Dynamics of groups of birational automorphisms of cubic surfaces and Fatou/Julia decomposition for Painlevé 6

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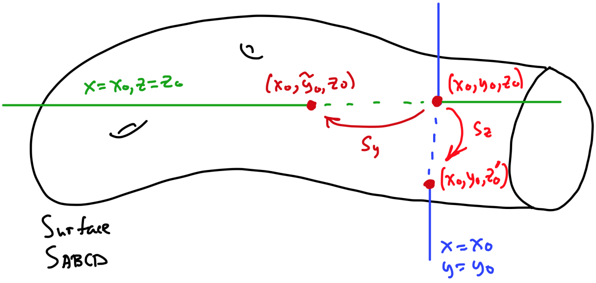

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Every line parallel to the x-axis intersects $S_{A,B,C,D}$ at two points and one can therefore define an involution $s_x : S_{A,B,C,D} \to S_{A,B,C,D}$ that switches them:

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Two further involutions $s_y : S_{A,B,C,D} \to S_{A,B,C,D}$ and $s_z : S_{A,B,C,D} \to S_{A,B,C,D}$ are defined analogously:

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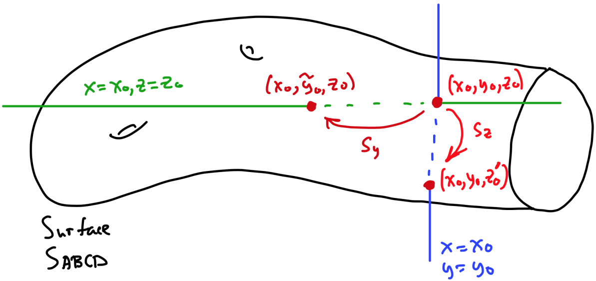

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Consider the group

$$\Gamma^* \equiv \Gamma^*_{A,B,C,D} := \langle s_x, s_y, s_z \rangle \leq \mathrm{Aut}(S_{A,B,C,D}),$$

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Consider also the finite index subgroup

$$\Gamma \equiv \Gamma_{A,B,C,D} := \langle g_x, g_y, g_z \rangle < \Gamma^*,$$

where $g_x := s_z \circ s_y$, $g_y := s_x \circ s_z$, and $g_z := s_y \circ s_x$.

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See, e.g. Cantat Bers and Hénon, Painlevé and Schrödinger 2009.

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Open sets where the group action is properly discontinuous for certain parameters: Hu-Tan-Zhang 2018, Bowditch 1998.

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More on that later...

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It is pretty easy to show that this holds for the Picard Parameters $(A, B, C, D) = (0, 0, 0, 4)$. Does it hold for any others?

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The core of this paper lies in the interplay between these two dichotomies.

Moreover, the techniques we develop are quite flexible and should be useful for several other problems.

Fatou-Julia dichotomy

The *Fatou set* of the group action Γ is defined as

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Julia set is always non-empty, but Fatou set can be empty for some parameters.

Basic results on the Julia set:

Theorem A: For any parameters (A, B, C, D) there exists $p \in \mathcal{J}_{A,B,C,D}$ such that

$$\overline{\Gamma(p)} = \mathcal{J}_{A,B,C,D},$$

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Theorem B: For any choice of parameters (A, B, C, D) there is a dense set $\mathcal{J}^{\#}_{A,B,C,D} \subset \mathcal{J}_{A,B,C,D}$ such that for every $p \in \mathcal{J}^{\#}_{A,B,C,D}$ there exists $\gamma \in \Gamma$ such that $\gamma(p) = p$ and

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It turns out that the set of points p with a hyperbolic stabilizer $\gamma \in \Gamma$ may have closure strictly smaller than $\mathcal{J}_{A,B,C,D}$.

Basic results on the Julia set:

Theorem A: For any parameters (A, B, C, D) there exists $p \in \mathcal{J}_{A,B,C,D}$ such that

$$\overline{\Gamma(p)} = \mathcal{J}_{A,B,C,D},$$

i.e., there is a dense orbit of Γ in $\mathcal{J}_{A,B,C,D}$.

