Deformation of klt singularities.

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Convention

For simplicity, all rings & schemes are ess.o.f.t/$\mathbb{C}$

1'st half ⋯ recall some notions & known results

2'nd half ⋯ main results.

§1. Deformation Problem

Def

$\mathcal{P}$: property for Noe local rings (eg $\mathcal{P}$ = "regular", "normal", "CM" etc...)

Today, we say that "$\mathcal{P}$ deforms" if

⌜ $\forall (A, \mathfrak{m})$: Noe local ring,

$\forall r \in \mathfrak{m}$: non-zero divisor

If $A/(r)$ has $\mathcal{P}$, then so does A. ⌟

eg

- "regular" deforms. (i.e $A/(r)$: regular $\Rightarrow$ so is A)
- (R_n), (S_n), CM, reduced, normal, Gor, etc... deforms.

Recall

$(A,\mathfrak{m})$: Gor $\overset{\text{def}}{\Longleftrightarrow}$ A : CM & $\omega_A \cong A$

↑ canonical module

($\doteqdot$ something like $\Omega^{\wedge d}_{A/\mathbb{C}}$)

- "$\mathcal{P}$ deforms" is very useful. ($\ddot{\smile}$ $\dim A/(r) \leftarrow \dim A$)

- But, unfortunately, some $\mathcal{P}$ do not deform.

Fact 1

"$\mathbb{Q}$-Gorenstein" does NOT deform.

Recall

$(A,\mathfrak{m})$: norm local is $\mathbb{Q}$-Gor

$\overset{\text{def}}{\Longleftrightarrow}$ $\exists n>0$, $(\omega_A^{\otimes n})^{**} \cong A$

($(-)^{**}$ … reflexive hull) $\left(n=1 \rightarrow (\omega_A^{\otimes 1})^{**} \cong \omega_A \right)$

Ex2 (of Fact 1) (cf [Singh'99])

$P := \mathbb{C}[a,b,c,d,e]$, pol. ring w/ 5-variables

$$M := \begin{pmatrix} a^2+e^5 & b & d \\ c & a^2 & b^2-d \end{pmatrix} \in M_{2,3}(P)$$

$I = I_2(M) \subseteq P$: ideal gen'ed by all 2-minors of M

(that is, $I = (f, g, h) \subseteq P$,

where

$$f := \det\begin{pmatrix} a^2+e^5 & b \\ c & a^2 \end{pmatrix} = a^2(a^2+e^5) - bc$$

$$g := \det\begin{pmatrix} b & d \\ a^2 & b^2-d \end{pmatrix} = b(b^2-d) - a^2 d$$

$$h := \det\begin{pmatrix} a^2+e^5 & d \\ c & b^2-d \end{pmatrix} = (a^2+e^5)(b^2-d) - cd \text{)}$$

$m := (a,b,c,d,e) \subseteq P$: maximal ideal.

$$R := (P/I)_m$$

$\rightsquigarrow$ $R/(e) \cong \bigoplus_{n \geq 0} H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\llcorner nE \lrcorner))$

w/ $E := \frac{1}{2}[0] + \frac{1}{2}[1] + \frac{1}{4}[\infty]$: $\mathbb{Q}$-div on $\mathbb{P}^1$

$\rightsquigarrow$ $\mathbb{Q}$-Gor.

R: NOT $\mathbb{Q}$-Gor.

§2. Singularities in MMP

MMP (minimal model program)

= A program s.t input ⋯ A normal var X

output ⋯ A normal var Y

s.t Y is "minimal"

& Y is birational to X

(similar.)

☆ Even if X is smooth,

Y may admit some singularities

"Singularities in MMP"

= several classes of singularities

which may appear here

eg terminal sing, can'l, klt, lc, rational, Du Bois

etc...

