Upper bounds for the moduli of polynomial-like maps

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https://arxiv.org/abs/2202.01282

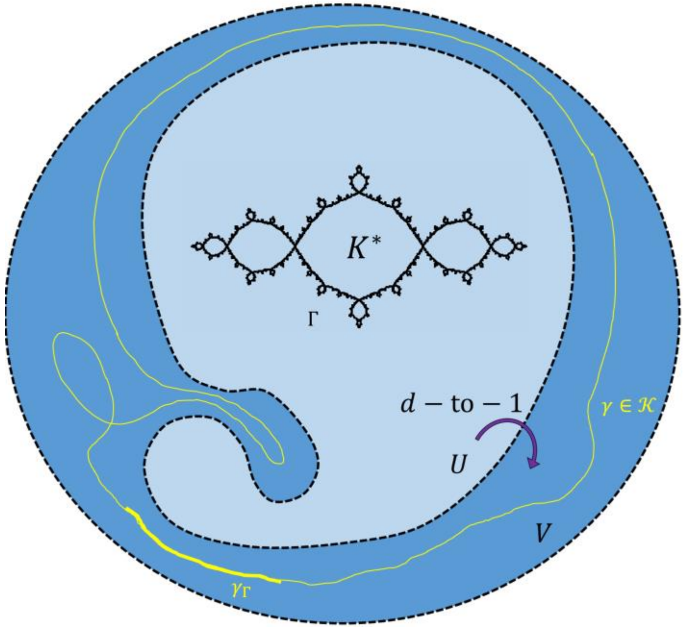
https://arxiv.org/abs/2205.03157

1. Background

Renormalization and the PLY inequality

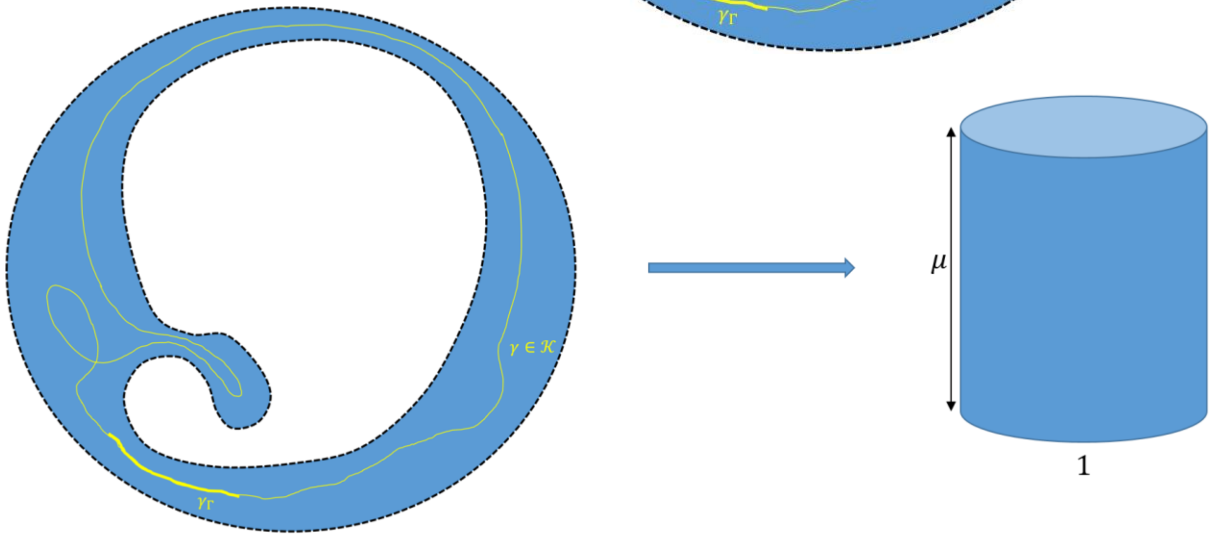
Polynomial-like maps (Douady-Hubbard)

- Let U, V be Jordan disks with $\overline{U} \subset V$, and $f: U \to V$ a degree d proper holomorphic map. Then f is called a polynomial-like (PL) map.



- $d = 1$ means a repelling fixed point.

