Hilbert Schemes

Goal: Study families of ideals/varieties with some prescribed invariants. Construct a moduli space parameterizing these ideals/varieties

Notation: k algebraically closed field

Ex Monic polynomials of degree d in $k[x]$

$$\{(x^d + a_{d-1}x^{d-1} + \dots + a_0)\} \longleftrightarrow \mathbb{A}^d$$

varying

$(a_0, \dots, a_{d-1})$

Ex Homogenous polynomials of degree d in $k[x,y]$

$$\{(a_d x^d + a_{d-1}x^{d} y + \dots + a_0 y^d)\} \longleftrightarrow \mathbb{P}^d$$

$$\longrightarrow [a_0 : \dots : a_d]$$

Ex Homogenous polynomials of deg. d in $k[x_0, \ldots, x_n]$
(Hypersurfaces in $\mathbb{P}^n$ of degree d)

$$\sum_{i_0+\cdots+i_n=d} a_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n} \longleftrightarrow \mathbb{P}^{\binom{n+d}{d}-1}$$

$$\longleftrightarrow [a_{i_0 \cdots i_n}]$$

hypersurface

Thm [Grothendieck] Let $S = k[x_0, \dots, x_n]$.
There is a projective k-scheme $\mathrm{Hilb}^P(\mathbb{P}^n_k)$ that parameterizes closed subschemes $Z \subseteq \mathbb{P}^n_k$ with Hilbert polynomial P:

$$\{\text{closed } Z \subseteq \mathbb{P}^n_k : Z \text{ has Hilbert polynomial } P\}$$

$$\Big\updownarrow \quad Z = \mathrm{Proj}(S/I)$$

$$= \left\{ I \subseteq S \text{ homogenous ideals} : \begin{array}{l} \bullet\ I \text{ saturated} \\ \bullet\ [S/I]_i \text{ has rank } P(i) \\ \quad \text{for } i \gg 0 \end{array} \right\}$$

- $[Z]$, $[I]$ to denote the point in the Hilbert scheme.

Defn Consider the functor

$$\underline{\mathrm{Hilb}}^{P}_{\mathbb{P}^n_k} : (\mathrm{Schemes}/k)^{op} \longrightarrow \mathrm{Sets}$$

$$T \longmapsto \left\{ Z \subseteq \mathbb{P}^n_T \text{ closed} : \begin{array}{l} \bullet\ Z \to T \text{ is flat} \\ \bullet\ Z_t \text{ has Hilbert polynomial } P(t) \ \forall t \end{array} \right\}$$

$\mathbb{P}^n_k \times_k T$

Thm $\underline{\mathrm{Hilb}}^{P}_{\mathbb{P}^n_k}$ is represented by a projective scheme $\mathrm{Hilb}^P(\mathbb{P}^n_k)$.

$$\underline{\mathrm{Hilb}}^{P}_{\mathbb{P}^n_k}(T) = \mathrm{Hom}(T, \mathrm{Hilb}^P(\mathbb{P}^n_k))$$

M module over R $\qquad \mathcal{O}_{T,f(p)} \xrightarrow{\text{flat}} \mathcal{O}_{Z,p}$

$K \hookrightarrow L$ R-module map

M is flat over R $\quad K \otimes_R M \hookrightarrow L \otimes_R M$

Ex Let P be the Hilbert polynomial of a degree d hypersurface in $\mathbb{P}^n_k$.

$$\underbrace{\mathrm{Hilb}^P(\mathbb{P}^n_k)}_{\uparrow \text{ smooth}} = \mathbb{P}^{\binom{n+d}{d}-1}$$

Fact • $\mathbb{P}^n_k$ can be replaced with X projective (even quasi-projective)

