# Rational Surface Automorphisms
# Real and Complex dynamics

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# Rational Surfaces

A rational surface is a surface birationally equivalent to a projective plane $\mathbf{P}^2$.

If $X$ is a rational surface, then there is a sequence of blowups of a point :

$$\pi : X = X_n \xrightarrow{\pi_n} X_{n-1} \xrightarrow{\pi_{n-1}} \cdots \xrightarrow{\pi_2} X_1 \xrightarrow{\pi_1} X_0 = \mathbf{P}^2$$

where $\pi_i : X_i \to X_{i-1}$ is a blowup of a point $p_i \in X_{i-1}$.

## Generators of Cohomology group

- $e_0 = \pi^*[\ell]$ where $\ell$ is a line in $\mathbf{P}^2$
- $e_i$ is the cohomology class of the total transform of the exceptional curve over $p_i$.
- $e_0.e_0 = 1, e_i.e_i = -1, e_i.e_j = 0$

If $f$ is an automorphism on $X$,

$$h_{top}(f) = \log \lambda(f)$$

where $\lambda(f) =$ the spectral radius of thee induced action $f^*|H^{1,1}(X, \mathbb{C}) \to H^{1,1}(X, \mathbb{C}))$ is the *dynamical degree* of $f$.

## Theorem (Nagata)

*If $F : X \to X$ is an automorphism on a rational surface $X$ with $\lambda(f) > 1$, then there is a birational map $f : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C})$ such that $f = \pi \circ F \circ \pi^{-1}$*

$$\begin{array}{ccc} X & \xrightarrow{F} & X \\ \downarrow \pi & & \downarrow \pi \\ \mathbf{P}^2(\mathbb{C}) & \overset{f}{\dashrightarrow} & \mathbf{P}^2(\mathbb{C}) \end{array}$$

# Birational Maps on $\mathbf{P}^2$

$f$ is a birational map on $\mathbf{P}^2$:

- $f = [f_1 : f_2 : f_3]$ where $f_i' s$ are homogeneous polynomials of the same degree.
- There is a rational inverse.
- There is no common divisor of $f_1, f_2, f_3$.
- There are points of indeterminacy :

$$\mathcal{I}(f) = \cap_i \{f_i = 0\}$$

- There are exceptional curves which map to points.

$$\mathcal{E}(f) = \{Det(Df) = 0\}$$

### Theorem (Nœther Decomposition)

*If* $f : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C})$ *is a birational map, then* $f$ *can be written as a composition of the Cremona Involution* $J$ *and automorphisms on* $\mathbf{P}^2(\mathbb{C})$*.*

$$f = L_0 \circ J \circ L_1 \circ \cdots \circ L_{k-1} \circ J \circ L_k, \quad L_j \in Aut(\mathbf{P}^2(\mathbb{C}))$$

### The Cremona Involution

$$J : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C})$$

$$J : [x_1 : x_2 : x_3] \mapsto [\frac{1}{x_1} : \frac{1}{x_2} : \frac{1}{x_3}] = [x_2 x_3 : x_1 x_3 : x_1 x_2]$$

$J$ is not defined at three points $[1:0:0], [0:1:0], [0:0:1]$

The Cremona involution $J$ lifts to an automorphism on a rational surface $X = B\ell_P \mathbf{P}^2(\mathbb{C})$ where
$P = \{e_1 = [1:0:0], e_2 = [0:1:0], e_3 = [0:0:1]\}$

If $f = L \circ J$ with $L \in Aut(\mathbf{P}^2(\mathbb{C}))$,

- Each line $\{x_i = 0\}$ maps to a point $p_i = Le_i$
- Each point $e_i$ blows up to a line $L\{x_i = 0\}$.
- If there are three positive integers and a permutation $\sigma \in S_3$ such that

$$\begin{cases} f^{n_i - 1}(p_i) = e_{\sigma(i)}, \text{and} \\ \dim f^j(p_i) = 0 \text{ for all } 0 \leq j \leq n_i - 1 \end{cases}$$

  then $f$ lifts to an automorphism an a rational surface.

# Orbit data of $f = L \circ J$

## Theorem (Nagata)

*Suppose $F : X \to X$ is an automorphism with positive entropy on a rational surface $X$.*
*Then there is a natural identification between the induced action $F^* : H^{1,1}(X) \to H^{1,1}(X)$ and an element of a Coxeter group generated by certain reflections.*

Suppose there are $n_1, n_2, n_3 \in \mathbb{N}$ and a permutation $\sigma \in S_3$ such that

- $f^{n_i}\{x_i = 0\} = e_{\sigma(i)}$
- Dim $(f^j\{x_i = 0\}) = 0$ for all $1 \leq j \leq n_i$, $i = 1, 2, 3$

We call these numerical information **Orbit data** for $f$.

We say the orbit data is *realizable* if there is a birational map with the orbit data.