- The fundamental annulus $A = V \setminus \overline{U}$ is isomorphic to a Euclidean cylinder of height μ and circumference 1. Set $\mathrm{mod}(A) = \mu$. This is the modulus of the PL map f.



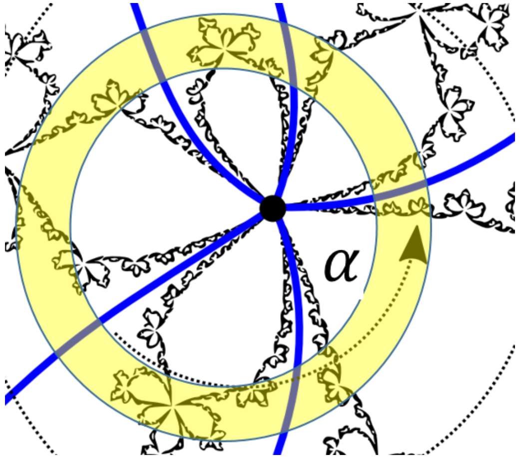
Straightening and renormalization

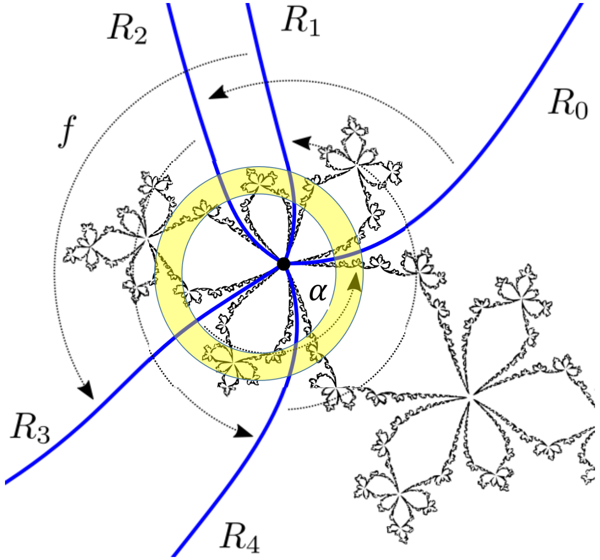
- **Douady-Hubbard straightening theorem**: *if $f: U \to V$ is a PL map with PL set $K^* = \{z \in U \mid f^n(z) \in U \ \forall n \geq 0\}$, then f is topologically (in fact, hybrid) conjugate to a polynomial.*
- **Renormalization**: a rational function $f: \mathbb{P}^1 \to \mathbb{P}^1$ is renormalizable of period p if $f^p: U \to V$ is PL with connected PL set K^*, and $K^* \neq K(f)$.
- **A priori bounds**: lower bounds on $\operatorname{mod}(V \setminus U)$.
- Typical applications: local connectivity, triviality of fibers, zero measure, ...

Pommerenke-Levin-Yoccoz (PLY) inequality

- **Theorem (PLY).** *Let α be a repelling fixed point of a degree d polynomial P with connected filled Julia set $K(P)$. If $K(P) \setminus \{\alpha\}$ consists of q components, then* $\log|f'(\alpha)| \leq \frac{2\log d}{q}$.
- Here $\log|f'(\alpha)|/2\pi$ is the modulus of a fundamental annulus around α, and q can also be defined as the number of wedges formed by pairs of adjacent external rays landing at α.