Theorem B: For any choice of parameters (A, B, C, D) there is a dense set $\mathcal{J}^{\#}_{A,B,C,D} \subset \mathcal{J}_{A,B,C,D}$ such that for every $p \in \mathcal{J}^{\#}_{A,B,C,D}$ there exists $\gamma \in \Gamma$ such that $\gamma(p) = p$ and

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Theorem C: For any parameters A, B, C, D the Julia set $\mathcal{J}_{A,B,C,D}$ is connected.

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Studied by Dubrovin and Mazzocco 2000. Seem to play a significant role in several problems related to Mathematical-Physics and, in particular, on the study of Frobenius manifolds.

Results that hold for some parameters:

Theorem D: For the Picard Parameters $(A, B, C, D) = (0, 0, 0, 4)$ we have:

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Key idea for proof of Theorem E comes from work of Hu-Tan-Zhang 2018 and Bowditch 1998.

Locally discrete/non-discrete dichotomy

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For an action by a finite dimensional Lie group, local non-discreteness on some open set implies that the corresponding sequence of elements converges to the identity on all of M, i.e. that the action is globally non-discrete.

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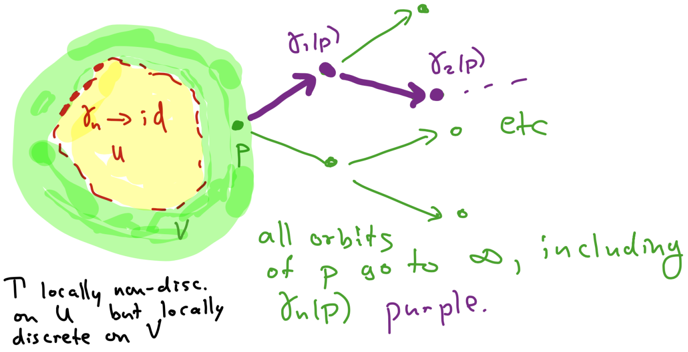
However, in our context the non-linearity of the mappings allow for local non-discreteness to occur on a proper open subset $U \subset M$.



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$\mathcal{N}_{A,B,C,D} =$
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$\mathcal{N}_{A,B,C,D}$ is open, $\mathcal{D}_{A,B,C,D}$ is closed, and both are invariant under $\Gamma_{A,B,C,D}$.

Co-existence of local discreteness and non-discreteness

Theorem F: There is an open neighborhood $\mathcal{P} \subset \mathbb{C}^4$ of the Markoff Parameters of the Dubrovin-Mazzocco Parameters such that both the locally non-discrete locus $\mathcal{N}_{A,B,C,D}$ and the locally discrete locus $\mathcal{D}_{A,B,C,D}$ are non-empty.

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Corollary to Theorem F: For every choice of parameters $(A, B, C, D) \in \mathcal{P}$ there is a set

$$\mathcal{B}_{A,B,C,D} \subset \partial\mathcal{N}_{A,B,C,D} = \partial\mathcal{D}_{A,B,C,D}$$

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We expect $\mathcal{B}_{A,B,C,D}$ to be “fractal” for typical parameters.

Local non-discreteness vs. producing dense orbits

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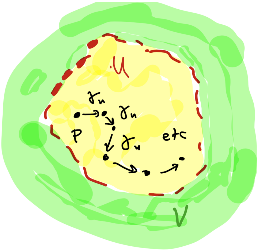

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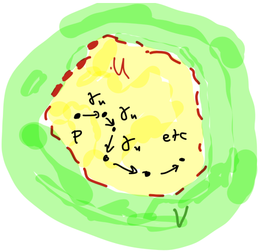

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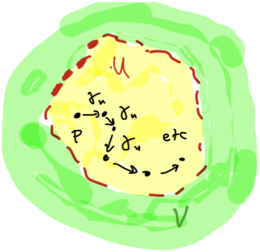

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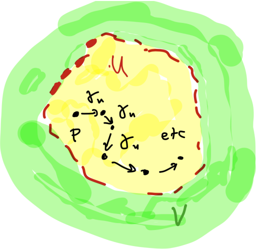

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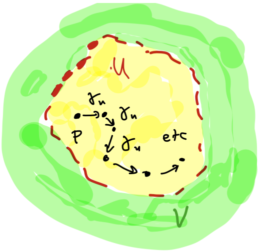

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Instead, we will use this to prove that $U \subset \mathcal{J}_{A,B,C,D}$ and then appeal to Theorem A about dense orbits in $\mathcal{J}_{A,B,C,D}$.