# Rational surface automorphisms with positive entropy: Construction

## Using invariant curves

- McMullen : Existence of maps with $n_1 = n_2 = 1, n_3 \geq 8$ with a cyclic permutation.
- Diller : construction of (almost all) quadratic maps fixing a cubic with one singular point.
- Blanc : construction of higher degree maps with a curve of fixed points.

## No periodic curve

- Lesieutre : involutions on a blowup of a cubic surface $\in \mathbf{P}^3(C)$ at sixpoints.

# Diller's construction

## Very explicit!!

$n_1 = 3, n_2 = 4, n_3 = 4$ and an identity permutation
Suppose $f$ properly fixes a cubic curve $C$ with a cusp.

$f \;=\; L \circ J \quad \text{where} \quad L \;=$

$$\begin{pmatrix} 2\alpha^5 + \alpha^4 - \alpha - 1 & 2\alpha^4 + 3\alpha^3 + 3\alpha^2 + 2 & -2\alpha^5 - 3\alpha^4 - 3\alpha^3 - 3\alpha^2 - \\ \alpha^5 - \alpha^3 & \alpha^4 + \alpha^3 & -\alpha^5 - \alpha^4 + 1 \\ \alpha^2 - 1 & \alpha + 2 & -\alpha^2 - \alpha \end{pmatrix}$$

total of each row $= 1$

$$J[x1 : x2 : x3] \;=\; [x2x3 : x1x3 : x1x2]$$

where

$$\chi(\alpha) = \alpha^6 - \alpha^4 - \alpha^3 - \alpha^2 + 1 = 0$$

$$f|_C : \gamma(t) \mapsto \gamma(1/\alpha t + b), \quad C_{reg} = \{\gamma(t), t \in \mathbb{C}\}$$

# Diller's construction

## Favorite Example

$n_1 = n_2 = 1, n_3 = 8$ and a cyclic permutation $1 \to 2 \to 3 \to 1$
Suppose $f$ properly fixes a cubic curve $C$ with a cusp.

$$f = L \circ J \quad \text{where}$$

$$L = \begin{pmatrix} 0 & 0 & 1 \\ \alpha^9 - \alpha^6 - \alpha^4 + 1 & 0 & -\alpha^9 + \alpha^6 + \alpha^4 \\ 0 & \alpha^3 + \alpha^2 & -\alpha^3 - \alpha^2 + 1 \end{pmatrix},$$

$$J[x1 : x2 : x3] = [x2x3 : x1x3 : x1x2]$$

where

$$\chi(\alpha) = \alpha^{10} + \alpha^9 - \alpha^7 - \alpha^6 - \alpha^5 - \alpha^4 - \alpha^3 + \alpha + 1 = 0$$

$$f|_C : \gamma(t) \mapsto \gamma(1/\alpha t + b), \quad C_{reg} = \{\gamma(t), t \in \mathbb{C}\}$$

In each orbit data, if it is realizable,

- $\chi(t)$ has exactly one real root $\lambda$ outside the unit circle, exactly one real root $1/\lambda$ inside the unit circle. All other roots are non-real complex numbers of modulus 1.
- Each root $\alpha$ of $\chi(t)$ determines a birational map $f_\alpha$ such that the multiplier of the restriction map is $1/\alpha$ and each coordinate function is in $\mathbb{Z}(\alpha)[x_1, x_2, x_3]$.
- The largest real root $\lambda$ of $\chi(t)$ is the dynamcial degree of $f = \lambda > 1$.
- there are two maps $f_\lambda, f_\lambda^{-1}$ with real coefficients.

$$\begin{cases} f_\lambda : \mathbf{P}^2(\mathbb{C}) \dashrightarrow \mathbf{P}^2(\mathbb{C}) \\ f_\lambda : \mathbf{P}^2(\mathbb{R}) \dashrightarrow \mathbf{P}^2(\mathbb{R}) \end{cases}$$

$f_\lambda$ lifts to an automorphism $F : X \to X$ on a rational surface. We have two related dynamics.

$$\begin{cases} F : X(\mathbb{C}) \to X(\mathbb{C}) \\ F_{\mathbf{R}} : X(\mathbb{R}) \to X(\mathbb{R}) \end{cases}$$

- $h_{top}(F) = \log \lambda$
- $h_{top}(F_{\mathbf{R}}) \leq h_{top}(F)$
- $F_{\mathbf{R}}$ has maximal entropy if $h_{top}(F_{\mathbf{R}}) = h_{top}(F)$