Modified figure from
Alex Kapiamba's preprint
https://arxiv.org/abs/2103.03211



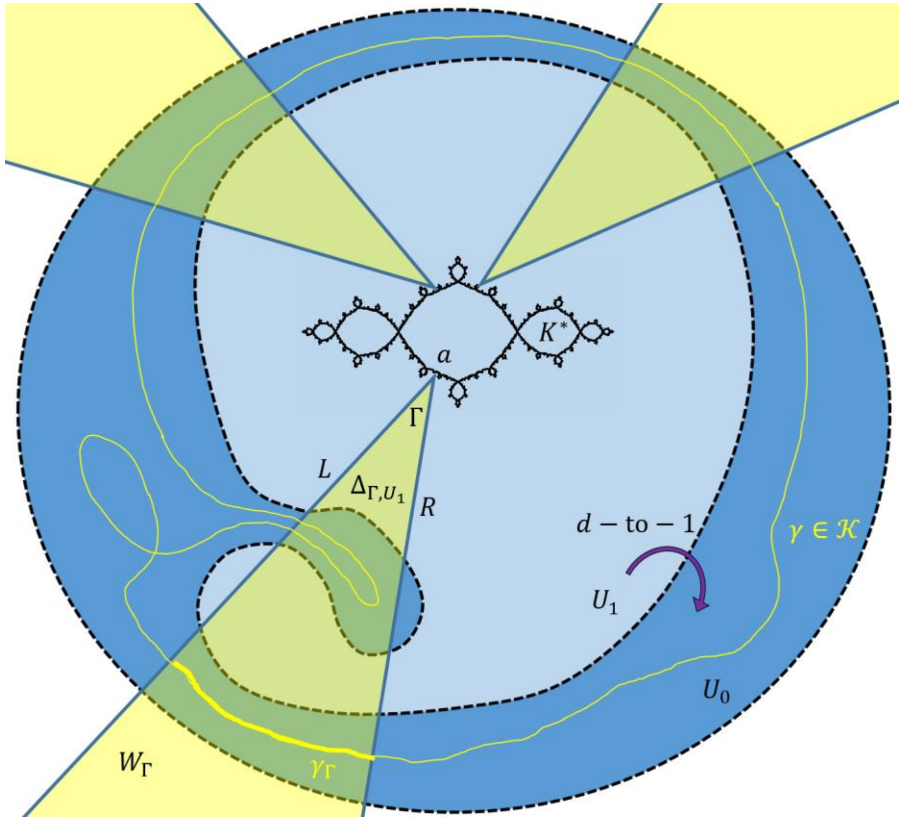


2. Statements of results

Upper bounds on the moduli of renormalizations

Wedges and an upper bound on $\mathbf{mod}(V \setminus \overline{U})$

- Let R, L be external rays for a degree d polynomial P landing at a. Then $\Gamma = R \cup L \cup \{a\}$ is a cut. It is periodic if $f^m(\Gamma) = \Gamma$ for some minimal m called the period of Γ.
- Suppose that all $\Gamma_i = P^i(\Gamma)$ (where $i = 0, \dots, m-1$) bound wedges W_i that are pairwise disjoint.