Ex $\mathrm{Hilb}^d X = \{ Z \subseteq X : \deg Z = d, \dim Z = 0 \}$

$(X = \mathbb{A}^n$ or $\mathbb{P}^n)$

$Z \subseteq \mathbb{P}^n$

• $[Z] \in \mathrm{Hilb}^P(\mathbb{P}^n)$

$[I] \in \mathrm{Hilb}^d(\mathbb{A}^n)$

Tangent Space

- $T_{[Z]}(\mathrm{Hilb}^P(\mathbb{P}^n_k))$

$$= \mathrm{Hom}(\mathrm{Spec}\, k[\varepsilon]/\varepsilon^2, \mathrm{Hilb}^P(\mathbb{P}^n_k))$$

$$= \{\tilde{Z} \subseteq \mathbb{P}^n_{k[\varepsilon]/\varepsilon^2} \text{ flat deformation of } Z \text{ over } k[\varepsilon]/\varepsilon^2\}$$

$$= H^0(Z, N_{Z/\mathbb{P}^n})$$

- $T_{[I]}(\mathrm{Hilb}^d \mathbb{A}^n) = \{\tilde{I} \subseteq S[\varepsilon]/\varepsilon^2 \text{ flat over } S \text{ and } \tilde{I}|_{\varepsilon=0} = I\}$

$$\{f + \varepsilon g : f \in I \text{ s.t. } \varphi(f) = \bar{g} \text{ in } S/I\}$$

$$\varphi$$

$$= \mathrm{Hom}_S(I, S/I)$$

Notation: $T(I) = T_{[I]}\mathrm{Hilb}^d \mathbb{A}^n$

Hilbert scheme of points (think $X = \mathbb{A}^n$ or $\mathbb{P}^n$)

Notation: X smooth, connected, quasi-proj of dim n.

$$\mathrm{Hilb}^d X = \{ Z \subseteq X : \deg Z = d, \dim Z = 0 \}$$

$$\mathcal{U} = \{ d \text{ distinct points of } X \} \subseteq \mathrm{Hilb}^d X$$

<u>Defn</u> $\mathcal{U}$ is open in $\mathrm{Hilb}^d X$ and its closure is an irreducible component of dim. nd (smoothable component)

- $\dim \mathrm{Hilb}^d X \geq nd$

<u>Thm</u> (Fogarty 68) $\mathrm{Hilb}^d X$ is connected.

X is a curve

$$\mathrm{Hilb}^d X = \mathrm{Div}^d X \quad \left(= \mathrm{Sym}^{(d)} X\right)$$

<u>Ex</u> $\quad X = \mathbb{P}^1 = \mathrm{Proj}(K[x,y])$

$$[(f)] \in \mathrm{Hilb}^d \mathbb{P}^1 \rightsquigarrow f = (\alpha_1 x - \beta_1 y) \cdots (\alpha_d x - \beta_d y)$$

$\mathrm{Hilb}^2 \mathbb{P}^1 \quad V(xy) \qquad\qquad V(x^2)$

<u>Prop</u> $\mathrm{Hilb}^d X$ is smooth.

① Suffices to show $\dim T_{[Z]} \mathrm{Hilb}^d X \leq d$

$$d \leq \dim \mathrm{Hilb}^d X \leq \dim T_{[Z]} \mathrm{Hilb}^d X \leq d$$

② We may assume $X = \mathbb{A}^1$

étale map $\mathbb{A}^1 \xrightarrow{p \,\mapsto\, \text{or}} X$

$\Rightarrow$ étale map $\mathrm{Hilb}^d \mathbb{A}^1 \longrightarrow \mathrm{Hilb}^d X$

Tangent Space

If there is a one-parameter degeneration from $V(I)$ to $V(J)$ i.e. a family $Z \subseteq \mathbb{P}^n \times \mathbb{A}^1$

- $Z \longrightarrow \mathbb{A}^1$ is Flat
- $Z_t \simeq V(I)$ for all $t \neq 0$
- $Z_0 = V(J)$

then $\dim_K T(I) \leq \dim_K T(J)$.