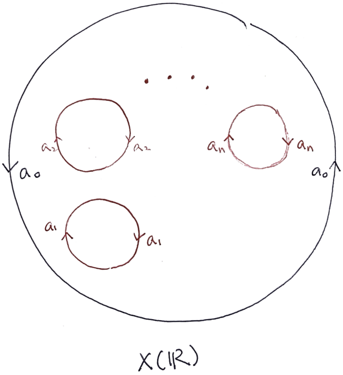
Let $C$ be a curve with one singularity.
For each orbit data (with few exceptions), there exists two real rational surface automorphisms properly fixing $C$. All centers of blowup are in $\mathbf{P}(\mathbb{R})$

# Real Rational Surfaces

$X(\mathbb{R}) =$ a blow up of $\mathbf{P}^2(\mathbb{R})$ along a finite set of points

$$X(\mathbb{R}) = \mathbf{P}^2(\mathbb{R}) \# \mathbf{P}^2(\mathbb{R}) \# \cdots \# \mathbf{P}^2(\mathbb{R})$$

the connected sum of $n+1$ copies of $\mathbf{P}^2(\mathbb{R})$



# Real Rational Surfaces

$$X(\mathbb{R}) = \text{ a blow up of } \mathbf{P}^2(\mathbb{R}) \text{ along a finite set of points}$$

$X(\mathbb{R}) \setminus D =$ a disk with $n+1$ twisted handles attached to the boundary.

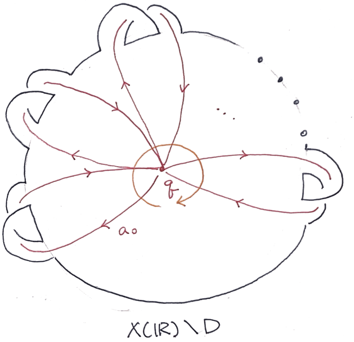



Figure: Each twisted handle is Möbius band

# Real Rational Surfaces

$$\begin{aligned} X(\mathbb{R}) &= \text{ a blow up of } \mathbf{P}^2(\mathbb{R}) \text{ along a set of } n \text{ points} \\ &= \underbrace{\mathbf{P}^2(\mathbb{R})\#\mathbf{P}^2(\mathbb{R})\#\cdots\#\mathbf{P}^2(\mathbb{R})}_{n+1 \text{ copies}} \end{aligned}$$

- $H_1(X(\mathbb{R})) = \langle a_0, a_1, \ldots, a_n | 2a_0 + 2a_1 + \cdots 2a_n = 0 \rangle$
  the finitely generated abelian group with one relation.
- $\pi_1(X(\mathbf{R}), q) = \langle a_0, a_1, \ldots, a_n | a_0^2 a_1^2 \cdots a_n^2 = 1 \rangle$
  the finitely generated NON-abelian group with one relation.
- $X(\mathbb{R})$ is a non-orientable surface!!

# Growth rate of homology classes (join work with J.Diller)

For all choices of realizable orbit data $n_1, n_2, n_3, \in S_3$ (Some can not have a map fixing a cusp cubic), we have the induced action on homology classes

If $F_R$ is associated with the orbit data $n_1 \leq n_2 \leq n_3$ and a cyclic permutation, then the spectral radius of $F_{\mathbf{R}*}$ is the largest modulus of roots of $\chi(t)$

$$\chi(t) = \frac{1}{t+1}\left[\phi(t) - (-t)^{n_1+n_2+n_3+1}\phi(1/t)\right]$$

where

$$\begin{aligned}\phi(t) &= (-1)^{n_1+n_2+n_3+1} + \frac{(-1)^{n_2+n_3+1}(t^2+1)t_1^{n_1}}{t-1} \\ &+ \frac{(-1)^{n_2+n_3}(t^3-t^2+3t+1)t^{n_3}}{t^2-1} + \frac{(t^2+1)t^{n_2}}{t+1}\end{aligned}$$

For an orbit data $n_1, n_2, n_3, \in S_3$, we have a formula for the characteristic polynomial $\chi_{\mathbb{C}}(t)$ of $F^*$ on $H^{1.1}(X(C))$.
For example, if $\sigma$ is a cyclic permutation,

$$\chi_{\mathbb{C}}(t) = t - t^{n_1+n_2+n_3} + (t-1)(t^{n_1}+1)(t^{n_2}+1)(t^{n_3}+1)$$

## Theorem (Diller - K)

- $\chi(F_{\mathbf{R}*})$ *is reciprocal and* $\chi(F_{\mathbf{R}*}) = \chi(F_{\mathbf{R}*}^{-1})$
- *There are rational surface automrophisms* $F$ *such that* $F_{\mathbf{R}}$ *has maximal entropy*
  - $1, 1, n \geq 8$ *with a cyclic permutation*
  - $2, 2, n \geq 6$ *with a cyclic permutation*
  - $2, 3, n \geq 6$ *with the identity permutation*
  - $2, 4, n \geq 5$ *with the identity permutation*
  - $1, 4, n \geq 6$ *with the transposition* $1 \leftrightarrow 2$
  - $1, 5, n \geq 4$ *with the transposition* $1 \leftrightarrow 2$
  - $1, n \geq 8, 2$ *with the transposition* $1 \leftrightarrow 2$
- *There is a complex rational surface automorphism such that all periodic cycles lie in the real locus.*

- There is a family of maps such that their real restriction $F_{\mathbf{R}}$ do not have maximal entropy.

$$\text{e.g.} \quad 3, 3, n \text{ with a cyclic permutation}$$

- We identified five orbit data such that $F^k_{\mathbf{R}*} = Id$ for some k.