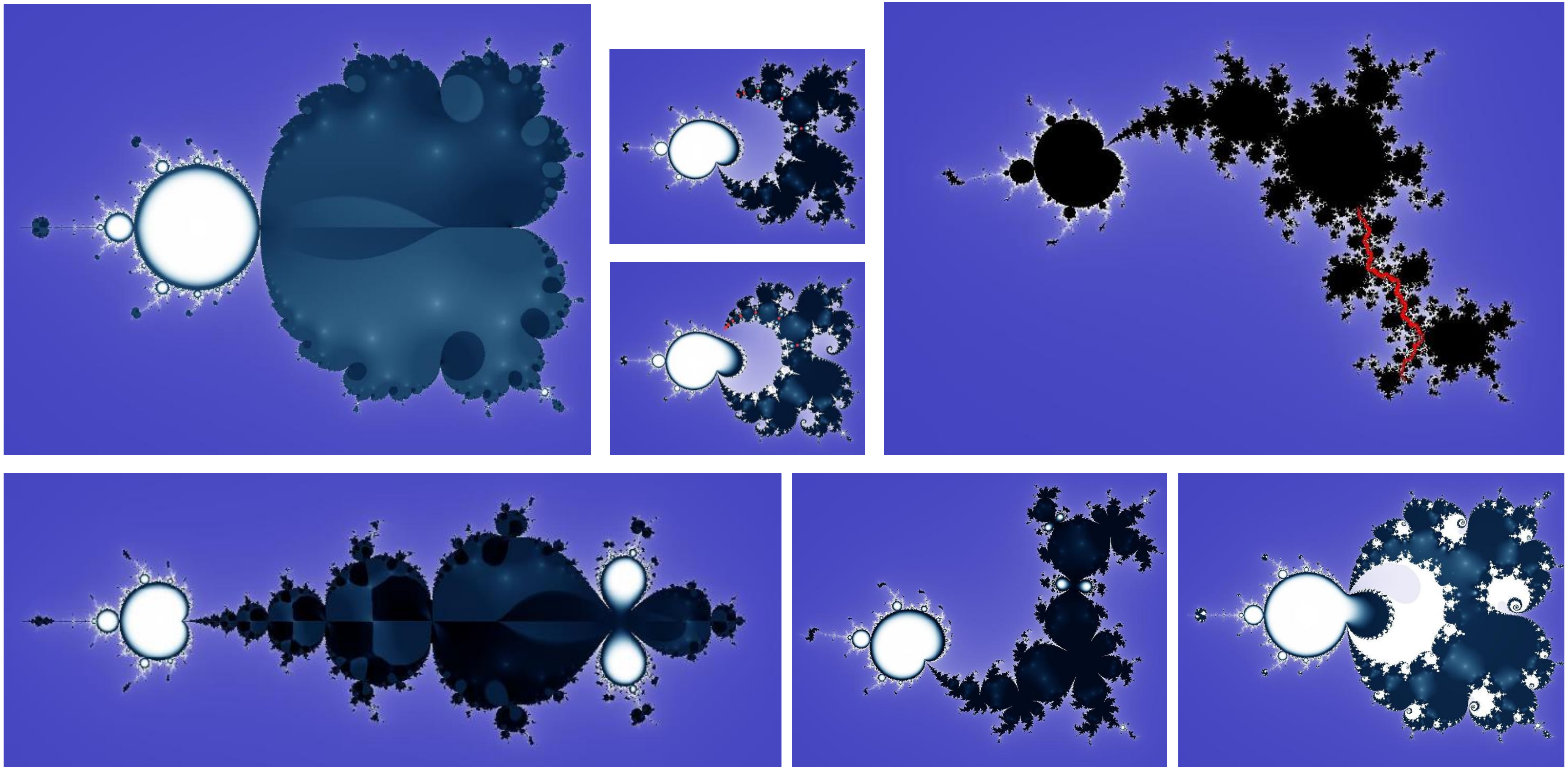
Theorem. *Let $P: U \to V$ be PL, and suppose that $P: W_i \cap U \to V$ are injective. Then*