Rmk

smooth $\Rightarrow$ terminal $\Rightarrow$ canonical $\Rightarrow$ klt $\Rightarrow$ lc $\Rightarrow$ $\mathbb{Q}$-Gor

$\Downarrow$ (under klt) $\quad$ $\Downarrow$ (under lc)

rat'l $\Rightarrow$ Du Bois

Def (klt)

- $(R, \mathfrak{m})$: norm, ess offt / $\mathbb{C}$.
- $X := \mathrm{Spec}(R)$

  $f : Y \longrightarrow X$: log res of sing of X.
- K_X : canonical divisor on X

  (K_Y) $\qquad$ (Y)

  (i.e K_X : Weil div on X s.t $\mathcal{O}_X(K_X) \cong \omega_X$)

(1) Assume $(R, \mathfrak{m})$: $\mathbb{Q}$-Gorenstein

($\Leftrightarrow$ K_X : $\mathbb{Q}$-Car divisor.

$\Rightarrow$ $\mathbb{Q}$-divisor "$f^* K_X$" on Y is well-def)

Then we define

$K_{Y/X} := K_Y - f^* K_X$: $\mathbb{Q}$-div on Y

(2) $(R, \mathfrak{m})$ is klt (resp. lc, can'l, term).

$\overset{\text{def}}{\Longleftrightarrow}$ (a) R: $\mathbb{Q}$-Gor. and

(b) every coefficient of $K_{Y/X}$ satisfies "> -1"

(resp. ≥ -1, > 0, ≥ 0)

Known results.

- [Nakayama'98] "terminal" deforms
- [Kawamata'99] "canonical" deforms.

Note

- In particular, $^{\forall}(A, \mathfrak{m})$: ess oft/$\mathbb{C}$ ~~$\mathbb{Q}$-Gor~~

$A/(r)$: can'l $\Rightarrow$ A: can'l $\Rightarrow$ A: $\mathbb{Q}$-Gor.

- Compare w/ Fact. 1

- [Elkik'78] "rational" deforms
- [Kovács-Schwede'16] "Du Bois" deforms.

essential

- "klt" deforms if the total space is $\mathbb{Q}$-Gor.

(i.e. $^{\forall}(A, \mathfrak{m})$: norm, ess oft/$\mathbb{C}$ & $\mathbb{Q}$-Gor.

(total space) $A/(r)$: klt $\Rightarrow$ A: klt.)

- [Kawakita'07] "lc" deforms if the total space is $\mathbb{Q}$-Gor.

Fact 3

"klt" does not deform.

$\because$ R : as in Ex.2

$\rightsquigarrow$ $R/(e)$: $\mathbb{Q}$-Gor & klt.

R : NOT $\mathbb{Q}$-Gor $\rightsquigarrow$ NOT klt

Q How about dFH-klt ?

- a generalization of "klt" to non $\mathbb{Q}$-Gor.
- X : dFH-klt $\overset{\text{def}}{\iff}$ $\exists \Delta \geq 0$: $\mathbb{Q}$-div on X s.t. $\begin{cases} K_X + \Delta : \mathbb{Q}\text{-Car} \\ (X, \Delta) : \text{klt pair}. \end{cases}$

◎ klt = $\mathbb{Q}$-Gor + dFH-klt

"F-reg" does not deform"

Fact 3' (cf. [Singh '99])

"dFH-klt" does not deform.

(!) R : as in Ex. 2

⇝ $R/(e)$: Q-Gor & klt ⇒ dFH-klt

R : NOT dFH-klt

Q Is the sing of this R so bad?

Philosophycally, R may admit some good property like klt

(Since $R/(e)$ is klt)

Answer R is "valuatively klt".

(explain later)

- vklt … { • gen'ion of klt to non $\mathbb{Q}$-Gor case.
  • weaker than dFH-klt
  • klt = $\mathbb{Q}$-Gor + vklt
  • (Main Thm) "vklt" deforms.

47~

§3. Main results.

Def ([de Fernex - Hacon]).

X: norm. var/$\mathbb{C}$, $f: Y \to X$: proper, birational w/ Y: normal.