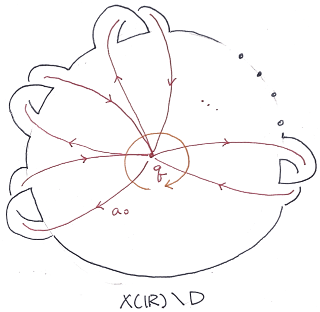
$$\text{e.g.} \quad \begin{array}{llll} 1, 4, 8, & \text{with a cyclic permutation} & : \text{ period} & = 180 \\ 2, 3, 5, & \text{with a cyclic permutation} & : \text{ period} & = 84 \end{array}$$

This does not mean $F_R$ has zero entropy.

We need better estimates

# Growth rate of homotopy classes (joint work with E. Klassen)

Recall that we do have a natural choice for a set of generators for the fundamental group for $X(\mathbb{R})$

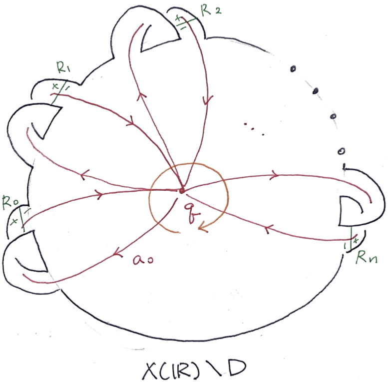



- We want to iterate the map. So let $q =$ a non-cusp fixed point (a saddle point) on the invariant cubic.
- We want to determine the image of each generator under the action $F_{\mathbf{R}*}$

# Reading curves

## Idea

- For each generator $\alpha$, find a curve $X_\alpha$ with $[X_\alpha] = \alpha$.
- *Calculate* the class $[F_{\mathbf{R}}(X_\alpha)]$ of the image curve
- With exceptional triangles and the invariant cubic, it is not hard to find $F_{\mathbf{R}}(X_\alpha)$



For each $i$, let $R_{\alpha_i}$ denote a line segment joining the two sides of the boundary of the handle traversed by the generator $\alpha_i$. Once we put the removed open disk $\Delta$ back, we can extend each line segment $R_{\alpha_i}$ to a simple closed curve (which we continue to denote by $R_{\alpha_i}$) with base point $\star \in \Delta$ such that

- $R_{\alpha_i}$ is a simple closed curve for all $i$,
- For each $i$, $R_{\alpha_i}$ intersects exactly one generator $\alpha_i$, and
- $\{R_{\alpha_i}\}$ are pairwise disjoint on $X \setminus \{\star\}$.

The curves $\{R_{\alpha_i}\}$ are referred to as *reading curves* for the generators $\{\alpha_i\}$.

With E. Klassen, we compute the induced $\pi_1$ action for real diffeomorphisms associated with birational maps fixing a cusp cubic with orbit data $n_1, n_2, n_3$ and a cyclic permutation.

- The image a generator under the induced action depends on the location of the base point on the cubic.
- There are only 6 different possibilities.

Eg. For $n_1 = n_2 = 1, n_3 = 8$

$$
\begin{aligned}
(f_{\mathbf{R}}^{-2})_* \quad : e &\mapsto e a_8^2 a_7^2 a_6^2 a_5^2 c_1 b_1 a_2^2 a_1 e \\
a_8 &\mapsto e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_7 &\mapsto b_1 a_2^2 a_1 e a_8^{-1} e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_6 &\mapsto b_1 a_2^2 a_1 e a_8^2 a_7^{-1} a_8^{-2} e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_5 &\mapsto b_1 a_2^2 a_1 e a_8^2 a_7^2 a_6^{-1} a_7^{-2} a_8^{-2} e^{-1} a_1^{-1} a_2^{-2} b_1^{-1} \\
a_4 &\mapsto b_1^{-1} a_3^{-2} a_4^{-2} c_1^{-1} a_5 c_1 a_4^2 a_3^2 b_1 \\
a_3 &\mapsto b_1^{-1} a_3^{-2} a_4^{-1} a_3^2 b_1 \\
a_2 &\mapsto e a_8^2 a_7^2 a_6^2 a_5^2 c_1 a_3 c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
a_1 &\mapsto e a_8^2 a_7^2 a_6^2 a_5^2 c_1 b_1 a_2^{-1} b_1^{-1} c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
b_1 &\mapsto b_1^{-1} a_3^{-2} c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
c_1 &\mapsto b_1 a_2^2 a_1 e a_8^2 a_7^2 a_6^2 c_1 a_4^2 a_3^2 b_1.
\end{aligned}
$$

$$\pi_1(X(\mathbb{R}), q) = \langle e, a_i, b_1, c_1 | e^2 a_8^2 a_7^2 a_6^2 a_5^2 c_1^2 a_4^2 a_3^2 b_1^2 a_2^2 a_1^2 = 1 \rangle$$