$$\mathrm{mod}(V \setminus \overline{U}) \le \frac{\log d}{\pi m}.$$

Application to cubic polynomials

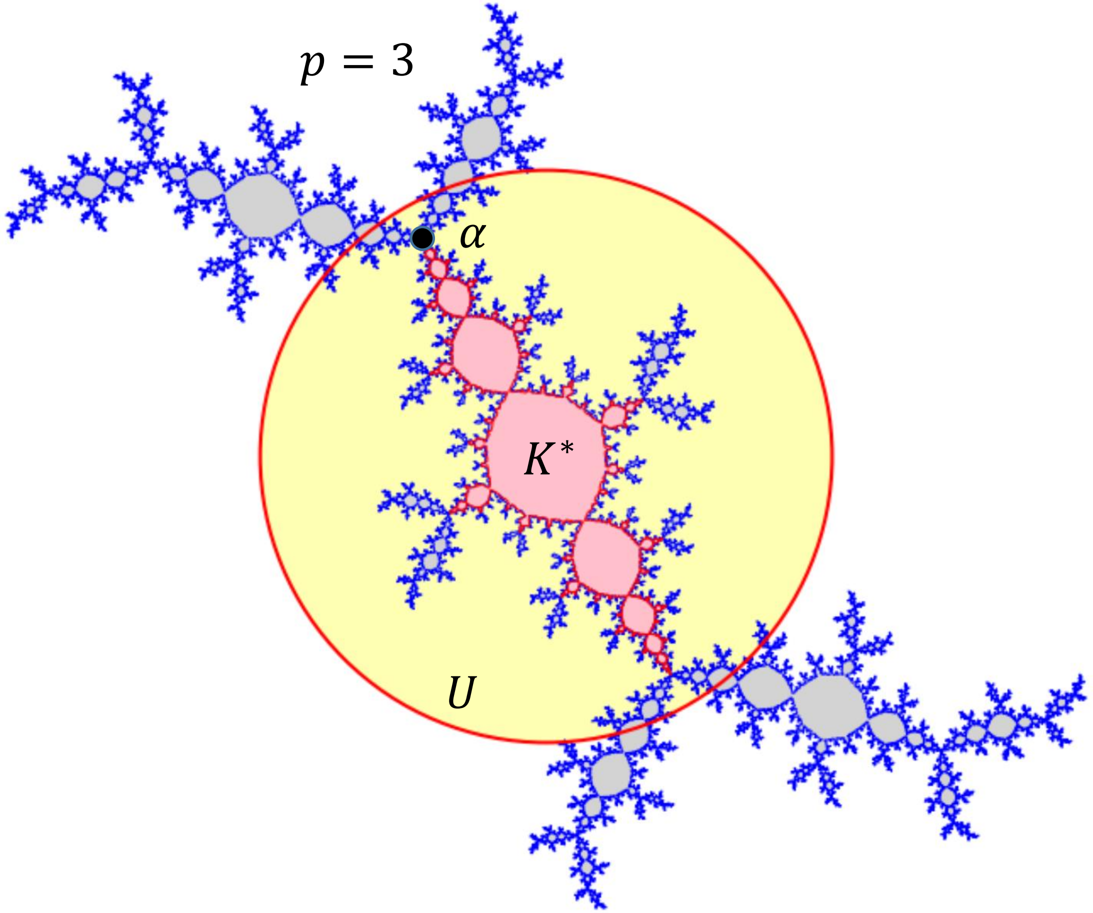
- Consider $f(z) = f_{\lambda,b}(z) = \lambda z + bz^2 + z^3$ with $|\lambda| \leq 1$.
- Outside of a central part (cubic analog of the filled main cardioid) in the space of such maps, all f admit degree 2 PL restrictions $f: U \to V, U \ni 0$. Set $K^* = \{z \in U | \, f^n(z) \in U \;\; \forall n = 1,2, \dots\}$ (PL set).
- **Theorem.** *Components of $K(f) \setminus K^*$ are contained in (pre)periodic wedges whose vertices are eventually mapped to repelling periodic points – unless $J(f)$ has positive measure and carries a measurable invariant line field*.
- The theorem also implies a parameter space statement describing the central parts of $\mathcal{F}_\lambda = \{f_{\lambda,b} \big| \, \lambda \text{ fixed}, \; b \in \mathbb{C}\}$.

Some parameter slices $\mathcal{F}_\lambda$ (plane of $a = b^2$)



Satellite renormalization for $f_c(z) = z^2 + c$

- **Theorem A.** *Let α be a repelling fixed point of f_c with $p \gg 1$ external rays landing at α. Let $f_c^p : U \to V$ be PL with the corresponding PL set $K^* \ni \alpha$. Then* $\mathrm{mod}(U \setminus K^*) \leq \frac{4\pi}{\log p}$.
- **Problem**: what is asymptotically the best upper bound?
- The annulus $U \setminus K^*$ is called the root annulus of K^*.



A general upper bound

- **Theorem**. *Let* $\mathcal{K} = \{K_0^*, \dots, K_{q-1}^*\}$ *be a cycle of PL sets for a rational function* f *such that at least* t *elements of* $\mathcal{K}$ *have spherical diameter at least* k *and a root annulus of modulus at least* m*. Then* $4tk^2 \leq e^{\frac{\pi}{m}}$ *provided that* t *is sufficiently large*.
- Cycles of PL sets are called PL cycles.
- A compactness argument + the Koebe ¼-theorem imply that there are many PL sets of big diameter in the satellite case.
- The upper bound for the quadratic satellite renormalization (Theorem A) can be generalized to rational maps of any degree.