(1) D: Weil div on X (Not nec $\mathbb{Q}$-Car)

$\rightsquigarrow$ $f^{\natural}(D)$: Weil div on Y

s.t. $\mathcal{O}_Y(-f^{\natural}D) = (\mathcal{O}_X(-D) \cdot \mathcal{O}_Y)^{**}$

(equivalently, $I := \mathcal{O}_X(-D) \subseteq K(X)$: fractional ideal

$$f^{\natural}(D) = \sum_{\substack{E \subseteq Y \\ \text{prime div.}}} v_E(I) \cdot E$$

(v_E: valuation w.r.t. $\mathcal{O}_{Y,\eta_E}$ DVR))

(2) D: Weil div on X coefficient wise

$$\rightsquigarrow f^*_{\mathrm{Weil}}(D) := \lim_{m \to \infty} \frac{f^{\natural}(mD)}{m} = \inf_{m \in \mathbb{N}_{>0}} \frac{f^{\natural}(mD)}{m}.$$

: $\mathbb{R}$-Weil div on Y

(3) $K_{Y/X}^{-} := K_Y - f^*_{\mathrm{Weil}}(K_X)$: $\mathbb{R}$-Weil on Y

$K_{Y/X}^{+} := K_Y + f^*_{\mathrm{Weil}}(-K_X)$: $\mathbb{R}$-Weil on Y

Rmk

- $K_{Y/X}^{+} \geq K_{Y/X}^{-}$
- In general, $K_{Y/X}^{+} \neq K_{Y/X}^{-}$
- If X : $\mathbb{Q}$-Gor $\longrightarrow$ well def

  then $K_{Y/X}^{+} = K_{Y/X}^{-} = K_{Y/X}$

  $(:= K_Y - f^*(K_X))$

Def ([dFH], [Chiecchio-Urbinati])

X : norm var, $f: Y \longrightarrow X$: log res

(1) X is dFH-klt $\overset{\text{def}}{\Longleftrightarrow}$ $K_{Y/X}^{-} > -1$

(2) X is valuatively klt $\underset{\text{def}}{\Longleftrightarrow}$ $K_{Y/X}^{+} > -1$

(or klt^{+})

11

Rmk

- dFH-klt $\Rightarrow$ vklt
- klt $\Leftrightarrow$ $\mathbb{Q}$-Gor & dFH-klt

  $\Leftrightarrow$ $\mathbb{Q}$-Gor & vklt.
- X : dFH-klt $\Leftrightarrow$ $\exists \Delta \geq 0$, s.t $\begin{cases} \cdot K_X + \Delta : \mathbb{Q}\text{-Car} \\ \cdot (X, \Delta) : \text{klt pair} \end{cases}$

(Q $\exists$? similar characterization of vklt sing?)

Def

X : norm var. is weakly vklt.

$\underset{\text{def}}{\Leftrightarrow}$ $\forall \Delta \leq 0$: anti-eff $\mathbb{Q}$-Weil on X

if $K_X + \Delta$ is $\mathbb{Q}$-Car

then (X, Δ) is (sub)-klt pair.

(Δ may not be eff)

Lem.4 (vklt $\fallingdotseq$ weakly vklt)

X : norm var/$\mathbb{C}$

(1) X : vklt $\Rightarrow$ X : weakly vklt

(2) $\forall \Gamma \geq 0$: Weil div on X

s.t. $\mathrm{Supp}(\Gamma) \supseteq \mathrm{Sing}(X)$

Then

X : vklt $\Longleftrightarrow$ $0 < {}^{\exists}\varepsilon \in \mathbb{Q}$

the pair $(X, \varepsilon\Gamma)$

is weakly vklt pair.

Thm A ([_ - Takagi])

- "vklt" deforms.
- "weakly vklt" deforms.

(A : normal local

$A/(r)$: vklt

$\Rightarrow A$: vklt)

$\rightsquigarrow$ answer to the last question in the first half.

$$\text{klt} = \underline{\mathbb{Q}\text{-Gor}} \ \& \ \underline{\text{dFH-klt}}$$

NOT deform

$$\text{klt} = \mathbb{Q}\text{-Gor} \ \& \ \text{vklt}$$

deforms

Cor

$\underline{R}$ defined in Ex. 2 is vklt
weakly vklt

$\because$ $R/(e)$: $\mathbb{Q}$-Gor-klt $\Rightarrow$ vklt

$\Downarrow$ ThmA

R : vklt.

