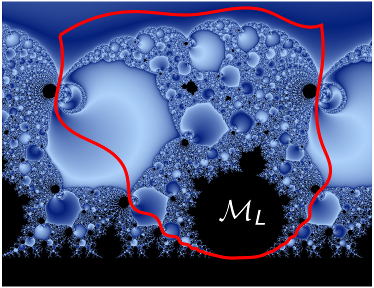

The size of the cage is asymptotically constant, so $\left| q - \frac{2\pi i}{\log \lambda} \right| < C$ for all $\lambda \in \mathcal{L}_{1/q}$.

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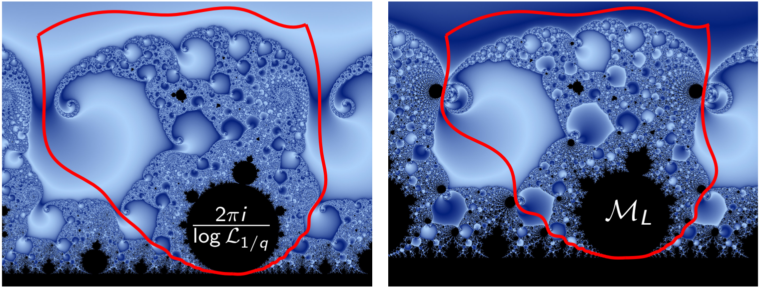

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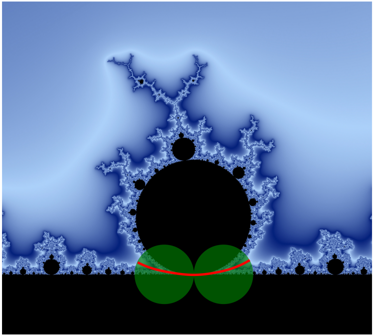
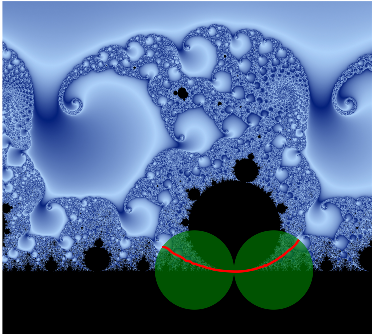
Thus

$$\left|\log \lambda - \frac{2\pi i}{q}\right| = \left|q - \frac{2\pi i}{\log \lambda}\right| \frac{|\log \lambda|}{q} \leq \left|q - \frac{2\pi i}{\log \lambda}\right| \cdot \frac{C}{q^2} \leq \frac{C}{q^2}.$$

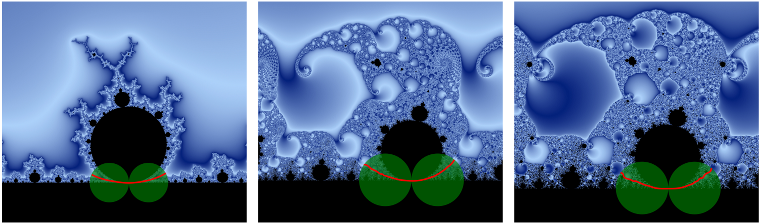
Douady and Hubbard used parabolic implosion to bound the rays landing at $e^{2\pi i p/q}$.





A priori, these bounds are unrelated.

The Lavaurs map L_0 at the root of $\mathcal{M}_L$ has a parabolic fixed point. We can use Douady and Hubbard's argument to bound the enriched parameter rays landing at L_0.



As $f^q_{e^{2\pi i/q}} \to L_0$, we can view the parabolic implosion near $f_{e^{2\pi i/q}}$ as a perturbation of the parabolic implosion near L_0.

It follows that the bounds on the parameter rays landing at $f_{e^{2\pi i/q}}$ converge to the bound on the enriched parameter rays landing at L_0.

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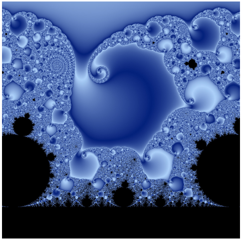
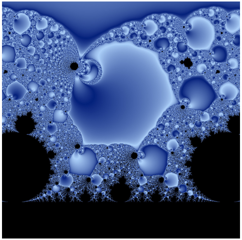
For a rational number p/q, denote $p/q = [0 : a_1, \ldots, a_n] \in \mathcal{Q}_n$ if

$$p/q = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}},$$

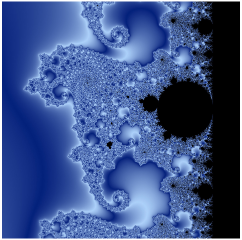
where each a_j is a positive integer.