Ex Gröbner degeneration: For any $I \subseteq S$ there is one-parameter degen. from $V(I)$ to $V(\mathrm{in}_> I)$

$$\dim T(I) \leq \dim T(\mathrm{in}_> I).$$

3. Suffices to show $\dim T_{[x^d]} \mathrm{Hilb}^d \mathbb{A}^1 = d$

- show $\mathrm{Hilb}^d \mathbb{A}^1$
- $\dim T_{[I]} \mathrm{Hilb}^d \mathbb{A}^1 \leq \dim T_{[\text{in } I]} \mathrm{Hilb}^d \mathbb{A}^1 \leq d$ (x^d)

4. $T(x^d) = \mathrm{Hom}_{K[x]}\left((x^d), K[x]/(x^d)\right)$

$$= \mathrm{span}_K \left\{ \begin{array}{l} x^d \mapsto 1, \\ x^d \mapsto x, \\ \vdots \\ x^d \mapsto x^{d-1} \end{array} \right\}$$

X is a surface

Thm (Fogarty 68) $\mathrm{Hilb}^d X$ is smooth and irreducible of dim. $2d$

- The map $\mathrm{Hilb}\, X \longrightarrow \mathrm{Sym}^{(d)} X = X^d / S_d$ is a resolution of singularities.

- Interesting (birational) geometry
  ($\mathrm{Hilb}^d \mathbb{P}^2$ a Mori dream space,
  Hilb^d(K3 surface) is Hyperkähler)
- Connections to representation theory
  (Construct a Heisenberg algebra by
  considering certain actions on $\bigoplus_{n \geq 1} H_\bullet(\mathrm{Hilb}^d X, \mathbb{C})$)
- Connections to combinatorics

① $\dim H_{2i}(\mathrm{Hilb}^d X, \mathbb{C})$ = # of partitions of d in $d-i$ parts

② Haiman's $n!$ theorem

X is a threefold

$\dim \mathcal{U} = 3 \cdot 4 = 12$

Ex $\mathrm{Hilb}^4 \mathbb{A}^3$ is singular

① $[\mathfrak{m}^2] \in \mathrm{Hilb}^4 \mathbb{A}^3$ $6 \cdot 3$

degree 4 $\shortparallel$

② $\dim T(I) = \dim \mathrm{Hom}_S(\mathfrak{m}^2, S/\mathfrak{m}^2) = 18$

$\mathfrak{m}^2 = (x,y,z)^2$

$$
\begin{aligned}
x^2 &\longmapsto \{x, y, z\} \\
xy &\longmapsto 0 \\
xz &\longmapsto 0 \\
y^2 &\longmapsto 0 \\
yz &\longmapsto 0 \\
z^2 &\longmapsto 0
\end{aligned}
$$

$e: x^2 \longmapsto x$ and everything else to 0.

[An S-module hom.]

Q) What can one say about the singularities of $\mathrm{Hilb}^d(X)$?

We may assume $X = \mathbb{A}^n$

On the tangent space to the Hilbert scheme of points in $\mathbb{A}^3$
(joint w/ Alessio Sammartano)

Notation $S := K[x_1, \ldots, x_n]$ $\left(K[x,y],\ K[x,y,z]\right)$

Def • $H_n^d = \mathrm{Hilb}^d(\mathbb{A}^n) = \{ I \subseteq S : \ell(S/I) = d \}$

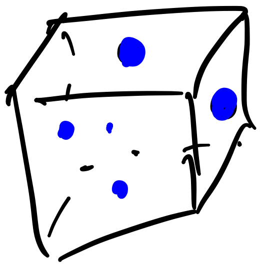
• $H_n^d \supseteq \overline{U = \{d\text{-distinct points in } \mathbb{A}^n\}}$ and $\dim U = nd$.