Co-existence of Julia set with interior and Fatou set

Let $\mathcal{P} \subset \mathbb{C}^4$ be the open neighborhood of the Markoff Parameters and of the Dubrovin-Mazzocco parameters given in Theorem F.

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Proposition: There is a countable union of real-algebraic hypersurfaces $\mathcal{H} \subset \mathbb{C}^4$ such that this if $(A, B, C, D) \in \mathbb{C}^4 \setminus \mathcal{H}$ then

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Co-existence of Julia set with interior and Fatou set

Let $\mathcal{P} \subset \mathbb{C}^4$ be the open neighborhood of the Markoff Parameters and of the Dubrovin-Mazzocco parameters given in Theorem F.

Let $U \subset \mathcal{N}_{A,B,C,D}$ be the explicit set constructed in the proof of Theorem F.

Proposition: There is a countable union of real-algebraic hypersurfaces $\mathcal{H} \subset \mathbb{C}^4$ such that this if $(A, B, C, D) \in \mathbb{C}^4 \setminus \mathcal{H}$ then

(P) any fixed point of any $\gamma \in \Gamma_{A,B,C,D} \setminus \{\mathrm{id}\}$ is in $\mathcal{J}_{A,B,C,D}$.

Theorem G: For any $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ we have:

$$U \subset \mathcal{J}_{A,B,C,D} \qquad \text{and} \qquad V_\infty \subset \mathcal{F}_{A,B,C,D}.$$

Corollary to Theorem G and Theorem A: For any $(A, B, C, D) \in \mathcal{P} \setminus \mathcal{H}$ there is a point $p \in U \subset \mathcal{J}_{A,B,C,D}$ such that

$$U \subset \overline{\Gamma(p)},$$

i.e. the orbit of p has closure of real dimension four.

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A motivating question: For what parameters is there an initial condition $p = (x_0, y_0, z_0) \in S_{A,B,C,D}$ whose orbit is dense in $S_{A,B,C,D}$?

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A new question: Are the Picard parameters $(A, B, C, D) = (0, 0, 0, 4)$ the only parameters for which there is an initial condition $p = (x_0, y_0, z_0) \in S_{A,B,C,D}$ whose orbit is dense in $S_{A,B,C,D}$?

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All of the mass tends to infinity.

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Proposition 1: Given $\epsilon > 0$, there is $K = K(\epsilon) > 0$ such that, if

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then the following hold:

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$$\sup_{p \in B_{\epsilon/2}(0)} \|\gamma(p) - p\| \leq \frac{K}{2^n}.$$

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Remark: It is "easy" to prove that any Fatou component for $\Gamma_{A,B,C,D}$ is Kobayashi hyperbolic.

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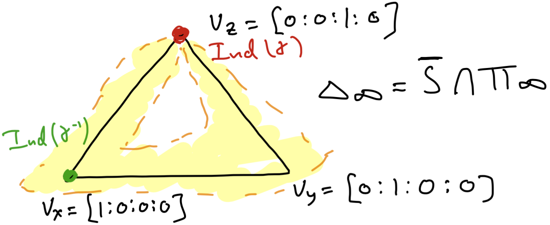

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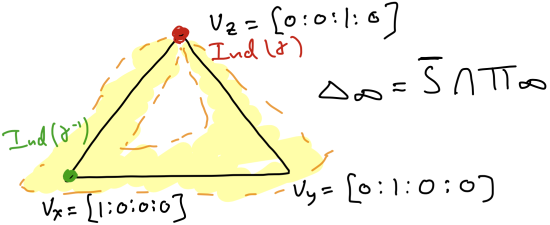