- We want to Calculate the growth rate.

$$\rho(f_{\mathbf{R}*}|_{\pi_1(X(\mathbb{R}))}) := \sup_{g\in G}\{\limsup_{n\to\infty}(\ell_G(f^n_{\mathbf{R}*}g))^{1/n}\}$$

  where $G$ is a set of generators, and $\ell_G(w)$ is the minimal length among all words representing $w$ with respect to $G$.

- $\pi_1(X(\mathbb{R}), q)$ is a non-abelian group with one relator.

- We want to ~~Calculate~~ Estimate the growth rate.

$$\rho(f_{\mathbf{R}*}|_{\pi_1(X(\mathbf{R}))}) := \sup_{g\in G}\{\limsup_{n\to\infty}(\ell_G(f_{\mathbf{R}*}^n g))^{1/n}\}$$

  where $G$ is a set of generators, and $\ell_G(w)$ is the minimal length among all words representing $w$ with respect to $G$.

- $\pi_1(X(\mathbb{R}), q)$ is a non-abelian group with a relator.
- For $\alpha \in \pi_1(X(\mathbb{R}), q)$, the minimum length $\ell_G(\alpha)$ is obtained by removing more than half-relators.

Theorem (E. Klassen -K)

*There are real quadratic rational surface automorphisms with maximal entropy such that the growth rate of homology classes is strictly smller than the growth rate of homotopy classes*

$$\begin{aligned} \textit{e.g.} \quad & n_1 = 1, n_2 = 3, n_3 = 9 \quad \textit{with a cylcic permutation} \\ & n_1 = 1, n_2 = 4, n_3 = 8 \quad \textit{with a cylcic permutation} \\ & n_1 = 1, n_2 = 4, n_3 = 5 \quad \textit{with a cylcic permutation} \\ & n_1 = 1, n_2 = 5, n_3 = 6 \quad \textit{with a cylcic permutation} \end{aligned}$$

The exponential homology growth rates for the first two cases above are zero.

# $1, 1, 8$ cyclic case

Fist, we examined the iterations under $f_{\mathbf{R}*}$

$\frac{1}{a4} ** \frac{1}{a4} ** \frac{1}{c1} ** \frac{1}{a5} ** \frac{1}{a6} ** \frac{1}{a6} ** \frac{1}{a7} ** \frac{1}{a7} ** \frac{1}{a8} ** \frac{1}{a8} ** \frac{1}{e} ** \frac{1}{a2} ** \frac{1}{b1} ** \frac{1}{a8} **$
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# 1, 1, 8 cyclic case