"Smoothable component"

	Smooth	Irreducible	Reducible	Singularities
H_1^d	✓	✓	—	—
H_2^d	✓	✓	—	—
H_3^d	$d \leq 3$	$d \leq 11$ (Douvropoulos, Jelisiejew, Nødland, Teitler 2017)	$d \geq 78$ (Iarrobino 1984)	Is the singular locus of a hypersurface in a smooth variety (Dimca, Szendrői 2009)
H_4^d	$d \leq 3$	$d \leq 7$	$d \geq 8$	Expected (?) to be messy
$\vdots$				
H_{16}^d	$d \leq 3$	$d \leq 7$	$d \geq 8$	Satisfies Murphy's law up-to retraction (Jelisiejew 2020)

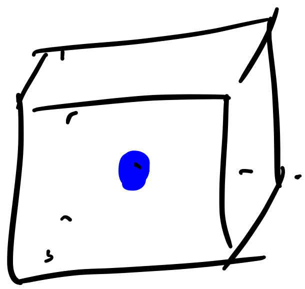
Q) For $I \in H^d_3$ how large is $\dim_k T(I)$?

$$\dim_k T(I) = \operatorname{Hom}(I, S/I)$$



$V(I)$

$\downarrow$ flat



$V(I')$ $\qquad \sqrt{I'} = (x,y,z)$

$\downarrow$ flat

$V(\mathrm{in}_> I')$ $\leftarrow$ monomial

- Monomial ideals I are on smoothable component so $\dim T(I) \geq 3d$

Ex Consider $I \in H_3^4$

$$\dim T(I) \le \dim T(m^2) = 18$$

Conjecture [Briancon - Iarrobino]

$[m^r] \in H_3^d$

Let $d = \ell(S/m^r)$ then

$d = \binom{r+2}{3}$

$$\boxed{\dim T(I) \overset{?}{\le} \dim T(m^r)} \text{ for all } I \in H_3^d$$

Theorem [R-S]

Let $d = \ell(S/m^r)$ then

$$\dim T(I) \le \frac{4}{3} \dim T(m^r) \text{ for all } I \in H_3^d$$

- Gives a bound on the dimension of H_3^d

<u>Conjecture</u> [Briancon - Iarrobino]

Let $d = \ell(S/m^r)$ then

$$\overset{\binom{r+2}{3}}{} \quad \dim T(I) \leq \dim T(m^r) \text{ for all } I \in H_3^d$$

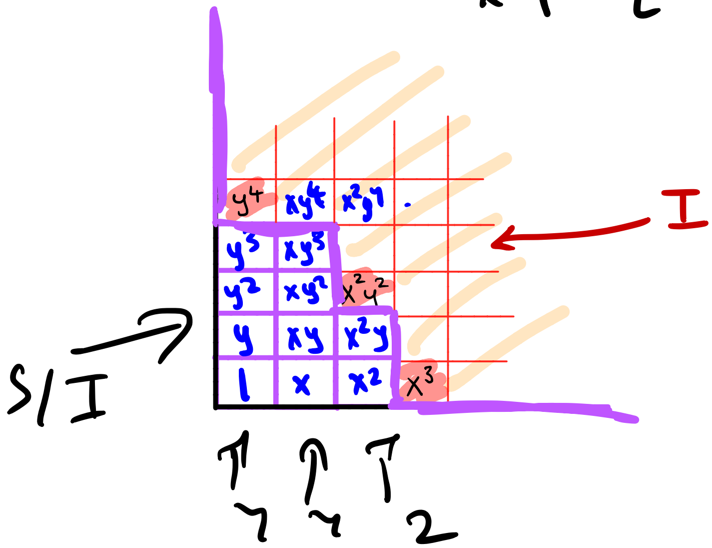
Suffices to prove conjecture for monomial I.

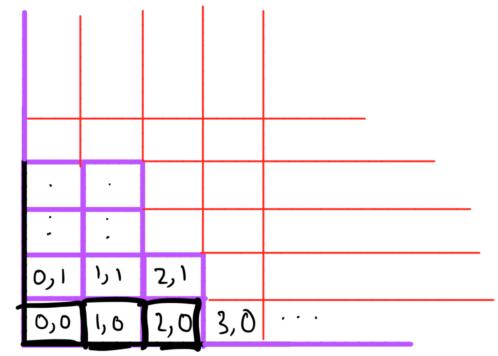
Monomial $I \subseteq K[x_1, \ldots, x_n]$ is $\mathbb{Z}^n$-graded

$$T(I) = \operatorname{Hom}(I, S/I) \text{ is } \mathbb{Z}^n\text{-graded}$$
$$= \bigoplus_{\alpha \in \mathbb{Z}^n} T(I)_\alpha$$