Cor. B

([Esnault-Viehweg '85] when $\dim T = 1$
[ST] for gen'l case)

$\pi : \mathcal{X} \to T$: flat proper morp w/ T : sm var/$\mathbb{C}$
$0 \in T$: closed pt

If · general fibre is $\mathbb{Q}$-Gor. &
· special fibre $\mathcal{X}_0 = \pi^{-1}(0)$ is klt.

then gen'l fibre is also klt

Note

In this Setting, $\mathcal{X}$ may not be $\mathbb{Q}$-Gor

prf of Cor. B

$\mathcal{X} \supseteq \mathcal{X}_0$: complete intersection

$\Downarrow$ repeated application of Thm A

$\mathcal{X}$ is vklt around $\mathcal{X}_0$

$\Downarrow$

gen'l fibre is vklt $\rightsquigarrow$ gen'l fibre klt $\square$

13

Cor. C $\left(\begin{array}{l} \dim(X_0) \geq 3 \rightsquigarrow \text{[Schlessinger '71]} \\ \dim(X_0) = 2 \rightsquigarrow \text{[EV'85] if } \dim T = 1 \\ \qquad\qquad\qquad\quad \text{[ST] in gen'l} \end{array}\right)$

$\pi : X \longrightarrow T$: flat proper morp w/ T : sm var/$\mathbb{C}$

$0 \in T$: closed pt

If special fibre is isolated quotient-sing

then so is a gen'l fibre.

(!) $\dim X_0 = 2$

(isolated) quot sing $\Leftarrow$ klt

apply Cor. B.

$\mathbb{Q}$ lc case

Thm. D ([ST])

$(R, \mathfrak{m})$: norm. ess of ft $/ \mathbb{C}$

$0 \neq r \in \mathfrak{m}$

If $R/(r)$: $\mathbb{Q}$-Gor, lc.

then R : valuatively lc.

($\Updownarrow$ def.

$f : Y \to X$: log res

X : vlc $\Leftrightarrow$ $K^{+}_{Y/X} \geq -1$)

Q What if $R/(r)$: vlc ?

(Not necessality $\mathbb{Q}$-Gor.)

([Kollár-Shepherd-Barron'88] if $\dim X_0 = 2$)

Cor. E

[Ishii '86] if X_0 : isolated. sing.
[ST] for gen'l case

$\pi : X \to T$: flat proper morp w/ T : sm curve ($\dim T = 1$), normal
$0 \in T$: closed pt

If
- general fibre is $\mathbb{Q}$-Gor. &
- special fibre X_0 is $\mathbb{Q}$-Gor, lc (resp. slc).

then gen'l fibre is also lc (resp. lc)

◎ Key pt for ThmA

Def

$m \geq 1$: integer

X : norm. var / $\mathbb{C}$ is m-weakly vklt

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall \Delta \leq 0$: $\mathbb{Q}$-div

if $m \cdot (K_X + \Delta)$: Cartier

then (X, Δ) : (sub) klt pair

weakly vklt $\Longleftrightarrow$ m-weakly vklt ($\forall m$)

- It is enough to show "m-weakly vklt"
  deform

Key Lem

$X = \mathrm{Spec}(R)$ norm, local, TFAE

(1) X is m-weakly vklt

(2) $\forall\, \mathfrak{b} \subseteq \mathcal{O}_X(mK_X)$: fract. ideal

$\Updownarrow$ $\quad \mathfrak{b} \subseteq \mathcal{J}(\omega_X, \mathfrak{b}^{\frac{m-1}{m}})$

multiplier submodule

(3) $\mathcal{O}_X(mK_X) = \mathcal{J}(\omega_X, \mathcal{O}_X(mK_X)^{\frac{m-1}{m}})$

Key Lem + "restriction thm" $\rightsquigarrow$ OK.

- vlc case ... we use "non lc ideal" instead of "mult. ideal"

- Thank you for listening!!

- Any comment or question is well come!!

  (I'm happy if you will kindly speak slowly due to my poor listening)

- $\underbrace{\mathcal{O}_X(mK_X)}_{(S_2)} \cdot \underbrace{R/(r)}_{\text{Car div}}$ is torsion free

  [Q pos. ch analog of vbKt?