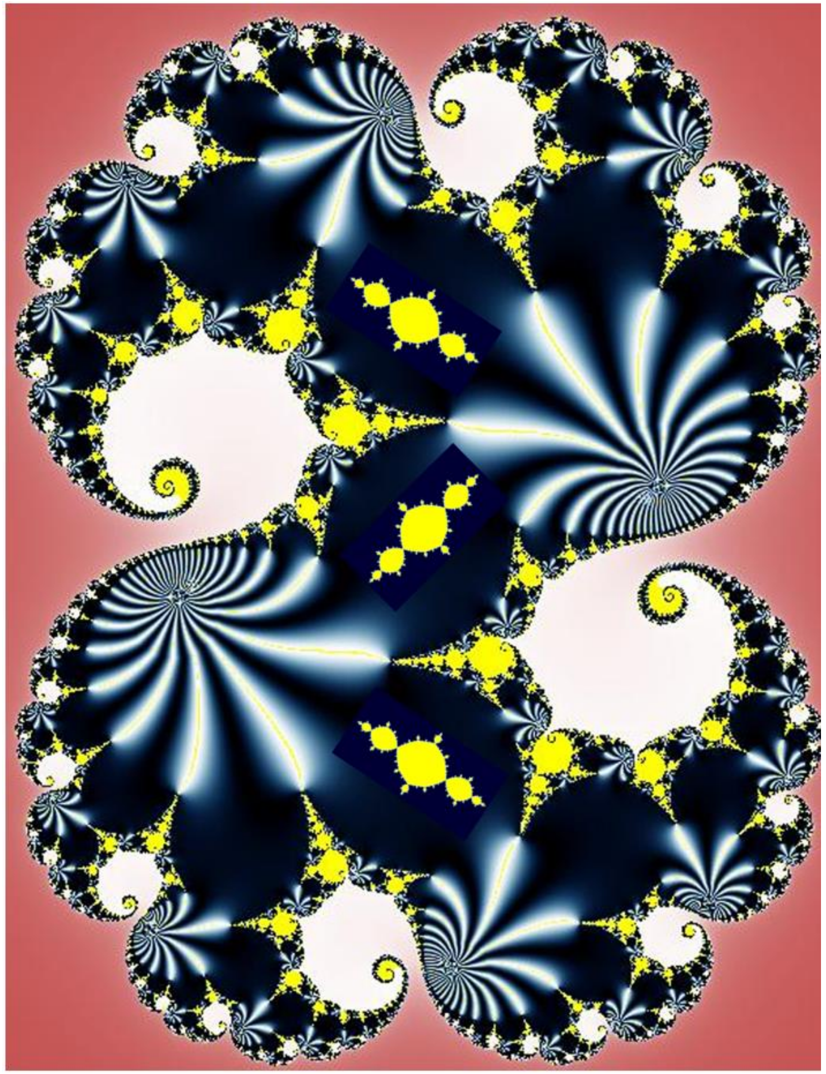
Non-null sequences of renormalizations

Theorem B. *Let $f_n \to f$ be degree $d \geq 2$ rational functions. Assume $f_n^{p_n}: U_n \to V_n$ are PL maps with connected PL sets K_n of period p_n, and $K_n \to K$ in the Hausdorff metric. If*

$$\mathrm{diam}(K) > 0, p_n \to \infty, \mathrm{mod}(V_n \setminus K_n) \nrightarrow 0,$$

then K is contained in a periodic parabolic domain of f.

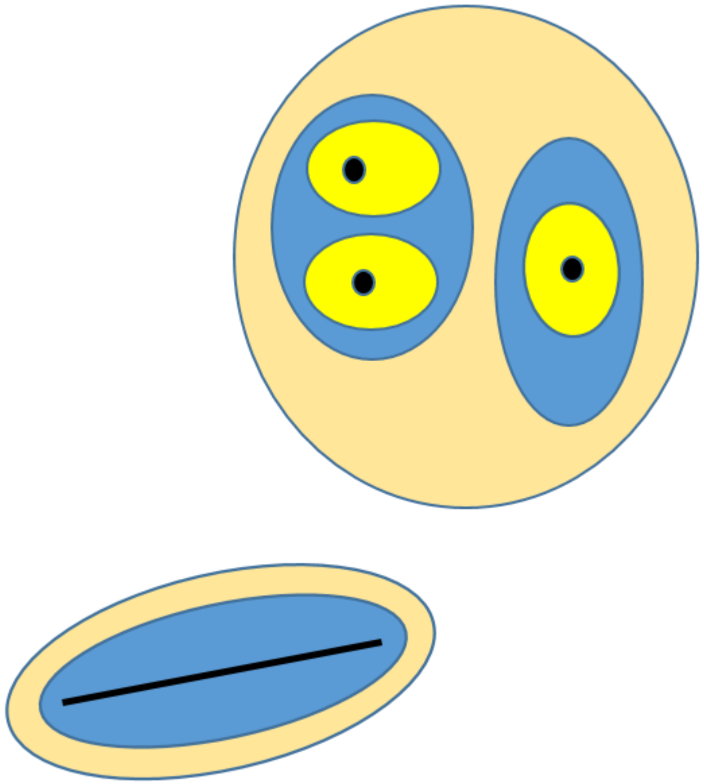
This can indeed happen (parabolic implosion).