Ex $\quad I = (x^3, x^2y^2, y^4) \subseteq K[x,y]$, $\dim T(I) = 20$

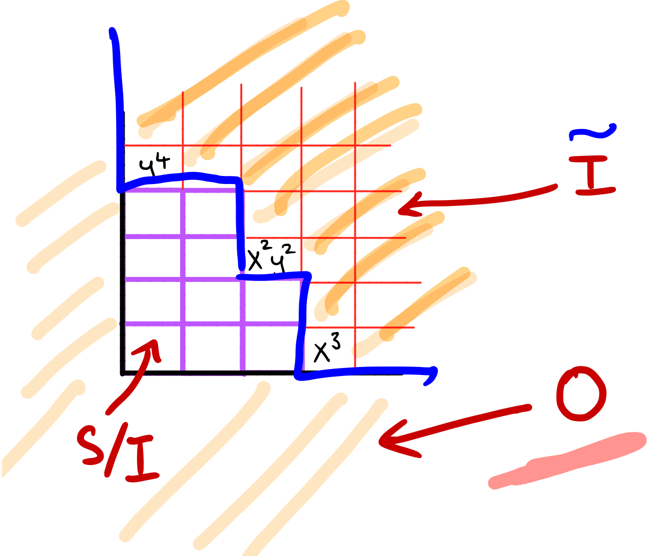
$$\ell(S/I) = \dim_K \operatorname{span}\{x^i y^j \notin I\} = 10$$





$\mathbb{N}^2 \longleftrightarrow$ Exponent vectors of monomials $\longleftrightarrow$ Point of $\mathbb{N}^2$ represented as a square.

Ex $\quad I = (x^3, x^2y^2, y^4)$, $\ell(S/I) = 10$, $\dim T(I) = 20$



Squares correspond to points in $\mathbb{Z}^2$

$\mathrm{Hom}(I, S/I)$

Some tangent vectors: $\tilde{I} + (-2,0) \setminus \tilde{I}$

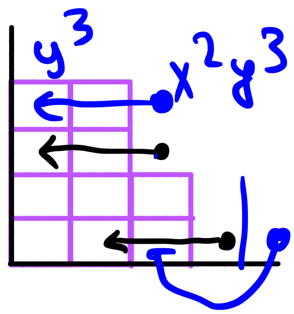
$x^3 \mapsto x$

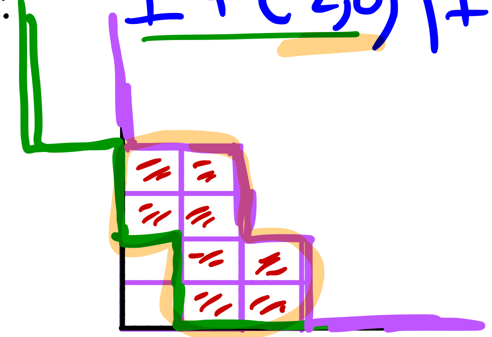
$x^2y^2 \mapsto y^2$

$y^4 \mapsto 0$

degree $(-2,0)$

$\cdot\,(x^{-2})$





All the monomials with non-zero image:

$x^2y^3 \mapsto y^3 \qquad x^3y^3 \mapsto xy^3$

$x^2y^2 \mapsto y^2 \qquad x^3y^2 \mapsto xy^2$

$x^3y \mapsto xy \qquad x^4y \mapsto x^2y$

$x^3 \mapsto x \qquad x^4 \mapsto x^2$

How do we encode all of this combinatorially?