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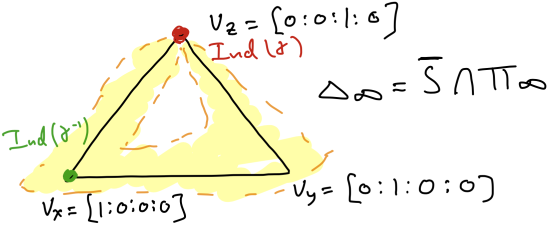

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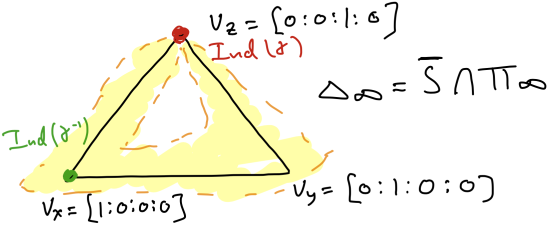

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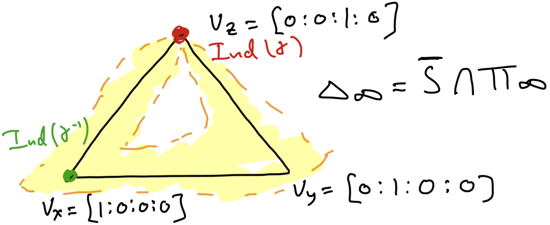

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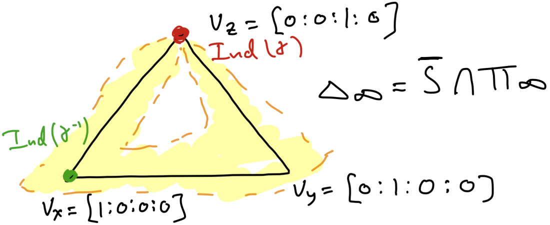

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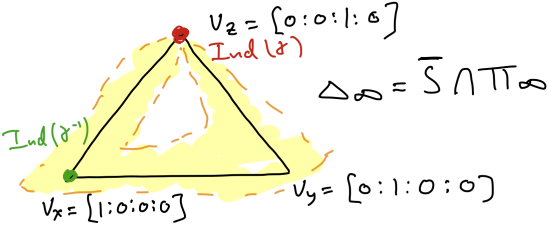

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Points in a (yellow) neighborhood of $\Delta_\infty \setminus \mathrm{Ind}(\gamma)$ are sent much closer to Δ_∞ by γ.

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Meanwhile Property (A) gives us rich enough combinatorics at infinity so that we can choose the γ_n so that if $q \in V$ is near infinity then $\gamma_n(q) \to \infty$.

Strategy of proof of Theorem G, Step 1:

Let $U \subset \mathcal{N}_{A,B,C,D}$ be the open set constructed in proof of Theorem F.
Used the Ghys strategy to construct it, so have plenty of mappings close to the identity on U.

Theorem H: Suppose that for some parameters A, B, C there is a point $p \in \mathbb{C}^3$ and $\epsilon > 0$ such that for any two vertices $v_i \neq v_j \in \mathcal{V}_\infty$, $i \neq j$, there is a hyperbolic element $\gamma_{i,j} \in \Gamma_{A,B,C}$ satisfying:

(A) $\mathrm{Ind}(\gamma_{i,j}) = v_i$ and $\mathrm{Attr}(\gamma_{i,j}) = v_j$, and

(B) $\sup_{z \in B_\epsilon(p)} \|\gamma_{i,j}(z) - z\| < K(\epsilon)$. (Constant from Prop. 1)

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Contradiction! Thus U does not intersect any unbounded Fatou component.

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$\square$ Theorem G (supposing Theorem K).

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This allows us to derive a contradiction to the fact that the Julia set is connected (Theorem C).

Thank you for listening!

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