Warning: the figure is fake.

Infinitely renormalizable sets

- Consider PL cycles $\mathcal{K}_n$ of f such that elements of $\mathcal{K}_{n+1}$ are proper subsets of those of $\mathcal{K}_n$.
- Let p_n be the period of $\mathcal{K}_n$. Then $p_{n+1}/p_n = q_n$, where $q_n \geq\ 2$ is the number of elements of $\mathcal{K}_{n+1}$ in every element of $\mathcal{K}_n$.
- The set $S = \bigcap_n \left(\bigcup_{K \in \mathcal{K}_n} K \right)$ is called an infinitely-renormalizable set for f.

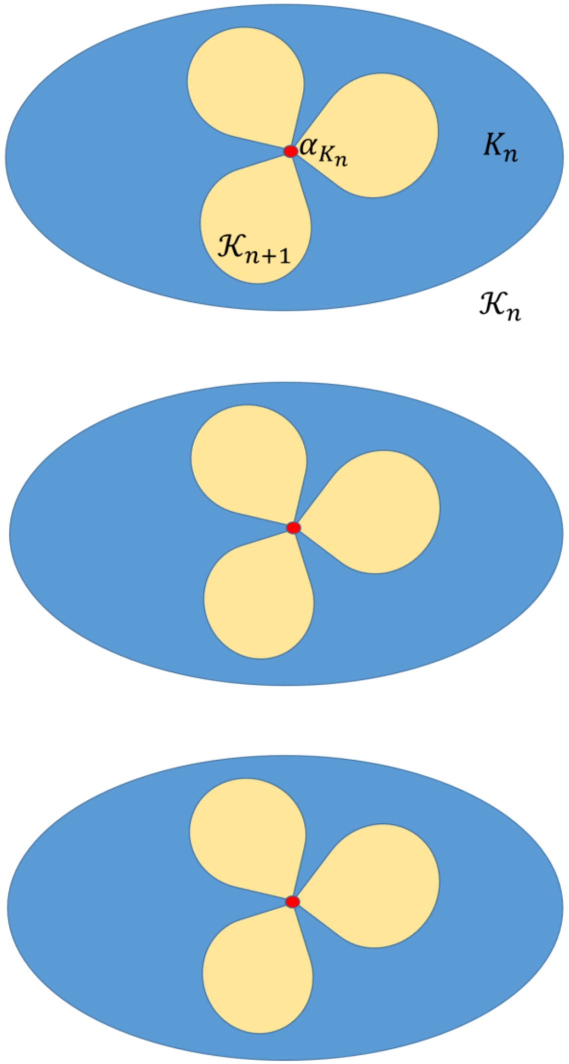


Corollary 1. *If S is not a Cantor set, then $\operatorname{mod}(U_n \setminus K_n^*) \to 0$ for any sequence $K_n \in \mathcal{K}_n$ such that $f^{p_n}: U_n \to V_n$ are the corresponding PL maps.*

No a priori bounds in the satellite case

Corollary 2. *Suppose that, for arbitrarily large n, s_n elements of $\mathcal{K}_{n+1}$ in a given $K_n \in \mathcal{K}_n$ have a unique common base point α_{K_n}.*