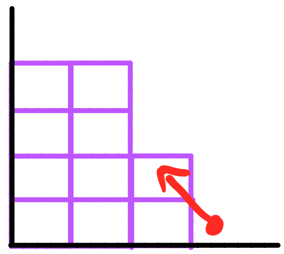
Defn: Let $\tilde{I} = \{(\alpha_1, \dots, \alpha_n) : x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in I\} \subseteq \mathbb{N}^n$

$x^3 \mapsto x^2y$

$x^2y^2 \mapsto 0$

$y^4 \mapsto 0$

degree $(-1,1)$

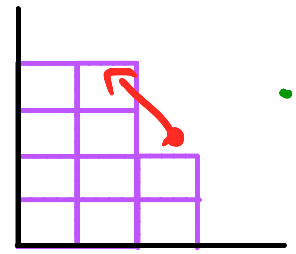


$x^3 \mapsto 0$

$x^2y^2 \mapsto xy^3$

$y^4 \mapsto 0$

degree $(-1,1)$



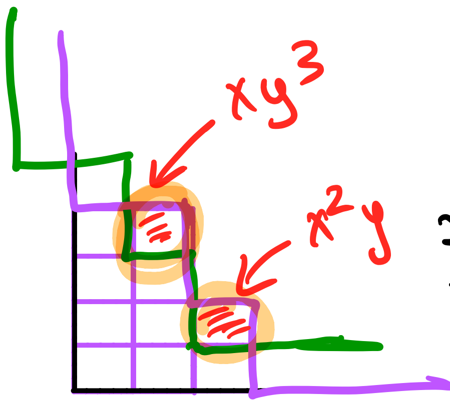
$\cdot\ (x^{-1}y)$

Non-zero images

$x^3 \mapsto x^2y$

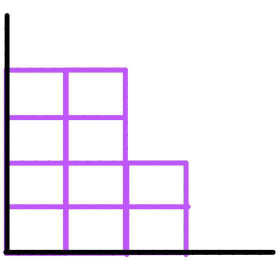
Non-zero images

$x^2y^2 \mapsto xy^3$

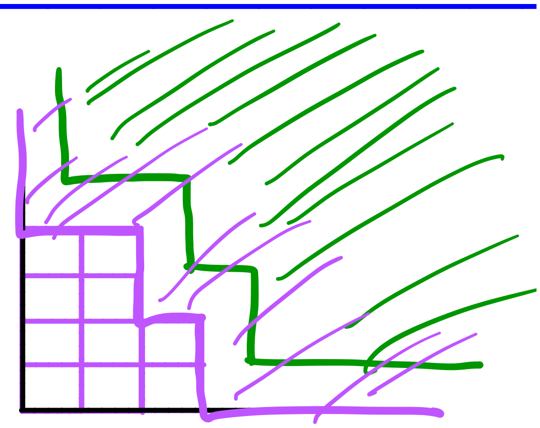


$\tilde{I} + (-1,1) \setminus \tilde{I}$

$I \to 0$



degree $(1,1)$

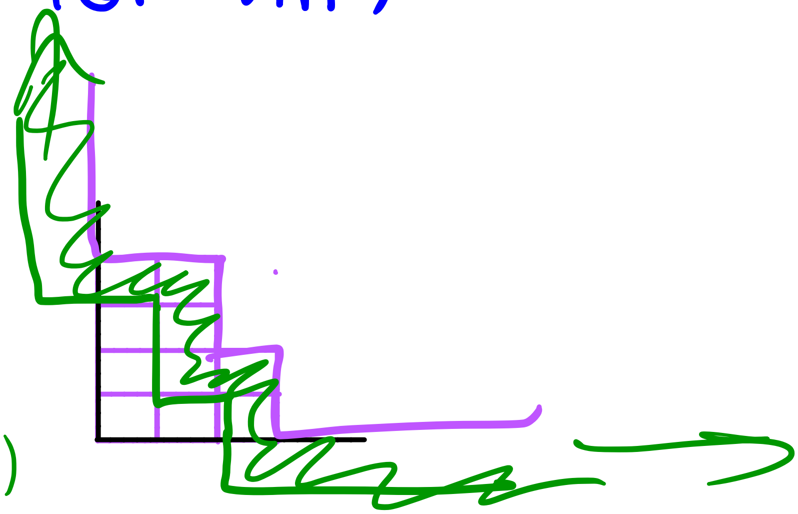


$\tilde{I} + (1,1) \setminus \tilde{I} = \emptyset$

$I \to S/I$

$x^i y^j \longmapsto x^{i+1} y^{j+1} \in I$

Prop The set of bounded components of $\tilde{I}+\alpha \setminus \tilde{I}$ forms a basis of $T(I)_\alpha$. (true for H^d_n)