## Something is going on here!!

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1/a4 ** 1/a4 ** 1/c1 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a8 **
1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 **
1/b1 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a3 ** 1/a4 ** 1/c1 **
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1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 **
1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a3 ** 1/a4 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 **
1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a3 **
1/a4 ** 1/c1 ** 1/a5 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/b1 **
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1/a4 ** 1/a4 ** 1/c1 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a7 **
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1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 **
1/a3 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a1 **
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1/a8 ** 1/a8 ** 1/e ** 1/b1 ** 1/a3 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 **
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1/a3 ** 1/a3 ** 1/a4 ** 1/a4 ** 1/c1 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 **
1/b1 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a3 ** 1/a4 ** 1/c1 **
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1/a3 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a3 ** 1/a4 ** 1/c1 **
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1/a5 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a6 **
1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 **
1/b1 ** 1/b1 ** 1/a3 ** 1/a3 ** 1/a4 ** 1/c1 ** 1/a5 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 **
1/e ** 1/b1 ** 1/a3 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a8 **
1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a3 ** 1/a4 ** 1/c1 ** 1/a5 ** 1/a5 ** 1/a6 ** 1/a6 **
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1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 ** 1/b1 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 **
1/b1 ** 1/b1 ** 1/a3 ** 1/a3 ** 1/a4 ** 1/a5 ** 1/a6 ** 1/a6 ** 1/a7 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a2 **
1/b1 ** 1/a8 ** 1/e ** 1/a1 ** 1/a2 ** 1/a2 ** 1/b1 ** 1/b1 ** 1/a3 ** 1/a7 ** 1/a8 ** 1/a8 ** 1/e ** 1/a1 **
```

There are ten reduced elements

$$\Gamma = \{\gamma_i \in \pi_1(X(\mathbb{R})), 1 \leq i \leq 10\}$$

and a subset $A$ of the set of ordered pairs

$$A \subset \{(i,j) | 1 \leq i, j, \leq 10\}$$

such that

- There are no relations between $\gamma_i$'s
- We say. $\gamma$ is $A$-admissible if $\gamma = \gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_n}$ with $(i_j, i_{j+1}) \in A, j = 1, \dots, n \pmod{n}$