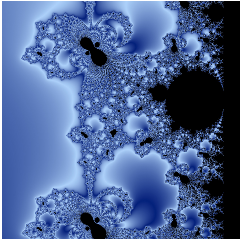
The families of "baby elephants" $p_n/q_n = [0 : n, 2]$





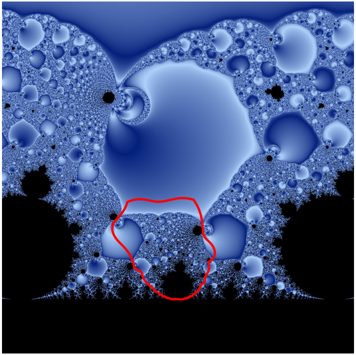
and "seahorses" $p_n/q_n = [0 : 2, n]$

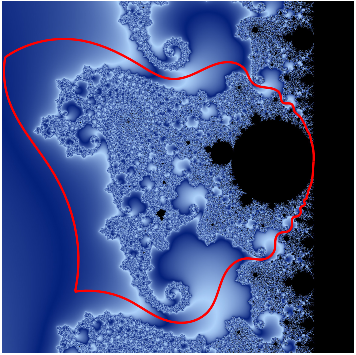




undergo similar parabolic implosions.

Using the same argument, we can control cages for these families and bound the size of the limbs by C_2/q^2 for some $C_2 > 0$.



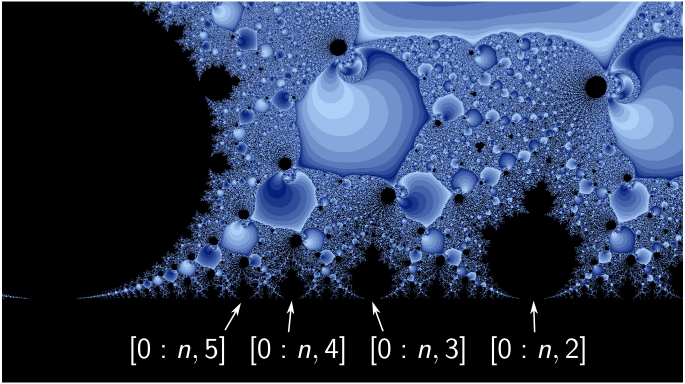


For $p/q \to [0:2,\infty], [0:\infty,2]$ we have C_2/q^2 bound.

For $p/q \to [0:3,\infty], [0:\infty,3]$ we have C_3/q^2 bound.

For $p/q \to [0:4,\infty], [0:\infty,4]$ we have C_4/q^2 bound.

$$\vdots$$



By doing a secondary parabolic implosion, corresponding to $p/q \to [0 : \infty, \infty]$, we can uniformly control C_j.

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Hence there is a uniform constant for all $p/q \in \mathcal{Q}_2$.

Theorem (K, in progress)

For all $n \geq 1$ there exists $\mathcal{C}_n > 0$ such that if $p/q \in \bigcup_{j=1}^{n} \mathcal{Q}_j$ and $\lambda \in \mathcal{L}_{p/q}$, then

$$\left| \log \lambda - \frac{2\pi i p}{q} \right| < \frac{\mathcal{C}_n}{q^2}$$

It remains to understand what happens when the length of the continued fraction expansion of p/q tends to infinity.

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For example, when p/q converges to

1. $[0 : 1, 1, 1, \ldots]$
2. $[0 : \infty, 1, 1, \ldots]$
3. $[0 : \infty, \infty, \infty, \ldots]$
4. $[0 : \infty, 2, \infty, 2, \ldots]$
5. $[0 : \infty, 1, \infty, 1, 1, \infty, 1, 1, 1, \infty, \ldots]$

Conjecture

If $p/q = [0 : a_1, \ldots, a_n]$ *and* $p'/q' = [0 : a'_1, \ldots, a'_n]$ *with* $a_j \leq a'_j$ *for all* $1 \leq j \leq n$, *then*

$$\sup_{\lambda \in \mathcal{L}_{p/q}} q^2 \left| \log \lambda - \frac{2\pi i p}{q} \right| \leq \sup_{\lambda \in \mathcal{L}_{p'/q'}} q'^2 \left| \log \lambda - \frac{2\pi i p'}{q'} \right|$$

It would follow from this conjecture that we only have to consider the case $p/q \to [0 : \infty, \infty, \ldots]$.