degree $(-1,-1)$

Cor $\tilde{I}+(i,j)\setminus\tilde{I}$ is unbounded if $i,j<0$.

Ex H_2^d is smooth (Based on Haiman's work)

Need to show $\dim T(I) = 2\,\ell(S/I)$

$$\boxed{T(I) = \underbrace{\bigoplus_{\substack{i\geq 0\\ j<0}} T(I)_{i,j}}_{T_{pn}(I)} \oplus \underbrace{\bigoplus_{\substack{i<0\\ j\geq 0}} T(I)_{i,j}}_{T_{np}(I)}} \oplus \underbrace{\cancelto{0}{\bigoplus_{\substack{i\geq 0\\ j\geq 0}} T(I)_{i,j}}}_{T_{pp}(I)} \oplus \underbrace{\cancelto{0}{\bigoplus_{\substack{i<0\\ j<0}} T(I)_{i,j}}}_{T_{nn}(I)}$$

$$\dim T_{pn}(I) = \dim T_{np}(I) = \ell(S/I)$$

- np, pn, pp, nn are "signatures"

$I \subseteq K[x,y,z]$ monomial

$$T(I) = \bigoplus_{\alpha \in \mathbb{Z}^3} T(I)_\alpha =$$

$$T_{ppn}(I) \oplus T_{pnp}(I) \oplus T_{npp}(I) \oplus T_{nnp}(I) \oplus T_{npn}(I) \oplus T_{pnn}(I) \cancel{\oplus T_{ppp}(I)} \cancel{\oplus T_{nnn}(I)}$$

0

0

$$T_{ppn}(I) = \bigoplus_{\substack{i \geq 0 \\ j \geq 0 \\ l < 0}} T_{ijl}(I)$$

$$T_{nnp}(I) = \bigoplus_{\substack{i < 0 \\ j < 0 \\ l \geq 0}} T_{ijl}$$

Thm [R-S] For any monomial point $I \in H_3^d$ we have

$$\dim_K T_{ppn}(I) = \dim_K T_{nnp}(I) + d$$

$$\dim_K T_{pnp}(I) = \dim_K T_{npn}(I) + d$$

$$\dim_K T_{npp}(I) = \dim_K T_{pnn}(I) + d$$

Cor I is smooth $\Longleftrightarrow$ $T_\sigma(I) = 0$ for $\sigma \in \{nnp, npn, pnn\}$

Cor I is smooth $\iff T_\sigma(I)=0$ for $\sigma \in \left\{ \begin{smallmatrix} nnp, npn \\ pnn \end{smallmatrix} \right\}$

Ex $(x^2, xy, y^3, xz^2, yz^2, z^4) \in H_3^{10}$ is singular

$xy \mapsto z^3$, others to 0 $(-1,-1,3)$

Cor [Behrend-Fantechi]

$\dim T(I) \equiv d \bmod 2$ for all monomial $I \in H_3^d$

Conj [Pandharipande-Okounkov]

$\dim T(I) \equiv d \bmod 2$ for all $I \in H_d^3$

Using the theorem it suffices to show

$$\dim T_\sigma(I) \le \dim T_\sigma(m^r)$$

$\dim T_\sigma(I) \le \dim T_\sigma(m^r)$ for $\sigma \in \{ppn, pnp, npp\}$

Missing.

<u>Thm</u> [R-S] For I Borel-fixed,

$\dim T_\sigma(I) \le \dim T_\sigma(m^r)$ for $\sigma \in \{ppn, pnp\}$

with equality in either case iff $I = m^r$.