- If $\gamma$ is $A$-admissible then $f_{\mathbf{R}*}\gamma$ is also admissible up to cyclic permutation.

The set $\Gamma_A$ of admissible cyclic words is invariant under $f_{\mathbf{R}*}\gamma$.

$$
\begin{aligned}
\gamma_1 &= a_2^{-1} b_1^{-1} a_8^{-1} e^{-1}, & \gamma_2 &= a_2^{-1} b_1^{-1} a_7^{-1} a_8^{-2} e^{-1} \\
\gamma_3 &= b_1^{-1} a_3^{-1} a_6^{-1} a_7^{-2} a_8^{-2} e^{-1} \\
\gamma_4 &= a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-1} a_7^{-1} a_8^{-2} e^{-1} \\
\gamma_5 &= a_2^{-1} b_1^{-2} a_3^{-1} a_6^{-1} a_7^{-2} a_8^{-2} e^{-1} \\
\gamma_6 &= b_1^{-1} a_3^{-1} a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
\gamma_7 &= a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-1} a_6^{-1} a_7^{-2} a_8^{-2} e^{-1} \\
\gamma_8 &= a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-2} a_4^{-1} a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
\gamma_9 &= a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-2} a_4^{-2} c_1^{-1} . a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
\gamma_{10} &= a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-2} a_4^{-1} c_1^{-1} a_5^{-2} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1} \\
\zeta_1 &= a_2 a_1, & \zeta_2 &= b_1 a_2^2 a_1 \\
\mu_1 &= a_1^{-1} a_2^{-2} b_1^{-2} a_3^{-1} a_5^{-1} a_6^{-2} a_7^{-2} a_8^{-2} e^{-1}
\end{aligned}
$$

$$A = \{(1,4),(1,7),(1,10),(2,8),(2,10),(3,2),(3,8),(3,9),(4,10),(5,2),(5,8),(5,9),(6,1),(6,5),(7,2),(8,1),(9,1),(9,2),(10,3),(10,5),(10,6)\}.$$

$$\begin{aligned}
f_{\mathbf{R}*}^{-2}\gamma_1 &= (e^{-1}a_1^{-1})\gamma_{10}\zeta_2(a_1e), & f_{\mathbf{R}*}^{-2}\gamma_2 &= (e^{-1}a_1^{-1})\gamma_{10}\zeta_1(a_1e)\\
f_{\mathbf{R}*}^{-2}\gamma_3 &= (e^{-1}a_1^{-1})\gamma_9\gamma_1(a_1e), & f_{\mathbf{R}*}^{-2}\gamma_4 &= (e^{-1}a_1^{-1})\mu_1\zeta_1(a_1e)\\
f_{\mathbf{R}*}^{-2}\gamma_5 &= (e^{-1}a_1^{-1})\gamma_8\gamma_1(a_1e), & f_{\mathbf{R}*}^{-2}\gamma_6 &= (e^{-1}a_1^{-1})\gamma_9\gamma_2(a_1e)\\
f_{\mathbf{R}*}^{-2}\gamma_7 &= (e^{-1}a_1^{-1})\mu_1\gamma_1(a_1e), & f_{\mathbf{R}*}^{-2}\gamma_8 &= (e^{-1}a_1^{-1})\gamma_7\gamma_2(a_1e)\\
f_{\mathbf{R}*}^{-2}\gamma_9 &= (e^{-1}a_1^{-1})\gamma_4(a_1e), & f_{\mathbf{R}*}^{-2}\gamma_{10} &= (e^{-1}a_1^{-1})\gamma_7(a_1e)
\end{aligned}$$

$$\gamma_3 = \zeta_2\gamma_7, \quad \gamma_5 = \zeta_1\gamma_7, \quad \text{and} \quad \gamma_6 = \zeta_2\mu_1$$

Since there is no relation, We can get the length growth by counting number of $\gamma_i$'s

$$V = \mathbb{R}^{10} \quad \text{with a basis } \{\gamma_1, \dots \gamma_{10}\}$$
$$W = \mathbb{R}^{|\Gamma_1|+|K|} \quad \text{with a basis } \Gamma_1 \cup K$$

## Theorem (Klassen-K)

*There are two linear maps $S, T : V \to W$ such that*

$$T \circ f_{\mathbf{R}*}|_{\Gamma_A} = S$$

*and there is a unique vector $v \in V$ such that $Sv = \lambda Tv$ where $\lambda$ is the dynamical degree of $f$*