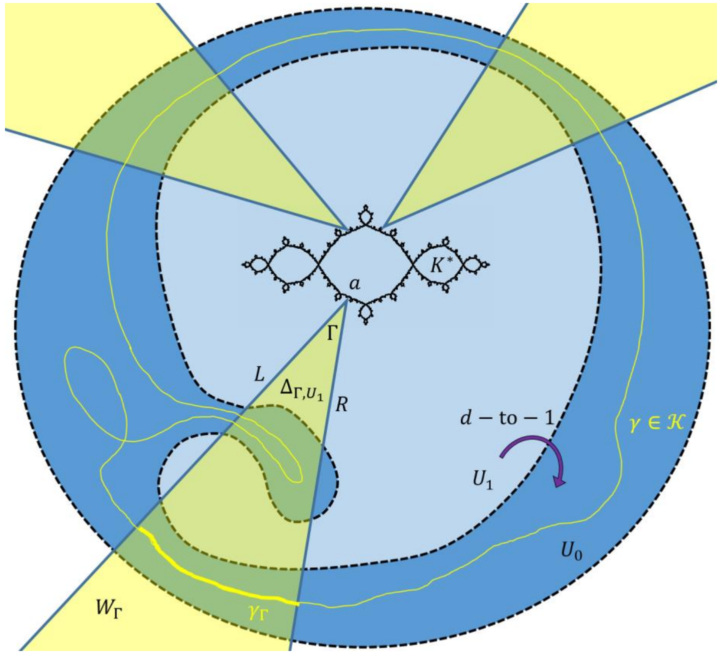
- *If the moduli of some fundamental annuli for $\mathcal{K}_n$ are $\geq m > 0$ and $s_n \to \infty$, then the moduli of all root annuli for $\mathcal{K}_{n+1}$ tend to zero.*
- *If every component of S containing a critical point is non-degenerate, then the moduli of all root annuli of $\mathcal{K}_{n+1}$ are $O\left(\frac{1}{\ln s_n}\right)$.*



3. Ideas of proofs

Extremal length, packing radius

Upper bound on $\mathbf{mod}(V \setminus \overline{U})$



Theorem. *Let $P: U \to V$ be PL, and suppose that $P: W_i \cap U \to V$ are injective. Then $\operatorname{mod}\left(V \setminus \overline{U}\right) \leq \frac{\log d}{\pi m}$.*

$\mathcal{L}$ = the set of curves in $A = V \setminus \overline{U}$ winding once.

$$\mathrm{EL}(\mathcal{L}) = \operatorname{mod}(A)^{-1}$$

$\mathcal{L}_i$ = the set of curves in $A \cap W_i$ connecting the boundary rays of W_i.

All $\mathcal{L}_i$ can be transferred to the same wedge W_0, where

$$\frac{\pi}{m \log d} \leq \operatorname{mod}\left({}^{W_0}/_{P^m}\right) \leq \frac{1}{\mathrm{EL}(\mathcal{L}_0)} + \cdots + \frac{1}{\mathrm{EL}(\mathcal{L}_{m-1})}.$$

An elementary inequality between the arithmetic mean and the harmonic mean now yields that

$$\mathrm{EL}(\mathcal{L}) \geq \mathrm{EL}(\mathcal{L}_0) + \cdots + \mathrm{EL}(\mathcal{L}_{m-1}) \geq \frac{m\pi}{\log d}.$$

A general upper bound

Theorem. *If at least $t \gg 1$ elements of $\mathcal{K}$ have spherical diameter at least k and a root annulus of modulus at least m, then $4tk^2 \leq e^{\frac{\pi}{m}}$.*

Consider continua Z_i and Jordan domains $U_i \supset Z_i$, $i = 1, \ldots, t$ such that $\mathrm{diam}(Z_i) \geq k$, $\mathrm{mod}(U_i \setminus Z_i) \geq m$ and $z_i \notin U_j$ for $j \neq i$. Then $4tk^2 \leq e^{\frac{\pi}{m}}$ if t is sufficiently large.

This is based on the following lemma.

Lemma. *Let $U \subset \mathbb{P}^1$ be a topological disk, and $Z \subset U$ a continuum of spherical diameter k. Then, for every point $z \in Z$, the round disk of spherical radius $\rho = 4ke^{-\frac{\pi}{2m}}$ is contained in U if m is sufficiently small.*

The lemma follows from the solution of the Teichmüller extremal problem.