We observed that the same phenomenon occurs in other orbit data.

$$T \circ f_{\mathbf{R}*}|_{\Gamma_A} = S$$

where $S, T : V \to W$ are linear.
Is $f_{\mathbf{R}*}|_{\Gamma_A}$ (almost) linear?

There are ten A-admissible words $S = \{s_1, \ldots, s_{10}\}$ where the action. $f_{\mathbf{R}*}|_{SP^+S}$ on the positive span of $S$ is "Linear"

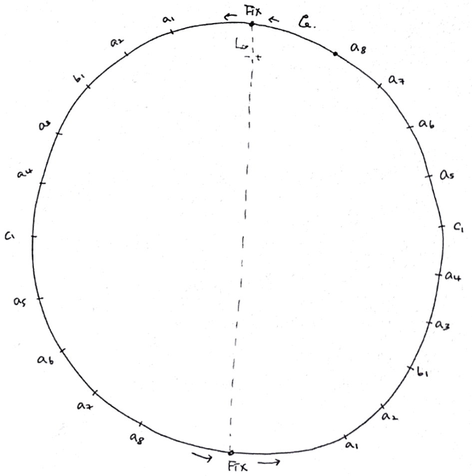
$$s_1 = \gamma_1\gamma_{10}\gamma_5 \cdots, \gamma_8$$

$$\vdots$$

$$
\begin{aligned}
f_{\mathbf{R}*}|_{SP^+S} : \; & s_1 \mapsto s_1 + s_2 + s_3 + s_4 + s_5 \\
& s_2 \mapsto s_1 + s_6 + s_7 \\
& s_3 \mapsto s_1 + s_2 + s_8 + s_3 + s_4 + s_5 \\
& s_4 \mapsto s_1 + s_2 + s_4 + s_{10} + s_6 + s_9 \\
& s_5 \mapsto s_1 + s_4 + s_5 + s_7 \\
& s_6 \mapsto s_1 + s_6 + s_9 + s_5 + s_7 \\
& s_7 \mapsto s_1 + s_2 + s_8 + s_9 \\
& s_8 \mapsto s_1 + s_2 + s_4 + s_{10} + s_6 + s_9 + s_5 + s_7 \\
& s_9 \mapsto s_1 + s_2 + s_8 + s_3 + s_4 + s_{10} + s_6 + s_9 \\
& s_{10} \mapsto s_1 + s_4 + s_5
\end{aligned}
$$

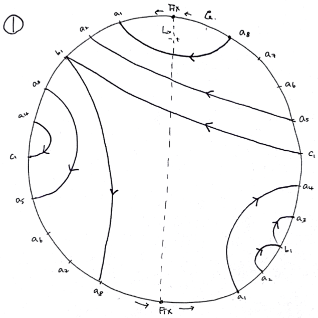
# Sequence of Admissible Words

Since the cubic is invariant and all base loci lie between two fixed points on the cubic, the $X(\mathbb{R})$ can be drawn as following:

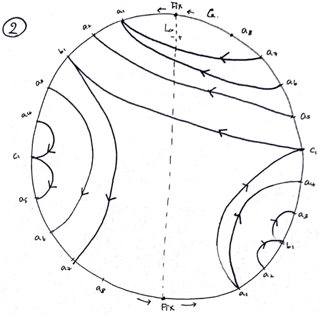


# Sequence of Admissible Words

Starting with an admissible word $\gamma$, under $f^2_{\mathbf{R}*}$ we see



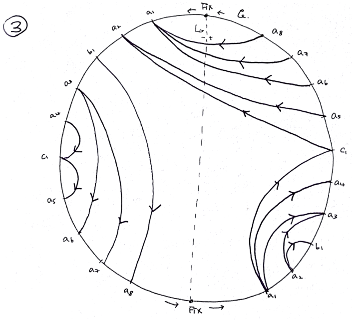
$\gamma_1 \gamma_{10} \gamma_6$



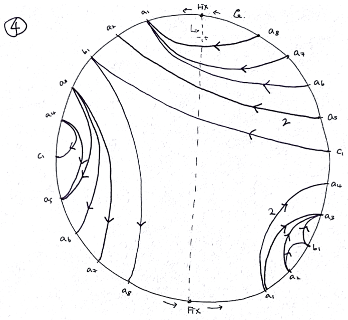
$\gamma_{10} \gamma_3 \gamma_9 \gamma_2$

# Sequence of Admissible Words

Starting with an admissible word $\gamma$, under $f^2_{\mathbf{R}*}$ we see



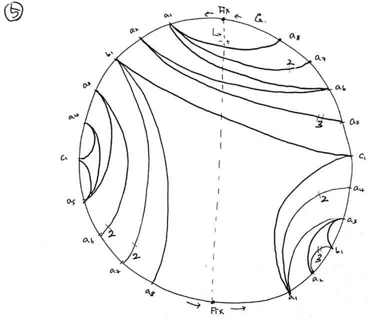

$\gamma_1 \gamma_4 \gamma_{10} \gamma_5 \gamma_9$



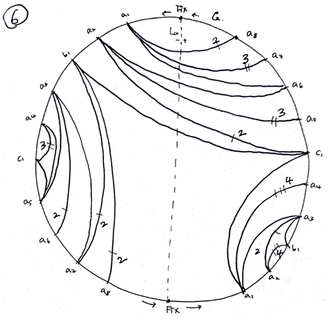

$\gamma_1 \gamma_4 \gamma_{10} \gamma_6 \gamma_5 \gamma_8$

# Sequence of Admissible Words

Starting with an admissible word $\gamma$, under $f_{\mathbf{R}*}^2$ we see



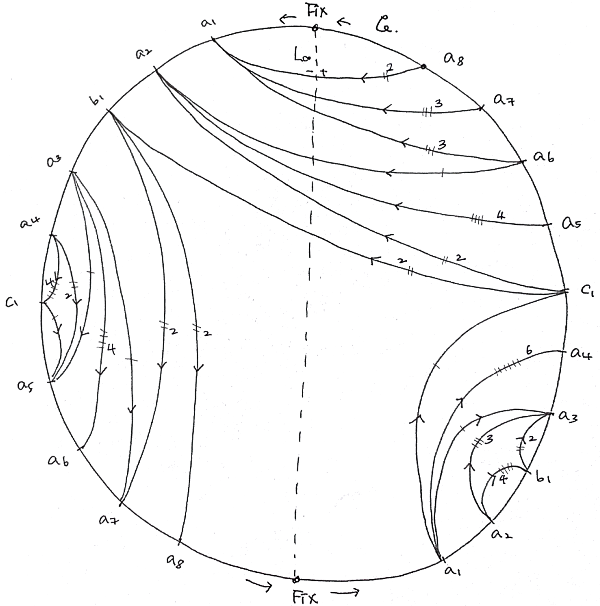

$\gamma_1\gamma_7\gamma_2\gamma_{10}\gamma_6\gamma_5\gamma_9\gamma_2\gamma_8$




$\gamma_1\gamma_4\gamma_{10}\gamma_5\gamma_2\gamma_{10}\gamma_6\gamma_1\gamma_{10}\gamma_5$

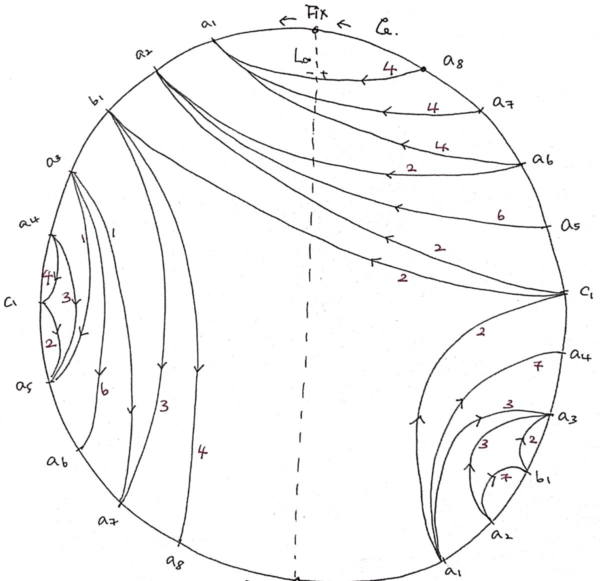
# Sequence of Admissible Words

Starting with an admissible word $\gamma$, under $f_{\mathbf{R}*}^2$ we see

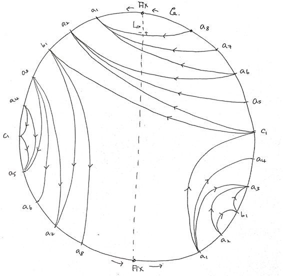


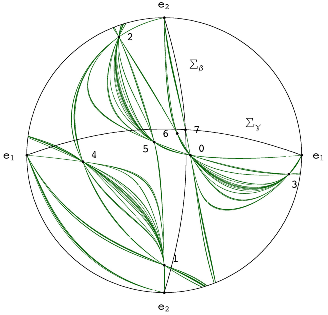
# Sequence of Admissible Words

Starting with an admissible word $\gamma$, under $f^2_{\mathbf{R}*}$ we see



It "seems" that with any choice of an initial word, we get the same picture with increasing weight (the number of arcs).






- [Kitchens-Roeder] Is this a Plykin attractor?
  - We know this grows exponentially.
  - There is a repelling fixed point whose basin has full Area. [Bedford-K]
- Are we seeing a hyperbolic set?
- What do you see?

# Thank you!!

감사합니다!